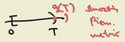
Sections 12 - 14

$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ on $t\in[0,T)$ ← maximal time

then $\boxed{\lim_{t\to T} \sup_M |Rm| = \infty.}$ $T<\infty$.

Sketch of proof: Suppose $\sup_M |Rm|_t \le C$ $\forall t\in[0,T)$

→ then show $g(t) \xrightarrow{C^\infty} g(T)$ as $t\to T^-$.



$\log g_t(V,V) - \log g_0(V,V)$

$= \int_0^t \frac{\partial}{\partial \tau} \log g_\tau(V,V)\, d\tau = \int_0^t \frac{1}{g_\tau(V,V)} \left(-2Ric_{g(\tau)}(V,V)\right) d\tau$ $\quad |V|^2$

$= \int_s^t -2Ric_{g(\tau)}\left(\frac{V}{|V|}, \frac{V}{|V|}\right) d\tau \le C|t-s|.$

- $R_{max} \nearrow \infty$. ✓ (Hamilton). as $t\to T$
  sum Ric

$\frac{\partial R_{min}}{\partial t} = \underbrace{\Delta R_{min}}_{\oplus} + \underbrace{2|Ric|^2}_{\ge \frac{2}{3}R_{min}} \Rightarrow R_{min} \ge \frac{C}{\hat{C} - \frac{2}{3}t}.$

$T \le \frac{\hat{C}}{\frac{2}{3}}$

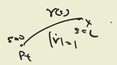
- $\boxed{\lim_{t\to T} \frac{R_{min}}{R_{max}} = 1.}$

$Ric \ge \varepsilon R g.$

$\forall \eta > 0$, $\exists C(\eta)$ s.t.

$|\nabla R|^2 \le \frac{1}{4}\eta^4 R^3 + C(\eta) \le$

$\Rightarrow |\nabla R| \le \frac{1}{2}\eta^2 R^{3/2} + \tilde{C}(\eta)$



→ $|R(p_t) - R(x)| = \left|\int_0^L \frac{d}{ds} R(\gamma(s))\, ds\right|$

$R_{max}(t)$ $\quad \le \int_0^L |\langle \nabla R, \dot\gamma\rangle|\, ds$

$\le L \cdot \max|\nabla R| \le L\left(\frac{1}{2}\eta^2 R^{3/2} + C(\eta)\right)$

$\le L\left(\frac{1}{2}\eta^2 R_{max}^{3/2} + C(\eta)\right)$

$\le L\left(\frac{1}{2}\eta^2 R_{max}^{3/2} + \frac{1}{2}\eta^2 R_{max}^{3/2}\right)$

$= L\eta^2 R_{max}^{3/2}$ when $t>\theta$.

$\left|\frac{R_{min}}{R_{max}} - 1\right| < \eta.$

Known: $R_{max} \nearrow \infty$.

$\exists \theta < T$ $\quad \theta \quad t \quad T$

$x \in B(p_t, \frac{1}{\eta R_{max}^{1/2}})$ → $\le \eta R_{max}$ ⇒ $R \ge (1-\eta)R_{max}$ on B.

$Ric \ge \varepsilon R g$ on M

$\ge \varepsilon(1-\eta) R_{max}\, g$ on $B(p_t, \frac{1}{\eta R_{max}^{1/2}})$

$\text{diam}\, B \overset{?}{\le} \frac{\pi}{\sqrt{\varepsilon(1-\eta)R_{max}}}$

$\frac{1}{\eta R_{max}^{1/2}} \ge \frac{\pi}{2\sqrt{\varepsilon(1-\eta)R_{max}}}$

$\Leftrightarrow \frac{1}{\eta} \ge \frac{\pi}{2\sqrt{\varepsilon(1-\eta)}}$

$\eta\to 0 \downarrow \infty$ $\quad \downarrow \eta\to 0$ $\quad \frac{\pi}{2\sqrt{\varepsilon}} > 0.$

$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ $\quad (n=3)$.

- $\frac{\partial}{\partial t} Rm$, $\frac{\partial}{\partial t} Ric$, $\frac{\partial}{\partial t} R$
- $Ric \ge \varepsilon R g > 0$. ✓ (tensor MP)
- $\left|Ric - \frac{R}{3} g\right| \le C R^{1-\delta}$ → $\left|\tilde{Ric} - \frac{\tilde{R}}{3}\tilde{g}\right|^2_{\tilde{g}} = \tilde{g}^{ij}\tilde{g}^{kl}\left(\tilde{R}_{ik} - \frac{\tilde{R}}{3}\tilde{g}_{ik}\right)(\cdot\cdot)$
- $|\nabla R|^2 \le \eta R^3 + C(\eta)$. $\forall \eta > 0$
- $\sup|Rm| \nearrow \infty$ as $t\to T$.
- $\frac{R_{min}}{R_{max}} \sim 1$ ✓ as $t\to T$

