MATH 4033 • Spring 2019 • Calculus on Manifolds
Problem Set #1 • Regular Surfaces • Due Date: 24/02/2019, 11:59PM

Instructions: *You can form a group of 1 to 3 students to work on the homework, and submit one copy of your solution to Canvas. All students in the same group will receive the same score. Grouping can be different in different homework. The problem sets are typically more challenging than many exercises in the lecture note. If you get stuck, it is best to first complete some related exercises in the lecture notes before trying the problem sets (as a corollary, start* ***early*** *working on the problem sets).*

1. (10 points) Suppose $\{v_1, \cdots, v_n\}$ are vectors in $\mathbb{R}^n$ and define an $n \times n$ matrix A whose (i,j)-th entry A_{ij} is the usual dot product $\langle v_i, v_j \rangle$. Show that A is positive-definite if and only if $\{v_1, \cdots, v_n\}$ are linearly independent.

   [Recall: A is positive-definite means $\sum_{i,j} A_{ij} x_i x_j \geq 0$ for any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and $\sum_{i,j} A_{ij} x_i x_j = 0$ if and only if $x = 0$.]

2. (30 points) Consider a regular surface Σ in $\mathbb{R}^3$. Denote $\nu : \Sigma \to \mathbb{R}^3$ be a smooth map of unit normal vectors to Σ. Let $f : \Sigma \to (0, \infty)$ be a C^∞ function on Σ, and consider the set:

$$\hat{\Sigma} := \{p + f(p)\nu(p) : p \in \Sigma\}.$$

   (a) Suppose $F(u_1, u_2)$ is a C^∞ local parametrization of Σ.

      i. Show that $\dfrac{\partial \nu}{\partial u_i}(p) \in T_p\Sigma$.

      ii. Consider the linear map $h : T_p\Sigma \to T_p\Sigma$ defined by:

$$h\left(\frac{\partial F}{\partial u_i}\right) := \frac{\partial \nu}{\partial u_i}$$

      and extends linearly to all of $T_p\Sigma$. Show that h is self-adjoint with respect to the standard dot product, i.e.

$$\langle h(X), Y \rangle = \langle X, h(Y) \rangle \text{ for any } X, Y \in T_p\Sigma.$$

   (b) From (a), we denote $h\left(\frac{\partial F}{\partial u_i}\right) = \sum_j h_i^j \frac{\partial F}{\partial u_j}$. Consider the map $\hat{F}$ defined on the same domain as F:

$$\hat{F}(u_1, u_2) := F(u_1, u_2) + f(F(u_1, u_2))\nu(F(u_1, u_2)).$$

   Suppose $\hat{F}$ is a homeomorphism onto its image. Show that if the linear map:

$$\mathrm{id} + fh : T_p\Sigma \to T_p\Sigma$$

   is invertible for any $p \in \Sigma$, then $\hat{F}$ is a C^∞ local parametrization of $\hat{\Sigma}$.

3. (25 points) Let A be a 3×3 symmetric real matrix. Consider the set

$$\Sigma := \{x \in \mathbb{R}^3 : x^T A x = 1\}.$$

   Here $x \in \mathbb{R}^3$ is regarded as a column vector.

   (a) Show that Σ is a regular surface whenever $\Sigma \neq \emptyset$.

   (b) Suppose A is semi-positive definite (i.e. eigenvalues are non-negative). Show that Σ (whenever non-empty) is diffeomorphic to either a sphere, a cylinder, or a pair of planes.

4. (35 points) Let F_+ and F_- be the stereographic parametrizations of the unit sphere $\mathbb{S}^2$ as discussed in Example 1.5 of the lecture notes. Here we regard $\mathbb{C}$ as $\mathbb{R}^2$ by identifying $z = u + iv \in \mathbb{C}$ with $(u, v) \in \mathbb{R}^2$. Then, $F_+ : \mathbb{C} \to \mathbb{S}^2 \backslash \{(0, 0, 1)\}$ and its inverse can be expressed as:

$$F_+(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \qquad F_+^{-1}(x_1, x_2, x_3) = \frac{x_1 + x_2 i}{1 - x_3}$$

Here we use (x_1, x_2, x_3) for coordinates of $\mathbb{R}^3$ instead of (x, y, z) to avoid notation conflicts.

(a) Consider the south-pole stereographic parametrization $F_- : \mathbb{C} \to \mathbb{S}^2 \backslash \{(0, 0, -1)\}$. Find the explicit expressions of $F_-(z)$, where $z \in \mathbb{C}$, and $F_-^{-1}(x_1, x_2, x_3)$, where $(x_1, x_2, x_3) \in \mathbb{S}^2 \backslash \{(0, 0, -1)\}$.

(b) Verify that $F_-^{-1} \circ F_+(z) = \frac{1}{\bar{z}}$ and $F_+^{-1} \circ F_-(z) = \frac{1}{\bar{z}}$. Hence, by modifying one of F_+ and F_-, find a pair of C^∞ local parametrizations $\{G_+, G_-\}$ of $\mathbb{S}^2$ such that $G_-^{-1} \circ G_+(z) = \frac{1}{z}$ and $G_+^{-1} \circ G_-(z) = \frac{1}{z}$

(c) Consider the complex-valued function $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C} \backslash \{0\}$ such that $\alpha\delta \neq \beta\gamma$. Define a map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by:

$$\Phi(p) := \begin{cases} F_+(\alpha/\gamma) & \text{if } p = (0, 0, 1) \\ (0, 0, 1) & \text{if } p = F_+(-\delta/\gamma) \\ F_+ \circ f \circ F_+^{-1}(p) & \text{otherwise} \end{cases}$$

It can be checked that Φ is bijective (no need to show the detail).

i. Find an explicit expression of each of the following:

$$G_+^{-1} \circ \Phi \circ G_+(z) \qquad G_-^{-1} \circ \Phi \circ G_+(z) \qquad G_+^{-1} \circ \Phi \circ G_-(z) \qquad G_-^{-1} \circ \Phi \circ G_-(z)$$

State the domain of each of them.

ii. Show that Φ is smooth at the point $(0, 0, 1)$.

iii. Express the tangent map Φ_* at $(0, 0, 1)$ in matrix representation and show that it is invertible.