ω, g. $g_{i\bar j}$

$R_{i\bar j} = -\partial_i\partial_{\bar j}\log\det[g]$. $\omega = \sqrt{-1}\, g_{i\bar j}\, dz^i\wedge d\bar z^j$

$\rho := Ric(J\cdot,\cdot)$ $\omega = g(J\cdot,\cdot)$

$= \sqrt{-1}R_{i\bar j}\, dz^i\wedge d\bar z^j$

$\tilde\omega, \tilde g, \tilde\rho$

$\rho-\tilde\rho = \sqrt{-1}\partial\bar\partial\log\frac{\det[\tilde g]}{\det[g]} = \sqrt{-1}\partial\bar\partial f = d(\sqrt{-1}\bar\partial f)$. global.

$\Rightarrow [\rho]=[\tilde\rho]=2\pi c_1(M)$.

When Kähler-Einstein metrics exist?

· If $\exists\,\omega$ s.t. $Ric(\omega) = \lambda\omega$, $\lambda\in\mathbb{R}$,

then $c_1(M) = [Ric(\omega)] = [\lambda\omega] = \lambda[\omega]$.

nec. $c_1(M)>0 \overset{def}{\iff} \exists$ a positive-def. ω s.t. $\omega\in c_1(M)$. (1,1)-form $dz^i\wedge d\bar z^j$

$c_1(M)=0 \overset{def}{\iff} 0\in c_1(M)$.

$c_1(M)<0 \overset{def}{\iff} \exists$ a positive-def ω s.t. $-\omega\in c_1(M)$.

· sufficient?

$c_1(M)<0$ ✓ Aubin, Yau. early 70's.

$c_1(M)=0$ ✓ Calabi conjecture, solved by Yau in 1976.

$c_1(M)>0$ (2012) Chen-Donaldson-Sun. Tian

$Ric(\omega) = \lambda\omega$. $g_{i\bar j}$. $R_{i\bar j} = -\partial_i\partial_{\bar j}\log\det g$

$-\partial_i\partial_{\bar j}\log\det(g) = \lambda g_{i\bar j}$

Search? restrict on those in the same deRham class as $[\omega_0]$.

Assumption: $c_1(M) = [\lambda\omega_0]$

$[Ric(\omega_0)] = [Ric(\omega)] = [\lambda\omega]$. if

$[\omega]=[\omega_0] \iff \omega-\omega_0 = \sqrt{-1}\partial\bar\partial\varphi$? ($\partial\bar\partial$-lemma) d-exact $\iff$ $\partial\bar\partial$-exact.

Reduce the problem to:

$-\partial_i\partial_{\bar j}\log\det(\underbrace{\omega_0+\sqrt{-1}\partial\bar\partial\varphi}_{\omega}) = \lambda((g_0)_{i\bar j}+\partial_i\partial_{\bar j}\varphi)$.

$-\partial_i\partial_{\bar j}\log\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi)$

$= (R_0)_{i\bar j} - \partial_i\partial_{\bar j}F + \lambda\partial_i\partial_{\bar j}\varphi$. $[Ric(\omega)] = [Ric(\omega_0)] = \lambda[\omega_0]$

$(R_0)_{i\bar j} = \lambda(g_0)_{i\bar j} + \partial_i\partial_{\bar j}F$

$-\partial_i\partial_{\bar j}\log\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi)$

$= -\partial_i\partial_{\bar j}\log\det(\omega_0) - \partial_i\partial_{\bar j}F + \lambda\partial_i\partial_{\bar j}\varphi$.

$\Rightarrow -\log\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi)$

$= -\log\det(\omega_0) - F + \lambda\varphi$.

If $\partial_i\partial_{\bar j}f=0$ $\Rightarrow \Delta f = g^{i\bar j}\nabla_i\nabla_{\bar j}f = g^{i\bar j}\partial_i\partial_{\bar j}f = 0$

$\Rightarrow$ (∗) $\boxed{\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi) = e^{F-\lambda\varphi}\det(\omega_0)}$ $\nabla_i\nabla_{\bar j} = \partial_i\partial_{\bar j} - \cancel{\Gamma}$

complex Monge-Ampère equation. $f=c$.

$\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi) = e^{F-\lambda\varphi}\det(\omega_0)$. — $(*)_1$

$(*)_t$ $\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi) = e^{tF-\lambda\varphi}\det(\omega_0)$.

$(*)_0$ has solution.

$S := \{t\in[0,1] : (*)_t \text{ is solvable}\}$.