$\tilde{g} = \psi g$

$Ric_g = Ric_{\tilde{g}}$

$R_{\tilde{g}}(\tilde{g}) = \frac{1}{\psi} Ric_g \cdot \psi g = R_g g.$

$\tilde{Ric} \ge \varepsilon \tilde{R} \tilde{g}.$

$= \frac{1}{\psi(t)^2} g^{ij} g^{kl}\left(\tilde{R}_{ik} - \frac{\tilde{R}}{3}\tilde{g}_{ik}\right)(\cdot\cdot)$

$= \frac{1}{\psi(t)^2} \left|Ric - \frac{R}{3} g\right|^2$

$\le \frac{1}{\psi(t)^2} \cdot C R^{2-2\delta}$

$= \frac{1}{\psi(t)^2} \cdot C\, \psi(t)^{2-2\delta} \tilde{R}^{2-2\delta}$

$= \frac{C}{\psi(t)^{2\delta}} \tilde{R}^{2-2\delta}$

$\frac{\tilde{R}_{min}}{\tilde{R}_{max}} \to 1$ as $\tilde{t}\to\tilde{T}$. ✓

$\tilde{R}ic \ge \varepsilon \tilde{R} \tilde{g} > \varepsilon \tilde{R}_{min}(\tilde{t})\, \tilde{g}(\tilde{t})$ on M.

$\Rightarrow \tilde{d} \le \frac{\pi}{\sqrt{\varepsilon \tilde{R}_{min}}}$

$\tilde{V}(0) \ge \tilde{V} \le C\tilde{d}^3 \le C \cdot \frac{\pi^3}{\varepsilon^{\frac{3}{2}} \tilde{R}_{min}^{\frac{3}{2}}}$

$\Rightarrow \tilde{R}_{min}(t) \le \hat{C}$.

$\frac{R_{min}(t)}{R_{max}(t)} \to 1$ $\quad \tilde{R}_{max}(t) \le \hat{C}$

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g_0 = round metric of S^3.

Guess $g(t) = c(t)\, g_0$.

$\frac{\partial g(t)}{\partial t} = c'(t)\, g_0$

$-2Ric(g(t)) = -2Ric(g_0)$

$= \frac{-2\lambda g_0}{\theta}$

$\begin{cases} c'(t) = -2\lambda \\ c(0) = 1 \end{cases} \Rightarrow c(t) = 1 - 2\lambda t.$

$\nabla^{cg} = \nabla^g$ $\quad \Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \cdots)$

$Rm^{(1,1)}_{cg} = \nabla^{cg}_X \nabla^{cg}_Y \cdots$

$= Rm^{(1,1)}_g$

$Rm^{(4,0)}_{cg} = c\, Rm^{(4,0)}_g$ $\quad Ric(cg) = Ric(g)$.

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normalized Ricci flow

$\to \frac{\partial g}{\partial t} = -2Ric + \frac{2}{3} r g$ $\quad r(t) = \frac{1}{Vol(g(t))} \int_M R_{g(t)}\, d\mu_{g(t)}$.

① $\frac{\partial}{\partial t}\sqrt{\det g} = \frac{1}{2} tr_g\left(-2Ric + \frac{2}{3} r g\right)\sqrt{\det g}$ $\quad g^{ij} g_{ij}$

$= \frac{1}{2}\left(-2R + \frac{2}{3} r \cdot 3\right)\sqrt{\det g}$

$= (-R + r)\sqrt{\det g}$

$\frac{d}{dt}\int_M d\mu_{g(t)} = \int_M (-R + r)\, d\mu_{g(t)} = -\int_M R\, d\mu_{g(t)} + r\int_M d\mu_{g(t)} = 0$

② $\tilde{g}(\tilde{t})$ $\quad g(t)$

$\partial_{\tilde{t}}\tilde{g} = -2\tilde{Ric} + \frac{2}{3}\tilde{r}\tilde{g}$ $\quad \partial_t g = -2Ric$.

let $\tilde{g}(\tilde{t}) = \psi(t)\, g(t)$

given $\partial_t g = -2Ric$

$\frac{\partial \tilde{g}}{\partial \tilde{t}} = \left(\psi'(t) g(t) + \psi(t)(-2Ric_{g(t)})\right)\frac{dt}{d\tilde{t}}$

$= \psi'(t)\frac{dt}{d\tilde{t}} g(t) - 2\psi(t)\frac{dt}{d\tilde{t}} Ric_{g(t)}$

$= \underbrace{\psi'(t)\frac{dt}{d\tilde{t}} \cdot \frac{1}{\psi(t)}}_{= \frac{2}{3}\tilde{r}} \tilde{g}(\tilde{t}) - 2\underbrace{\psi(t)\frac{dt}{d\tilde{t}}}_{=1} Ric_{\tilde{g}(\tilde{t})}$

$\frac{d\tilde{t}}{dt} = \psi(t)$ ✓ $\quad \left(\frac{dt}{d\tilde{t}}\right)\frac{d}{dt}\log\psi(t) = \frac{2}{3}\tilde{r}$.

