MATH 2043 • Spring 2019 • Honors Analysis I
Problem Set #2 • Due Date: 15/03/2019, 11:59PM

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1. Let $\alpha > 0$ be an irrational number.
   (a) Show that the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by $f(m,n) = m\alpha + n$ is injective.
   (b) Define $S := \{m\alpha + n : m, n \in \mathbb{Z}\}$. Show that for any $x \in \mathbb{R}$ and $\varepsilon > 0$, there exist infinitely many elements $s \in S$ such that $|s - x| < \varepsilon$.
   Hint: use similar technique as in the $\{\sin(n)\}_{n=1}^{\infty}$ example. Consider the complex numbers $\left\{e^{2\pi i(n\alpha - [n\alpha])}\right\}$ where $n \in \mathbb{Z}$.

2. (a) Let S be a subset of $\mathbb{R}$ such that $S \neq \mathbb{R}$ and $S \neq \emptyset$. Prove that the following are equivalent:
   i. $\mathbb{R}\backslash S$ is open.
   ii. Whenever $\{x_n\}$ is a sequence in S such that $x_n \to L$ as $n \to \infty$, then we have $L \in S$.

   [Hence, we can take either one statement above as the definition of "S is closed".]
   (b) Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Show that the limit set $\text{LIM}\{x_n\}$ is closed.

3. (Exercise 2.32) Consider a function $f : \mathbb{R} \to \mathbb{R}$, and given that for any $\varepsilon > 0$, the set $S_\varepsilon := \{x \in \mathbb{R} : |f(x)| \geq \varepsilon\}$ is finite (remark: $\emptyset$ is considered as a finite set). Prove that $\lim_{x \to a} f(x) = 0$ for any $a \in \mathbb{R}$.

4. Consider a bounded function $f : \mathbb{R} \to \mathbb{R}$ and a fixed $a \in \mathbb{R}$. Define for each $r > 0$:

   $$m(r) := \inf\{f(x) : x \in (a-r, a+r)\backslash\{a\}\} \text{ and } M(r) := \sup\{f(x) : x \in (a-r, a+r)\backslash\{a\}\}.$$

   (a) Suppose (just for this part) that $a = 0$ and

   $$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   Find $m(r)$ and $M(r)$ for any $r > 0$. Justify your answers.
   (b) Why do $\lim_{r \to 0^+} m(r)$ and $\lim_{r \to 0^+} M(r)$ always exist?
   (c) Show that the following are equivalent:
   i. $\lim_{r \to 0^+} m(r) = \lim_{r \to 0^+} M(r)$.
   ii. $\lim_{x \to a} f(x)$ exists.

   (d) Define:

   $$S = \left\{L \in \mathbb{R} : \exists\{x_n\}_{n=1}^{\infty} \text{ in } \mathbb{R}\backslash\{a\} \text{ such that } \lim_{n \to \infty} x_n = a \text{ and } \lim_{n \to \infty} f(x_n) = L\right\}.$$

   Show that

   $$\sup_{r>0} m(r) = \inf S \text{ and } \inf_{r>0} M(r) = \sup S.$$

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that the map $n \mapsto a_n$ is injective. Define a function $f : \mathbb{R} \to \mathbb{R}$ by:

   $$f(x) := \sum_{\{n \in \mathbb{N} : a_n < x\}} \frac{1}{2^n}.$$

(a) Find the explicit formula (without summation) of the function $f(x)$ in the two cases (i) $a_n = n$; (ii) $a_n = \frac{(-1)^n}{n}$. In each case, write down the set of all discontinuities S_f where

$$S_f := \{c \in \mathbb{R} : f \text{ is not continuous at } c\}.$$

(b) Show that (in general) f is continuous at any $c \neq a_n$, and not continuous at any a_n. In other words $S_f = \{a_n : n \in \mathbb{N}\}$.

6. Consider a function $f : \mathbb{R} \to \mathbb{R}$. For each $a \in \mathbb{R}$ and $\delta > 0$, we define

$$\omega_f(a, \delta) := \sup \left\{ |f(x) - f(y)| : x, y \in (a - \delta, a + \delta) \right\}$$
$$\omega_f(a) := \inf_{\delta > 0} \omega_f(a, \delta)$$

In layman terms, $\omega_f(a, \delta)$ measures how f fluctuates in the interval $(a - \delta, a + \delta)$, and $\omega_f(a)$ is its infinitesimal fluctuation near a.

(a) Show that the set of discontinuities of f can be expressed as:

$$S_f = \bigcup_{n=1}^{\infty} \left\{ a \in \mathbb{R} : \omega_f(a) \geq \frac{1}{n} \right\}.$$

(b) Show also that for any $n \in \mathbb{N}$ the set

$$\Omega_n := \left\{ a \in \mathbb{R} : \omega_f(a) \geq \frac{1}{n} \right\}$$

is closed. (In real analysis jargon, we then say S_f is an F_σ set).

7. (Exercise 2.43) Prove that any continuous $f : \mathbb{R} \to \mathbb{R}$ that is periodic (i.e. $\exists T > 0$ such that $f(x + T) = f(x)$ for any $x \in \mathbb{R}$) must be uniformly continuous on $\mathbb{R}$.

8. (Exercise 2.46) Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a closed and bounded interval $[a, b]$. Given any $\varepsilon > 0$, we define recursively

$$c_0 := a$$
$$c_n := \sup\{c \in [a, b] : |f(x) - f(c_{n-1})| < \varepsilon \text{ for any } x \in [c_{n-1}, c]\} \quad \text{for } n \geq 1.$$

(a) Show that there exists $N \in \mathbb{N}$ such that $c_n = b$ for any $n \geq N$.

(b) Hence, given another proof of Theorem 2.4.1 (that continuous functions on a closed and bounded interval must be uniformly continuous on that interval).

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$|f(x) - f(y)| \geq |x - y| \text{ for any } x, y \in \mathbb{R}.$$

(a) Prove that any $c \in \mathbb{R}$ is bounded between $f(A)$ and $f(-A)$ where $A = |c - f(0)|$.

(b) Hence, show that f is bijective.

10. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for any $c \in \mathbb{R}$ the set $f^{-1}(c) = \{x \in \mathbb{R} : f(x) = c\}$ has exactly two elements.