Key: show S is both open and closed.

$\varphi_1\ \varphi_2\ \varphi_3 \cdots$; $t_1\ t_2 \cdots t_\infty$; $t_i\in S$. $\varphi_i \xrightarrow{C^\infty}\varphi_\infty$ satisfies $(*)_{t_\infty}$. $t_0\in S$

Given $t_0\in S$, $\Rightarrow \exists\varphi_0$ s.t. $\det(\omega_0+i\partial\bar\partial\varphi_0) = e^{t_0F-\lambda\varphi_0}\det\omega_0$

$\Phi(\varphi,t) := \log\det(\omega_0+i\partial\bar\partial\varphi) - \log\det(\omega_0) - tF + \lambda\varphi$

$\exists\varphi$ s.t. $\Phi(\varphi,t)=0 \iff t\in S$.

WANT: when $t\to t_0$, $\exists\varphi_t$ s.t. $\boxed{\Phi(\varphi_t,t)=0}$.

$f(x,y)=0$, $f(x(y),y)=0$

$\frac{\partial\Phi(\varphi_t+s\psi,t)}{\partial s}\Big|_{s=0}$

$= \Delta\psi + \lambda\psi$. $\nabla f \perp$ level set.

Need $\psi \mapsto \Delta\psi+\lambda\psi = 0 \overset{?}{\Rightarrow} \psi=0$ ($\in$ Ker) $\frac{\partial f}{\partial x}\neq 0$. $\nabla f = f_x i + f_y j$

$\int_M \psi\Delta\psi + \lambda\psi^2\, d\mu = 0 \Rightarrow \int_M \psi\nabla^i\nabla_i\psi + \lambda\psi^2 = 0$

$\Rightarrow -\int_M|\nabla\psi|^2 + \int_M\lambda\psi^2 = 0$.

$\Rightarrow 0\le\int_M|\nabla\psi|^2 = \lambda\int_M\psi^2$ ⊖

$\Rightarrow \int_M\psi^2\le0 \Rightarrow \psi=0$.

$C^{k,\alpha}$

$a^{ij}\partial_i\partial_j f + b^i\partial_i f + cf = h$. (+)

Schauder's estimates. If $a^{ij}\in C^{k,\alpha}$, $b^i\in C^{k,\alpha}$, $c\in C^{k,\alpha}$ and $h\in C^0$, $|h|<C$.

$\alpha\in(0,1)$ $\|\ \|_{C^{k,\alpha}} = \|\ \|_{C^k} + \sup_{x\neq y}\frac{|D^kf(x)-D^kf(y)|}{|x-y|^\alpha}$ $\Rightarrow |f|_{C^{k+2,\alpha}}\le C$

$\varphi\in C^{2,\alpha}$

$\log\det(\omega_0+\sqrt{-1}\partial\bar\partial\varphi) = \log\det(\omega_0) + tF - \lambda\varphi$.

∂_k: $tr(\partial_k\omega_0 + \sqrt{-1}\partial\bar\partial(\partial_k\varphi)) = \partial_k\log\det(\omega_0) + t\partial_kF - \lambda\partial_k\varphi$.

$\Rightarrow \Delta_\omega(\partial_k\varphi) + \lambda(\partial_k\varphi) = -tr_\omega\partial_k\omega_0 + \partial_k\log\det(\omega_0) + t\partial_kF$ $\omega = \omega_0 + i\partial\bar\partial\varphi$.

$\varphi\in C^{2,\alpha} \Rightarrow \begin{cases}\partial\varphi\in C^{1,\alpha}\\ \omega = \omega_0+i\partial\bar\partial\varphi \in C^{0,\alpha}\end{cases}$

Schauder $\Rightarrow \partial\varphi\in C^{2,\alpha} \Rightarrow \varphi\in C^{3,\alpha}$

$\Rightarrow \begin{cases}\partial\varphi\in C^{2,\alpha},\ \partial\bar\partial\varphi\in C^{1,\alpha}\\ \omega = \omega_0+i\partial\bar\partial\varphi \in C^{1,\alpha}\end{cases}$

Schauder $\Rightarrow \partial\varphi\in C^{3,\alpha} \Rightarrow \varphi\in C^{4,\alpha}$.

$t_n \to t_\infty$; $\hat S$

$(*)_{t_n}$: $\det(g_0+\partial\bar\partial\varphi_n) = e^{t_nF-\lambda\varphi_n}\det(g_0)$. $(\lambda<0)$

WANT: $t_\infty\in S$.