$\frac{d}{d\tilde{t}}\log\psi(t) = \frac{2}{3}\psi(t)\,\tilde{r}$ ✓

$\frac{R_{min}}{R_{max}} \to 1$ as $t\to T$. ✓

$\frac{\partial}{\partial t} Vol(g(t)) = \frac{\partial}{\partial t}\int_\Sigma d\mu_{g(t)}$

$= \int_\Sigma \frac{1}{2} tr_g(-2Ric)\, d\mu$

$= -\int_\Sigma R\, d\mu \le -\int_\Sigma R_{min}(t)\, d\mu$

$= -\underbrace{R_{min}(t)}_{\to\infty} Vol(g(t))$.

$\Rightarrow Vol\, g(t) \to 0$

$\max_M |Rm| \to \infty$. as $t\to T$.

$R_{max} \to \infty$.

$R_{min} \to \infty$.

$\frac{\partial}{\partial t} g(t) = \psi(t)\, g(t)$.

constant volume. $\quad \infty \quad Vol(g(t)) \to 0$.

Only thing left: higher-order estimates.

$|\tilde{R}m| \le C$. $\quad \left(\int |\tilde{\nabla}^k \tilde{R}m|^p\right)^{\frac{1}{p}} \le C_{kp}$.

$\Rightarrow |\tilde{\nabla}^k \tilde{R}m| \le C_{kp}$.

$\frac{\partial}{\partial t} g = -2Ric$ $\quad \underset{n\ge 4}{n=3}$ Hamilton. Wan-Xiong Shi.

$|\partial^{(k)} g(t) - \partial^{(k)} g(s)| \le \left\|\frac{\partial}{\partial t}\partial^{(k)} g(t)\right\|_\infty |t-s|$.

$= \left\|\partial^{(k)}\frac{\partial}{\partial t} g(t)\right\|_\infty |t-s|$

$= \|-2\partial^{(k)} Ric\|_\infty |t-s|$.

$\le C\|2\partial^{(k)} Rm\|_\infty |t-s|$

(M, J) $\quad J^2 = -id$.

$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ $\quad \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} \overset{?}{\pm} \sqrt{-1}\frac{\partial}{\partial y_i}\right)$.

$z_i = x_i + \sqrt{-1} y_i$.

g Riem. metric $\quad g(JX, JY) = g(X, Y)$ Hermitian

$\omega(X,Y) = g(JX, Y)$ ← 2-form.

$d\omega = 0 \overset{def}{\Leftrightarrow} g$ Kähler.

$\Leftrightarrow \nabla J = 0 \Leftrightarrow \nabla\omega = 0$. $\quad d\omega = 0 \Leftrightarrow \frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}$

$\Rightarrow R_{i\bar{j}} = Ric\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = -\partial_i\partial_{\bar{j}} \log\det[g_{kl}]$

$\tilde{g}$ Kähler metric

$\tilde{\omega}$

$\tilde{R}_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log\det(\tilde{g}_{k\bar{l}})$

$\tilde{R}_{i\bar{j}} - R_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log\left(\frac{\det\sqrt{\tilde{g}}}{\det g}\right) = \frac{\det\tilde{g}}{\det g}$ well-defined on M

global.

$[\tilde{g}] = A^T[\tilde{g}']A$

$[g] = A^T[g']A$

$ric(X,Y) = Ric(JX, Y)$

$(\tilde{ric} - ric)_{ij} = \partial_i\partial_{\bar{j}} f$.

$\tilde{ric} - ric = d(\bar{\partial} f)$.

$[\tilde{ric}] = [ric]$.

$\frac{[ric]}{2\pi} =: c_1(M)$

$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}_i} d\bar{z}^i$ $\quad \partial f = \frac{\partial f}{\partial z_i} dz^i$

$\partial + \bar{\partial} = d$.

$d(\bar{\partial}) = (\partial + \bar{\partial})\bar{\partial}$

$= \partial\bar{\partial} + \underbrace{\bar{\partial}\bar{\partial}}_{=0}$.

$d^2 = 0 \Leftrightarrow (\partial + \bar{\partial})^2 = 0$

$\Rightarrow \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}\bar{\partial} = 0$.

$d^2(dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n)$