MATH 2043 • Spring 2019 • Honors Analysis I
Tutorial #8

1. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function which is continuous on (a, b). Show that f is Riemann integrable on $[a, b]$ using each of the following methods:
   (a) Riemann criterion (Theorem 4.1.3)
   (b) definition of Darboux integrals
   (c) Lebesgue Theorem for Riemann integrals

2. (Exercise 4.8) Suppose f is Riemann integrable on $[a, b]$, $P = \{x_i\}_{i=0}^n$ is a partition of $[a, b]$, and for any i we have
$$\inf_{[x_{i-1},x_i]} f \le \phi_i \le \sup_{[x_{i-1},x_i]} f$$
where ϕ_i is a constant depending only on i. Show that
$$\lim_{\|P\|\to 0} \sum_{i=1}^n \phi_i(x_i - x_{i-1}) = \int_a^b f(x)\,dx.$$

3. Using any method discussed in class, prove that if $f : [0, 1] \to [0, 1]$ is Riemann integrable, then $h : [0, 1] \to [0, 1]$ defined by $h(x) = f(\sqrt{x})$ is also Riemann integrable.

1 (a) Let $|f(x)| \le B$ $\forall x \in [a,b]$.

$\forall \varepsilon > 0$, first choose $\delta' > 0$ s.t. $2\delta' \cdot 2B < \frac{\varepsilon}{3}$.

then consider $[a+\delta', b-\delta'] \subset (a,b)$.

f is cts on $[a+\delta', b-\delta'] \Rightarrow f$ is uniformly cts on $[a+\delta', b-\delta']$.

$\therefore \exists \delta'' > 0$ s.t. $x, y \in [a+\delta', b-\delta']$ and $|x-y| < \delta''$
$$\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3(b-a)}.$$

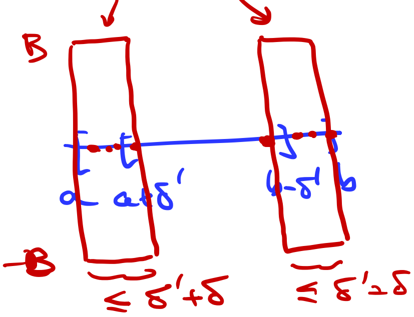
Take $\delta = \min\{\delta', \delta''\}$

then if $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a,b]$ such that $\|P\| < \delta$, then

$$\omega(P,\delta) = \Bigg(\sum_{[x_{i-1},x_i] \subset [a+\delta', b-\delta']} + \sum_{[x_{i-1},x_i] \not\subset [a+\delta', b-\delta']}\Bigg) \omega_{[x_{i-1},x_i]} \cdot (x_i - x_{i-1})$$

$\downarrow$ (as $|x_i - x_{i-1}| \le \delta''$)

$$\le \frac{\varepsilon}{3(b-a)} \cdot \sum_{\substack{[x_{i-1},x_i] \\ \subset [a+\delta', b-\delta']}} (x_i - x_{i-1}) + \underbrace{\left(2B \cdot (\delta+\delta')\right) \cdot 2}$$



$$\le \frac{\varepsilon}{3(b-a)} \cdot (b-a) + 8B\delta'$$

$$< \frac{\varepsilon}{3} + 8B \cdot \frac{1}{4B} \cdot \frac{\varepsilon}{3} = \varepsilon \quad \blacksquare$$

$\therefore$ By Riem. criterion, f is integrable on $[a,b]$.

(b) contained in (a).

(c) $\underset{\text{measure zero}}{\underbrace{S_f}} \subset \underset{\text{measure zero}}{\underbrace{\{a,b\}}}$ as f can only be discts at a, b.

2. $f:[a,b]\to\mathbb{R}$ is Riem. integrable
$\therefore \forall \varepsilon>0, \exists \delta>0$ s.t. $\|P\|<\delta \Rightarrow \left|S(P,\{y_i\},f)-\int_a^b f\right|<\varepsilon$
$\forall$ sample points $\{y_i\}$.

Denote $P=\{a=x_0<x_1\le\dots\le x_n=b\}$

regard $S(P,\{y_i\},f)$ as a function on $(y_1,\dots,y_n)$, (with P fixed)

then

$$\sup_{y_n\in[x_{n-1},x_n]}\cdots\sup_{y_1\in[x_0,x_1]}\underbrace{\left(f(y_1)\Delta x_1+\dots+f(y_n)\Delta x_n\right)}_{<\int_a^b f+\varepsilon}\le\int_a^b f+\varepsilon$$

$$\Rightarrow \sup_{[x_0,x_1]} f\cdot\Delta x_1+\dots+\sup_{[x_{n-1},x_n]} f\cdot\Delta x_n\le\int_a^b f+\varepsilon.$$

$$\Rightarrow \phi_1\Delta x_1+\dots+\phi_n\Delta x_n\le\sum_{i=1}^n\sup_{[x_{i-1},x_i]} f\cdot\Delta x_i\le\int_a^b f+\varepsilon.$$

Similarly, $\phi_1\Delta x_1+\dots+\phi_n\Delta x_n\ge\sum_{i=1}^n\inf_{[x_{i-1},x_i]} f\cdot\Delta x_i\ge\int_a^b f-\varepsilon.$

$$\Rightarrow \left|\sum_{i=1}^n\phi_i\Delta x_i-\int_a^b f\right|\le\varepsilon \qquad \boxtimes$$

3. Observe that $S_h\subset S_f^2=\{a^2: a\in S_f\}$.
Need to show $m(S_f^2)=0$.
Note that $S_f=(S_f\setminus\{0\})\cup\{0\}$, and $S_f^2\subset(S_f\setminus\{0\})^2\cup\{0\}$.
It suffices to show $(S_f\setminus\{0\})^2$ has measure zero.
Known: $m(S_f\setminus\{0\})=0$ (as f is Riem. integrable)
$\forall\varepsilon>0$, $\exists\{(a_i,b_i)\}_{i=1}^\infty$ s.t. $\sum_i(b_i-a_i)<\varepsilon/2$
and $\bigcup_{i=1}^\infty(a_i,b_i)\supset S_f\setminus\{0\}$. As $S_f\setminus\{0\}\subset(0,1]$,
we can assume $a_i\ge0$ $\forall i$ (by replacing (a_i,b_i) by $(0,b_i)$ if a_i is negative).

then $\bigcup_{i=1}^{\infty} (a_i^2, b_i^2) \supset (S_f \setminus \{0\})^2$.

and $\sum_i (b_i^2 - a_i^2) = \sum_i (b_i - a_i)\underbrace{(b_i + a_i)}_{\leq 2} \leq 2\varepsilon$.

$\therefore m((S_f \setminus \{0\})^2) = 0$

$$\Rightarrow \quad \underset{m=0 \Leftarrow}{S_h} \subset \underbrace{S_f^2}_{m=0} \underset{\Leftarrow}{\subset} \underbrace{(S_f \setminus \{0\})^2}_{m=0} \cup \underbrace{\{0\}}_{m=0}$$

$\Downarrow$

$\therefore$ h is Riem. integrable on $[0,1]$.