$\varphi_n \overset{?}{\to} \varphi_\infty$ C^∞.

$C_1=0$? $\|\varphi\|_{L^{p(\frac{n}{n-1})}} \le (Cp)^{1/p}\|\varphi\|_{L^p}$ $n=\dim M$ $\|\varphi\|_{L^{p(\frac{n}{n-1})^2}}$ Nash-Moser iteration.

① $|\varphi_n|\le C$.

$(\varphi_n)_{max} =: \varphi_n(x_n)$

At x_n: $\partial\bar\partial\varphi_n|_{x_n}\le 0$.

$\det(g_0+\partial\bar\partial\varphi_n)|_{x_n} \le \det g_0|_{x_n}$

$e^{t_nF(x_n)-\lambda\varphi_n(x_n)}\det g_0|_{x_n}$ $Ric(\omega_0)-\lambda\omega_0 = \sqrt{-1}\partial\bar\partial F$.

$\Rightarrow t_nF(x_n) - \lambda\varphi_n(x_n) \le 0$. $\lambda=-1$

$\Rightarrow \varphi_n|_{x_n} \le -t_nF(x_n) \le C$. $(\varphi_n)_{max}$

Similar $\Rightarrow (\varphi_n)_{min} \ge -\tilde C$ ⟶ $|\varphi_n|\le C$.

Schwarz. $f: \Sigma_1\to\Sigma_2$, g_1, g_2. $tr_{g_1}f^*g_2 = (g_1)^{ij}(f^*g_2)_{ij}$

$\Delta_{\Sigma_1}\log tr_{g_1}f^*g_2 \ge \cdots$

$g := g_0 + \partial\bar\partial\varphi$; g_0

$\Delta_g\log tr_{g_0}g = \frac{\Delta_g tr_{g_0}g}{tr_{g_0}g} - \frac{g^{p\bar q}(\nabla_p tr_{g_0}g)(\nabla_{\bar q}tr_{g_0}g)}{(tr_{g_0}g)^2}$ $g_0=\delta$, $g=\lambda_i\delta_{ij}$

$\ge -B\,tr_g g_0 - \frac{g_0^{i\bar j}R_{i\bar j}}{tr_{g_0}g} \ge -B\,tr_g g_0 - \frac{tr_{g_0}g+C}{tr_{g_0}g} \ge -A\,tr_g g_0$. $\Sigma\frac{1}{\lambda_i}$ $\Sigma\lambda_i$

$\frac{\det(g_0+\partial\bar\partial\varphi_n)}{\det g} = e^{t_nF-\lambda\varphi_n}\det(g_0)$

$R(g)_{i\bar j} = -\partial_i\partial_{\bar j}\log\det(g) = -\partial_i\partial_{\bar j}(t_nF-\lambda\varphi_n+\log\det g_0)$

$= -t_n\partial_i\partial_{\bar j}F + \lambda(\partial_i\partial_{\bar j}\varphi_n) + R(g_0)_{i\bar j}$.

$g = g_0+\partial\bar\partial\varphi_n$

$\Delta_g\log tr_{g_0}g \ge -A\,tr_g g_0$ — (∗).

$g = g_0+\partial\bar\partial\varphi$

tr_g: $n = tr_g g_0 + \Delta_g\varphi$. — (∗∗)

$\Delta_g(\underbrace{\log tr_{g_0}g - (A+1)\varphi}_{Q}) \ge \cancel{-A\,tr_g g_0} - An + (A+1)tr_g g_0$

$= tr_g g_0 - An$.

$Q_{max} := Q(x_{max})$. $\lambda_i\delta_{ij}$

$0 \ge \Delta_g Q \ge tr_g g_0 - An|_{x_{max}} \Rightarrow tr_g g_0|_{x_{max}} \le An$. $\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_n}\le An$

$\det(g_0+\partial\bar\partial\varphi) = e^{t_nF-\lambda\varphi_n}\det g_0$. $\lambda_i\le C$. at x_{max}.

$\Rightarrow \lambda_1\cdots\lambda_n \le C$. $\lambda_i\ge\frac{1}{\tilde C}$

logic $\Rightarrow$ $tr_{g_0}g\le C$ on M $\Rightarrow$ $\lambda_i\le\tilde C$ on M.

$Cg_0 \ge g_0+\partial\bar\partial\varphi_n \ge \frac{1}{C}g_0$.

Next: C^3 $\|\varphi\|_{C^3}$ (Yau)

$C^{2,\alpha}$ (Evans-Krylov).