Poincaré Conjecture (USD 1M) ~1891. 1982 2002–03 Perelman

Richard Hamilton

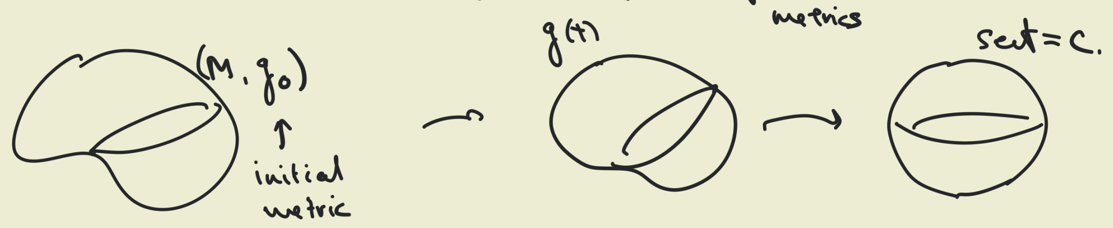
M^3 closed 3-manifold and $\pi_1(M^3)=0$ (simply-connected)

$\Rightarrow M^3 \cong S^3$.

Ricci flow (due to Hamilton 1982)

(RF) $\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t))}$ $(M^3, g(t))$ smooth family of Riem. metrics

$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, $\boxed{g(0)=g_0}$



(1982, Hamilton)

Given (M, g_0) has $\mathrm{Ric}(g_0) > 0$, then under (RF)

① $\mathrm{Ric}(g(t)) > 0$,

② $\frac{1}{\mathrm{Vol}(t)^{2/3}} g(t) \xrightarrow{C^\infty} g_\infty$ (const. sect.) as $t \to T_{max}$

$\mathrm{Vol}_{cg} = c^{\frac{n}{2}} \mathrm{Vol}_g$

Cor: M^3, closed, $\mathrm{Ric}(g_0) > 0$ $\Rightarrow M^3 \cong S^3/\Gamma$. $(M, g(t))$

Outline:

- $|\mathrm{Ric}_{g(t)} - \frac{R}{3} g| \le C R^{1-\delta}$ (indep. of t) $\forall t \in (0,T)$, $\exists \delta \in (0,1)$

$Rm = \pm \frac{R}{?} g \otimes g$? $\pm ? (R_{ic} - \frac{R}{3} g) \otimes g$.

Conseq. $\tilde g := \frac{1}{V(t)^{2/3}} g$

$|\widetilde{\mathrm{Ric}}(\tilde g) - \frac{\tilde R}{3} \tilde g|^2 = \tilde g^{ij} \tilde g^{kl} (\tilde R_{ik} - \frac{\tilde R}{3} \tilde g_{ik})(\tilde R_{jl} - \frac{\tilde R}{3} \tilde g_{jl})$ Einstein $\overset{n=3}{\Rightarrow}$ sect = C

$= V(t)^3 \; g^{ij} g^{kl} (R_{ik} - \frac{R}{3} g_{ik})(R_{jl} - \frac{R}{3} g_{jl})$

$\tilde R = V(t)^{2/3} R$

$= V(t)^3 \; |\mathrm{Ric}(g) - \frac{R}{3} g|^2 \le V(t)^{3/2} R_{g(t)}^{2-2\delta}$

$= V(t)^3 \left(\frac{1}{V(t)^{2/3}} \tilde R\right)^{2-2\delta} = \underbrace{V(t)^{+?\delta}}_{\downarrow 0} \underbrace{\tilde R^{2-2\delta}}_{\le C}$

- $|\nabla R| \le \cdots$

$|R(p) - R(q)| \le \sup |\nabla R| \, d(p,q)$.

$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$

$\frac{\partial}{\partial t} R_{ijkl} = \cdots$ $\qquad \frac{\partial}{\partial t} R_{ij} = \cdots$

$\frac{\partial}{\partial t} R = \Delta R + 2|\mathrm{Ric}|^2$ $\qquad \Delta = g^{ij} \nabla_i \nabla_j$

$R_{min}(t) := \min_{x \in M} R(x,t) = R(x_t, t)$

(parabolic max. principle)

$\frac{d}{dt} R_{min}(t) = \frac{\partial}{\partial t} R \big|_{(x_t,t)} = \Delta R \big|_{(x_t,t)} + 2|\mathrm{Ric}|^2_{(x_t,t)}$

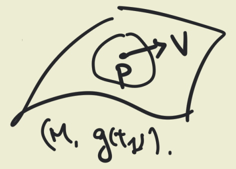
$\ge 0 + 0 \ge 0$. $++$

If $R(g(0)) \ge C \Rightarrow R(g(t)) \ge C$.

$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} - Q_{ij}$

$R_{ij}(0) > 0 \Rightarrow$ WANT: $R_{ij}(t) > 0$

Assume $\exists\, t_1 > 0$ (first time), $p \in M$, $V \in T_pM$



such that $\mathrm{Ric}_{g(t_1)}(V, \cdot) = 0$.

parallel transport V in neigh. of p. (w.r.t. $g(t_1)$)

at t_1, p: $0 \ge \frac{\partial}{\partial t}(R_{ij} V^i V^j) = (\frac{\partial}{\partial t} R_{ij}) V^i V^j = (\Delta R_{ij} - Q_{ij}) V^i V^j$

$R_{ij}(0) > \varepsilon$ $\qquad R_{ij}(t) > \frac{\varepsilon}{2}$.

$\underbrace{\mathrm{Ric}(V,V)}_{=0 \text{ at } (p,t_1)}$ $= \underbrace{\Delta (R_{ij} V^i V^j)}_{\ge 0} - Q_{ij} V^i V^j$

$g^{kl} \nabla_k \nabla_l (R_{ij} V^i V^j)$ $\qquad \mathrm{Ric}(V,V) = \begin{cases} 0 & \text{at } p. \\ \ge 0 & \text{other points.} \end{cases}$

$\ge -Q_{ij} V^i V^j$

$\boxed{\mathrm{Ric}(V, \cdot) = 0}$ $= \left(-6 g^{kl} R_{ik} R_{lj} + 3R R_{ij} - (R^2 - 2|\mathrm{Ric}|^2) g_{ij}\right) V^i V^j$

$R_{ij} V^i = 0 \quad \forall j$.

the tensor $Q_{ij} = 6S_{ij} - 3RR_{ij} + (R^2 - 2S)g_{ij}$, ($S_{ij} = R_{ik}R_{kj}$, $S = |\mathrm{Ric}|^2$, $n=3$) $\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} - Q_{ij}$ HW.

top corner is

$= \underbrace{(-R^2 + 2|\mathrm{Ric}|^2)}_{\ge 0} \underbrace{g_{ij} V^i V^j}_{|V|^2 \ne 0}$. (t_1, p)

$\mathrm{Ric} \sim \begin{bmatrix} \cancel{\lambda}^{0} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $\qquad R = \lambda + \mu + \nu$.

$(-R^2 + 2|\mathrm{Ric}|^2) = -(\cancel{\lambda} + \mu + \nu)^2 + 2(\cancel{\lambda^2} + \mu^2 + \nu^2) \ge 0$.

$\frac{\partial}{\partial t} T_{ij} = \Delta T_{ij} + X^k \nabla_k T_{ij} + N_{ij}$

Given: (If $T_{ij} V^i = 0$, then $N_{ij} V^i V^j \ge 0$.)

then: $T_{ij}(0) > 0 \Rightarrow T_{ij}(t) > 0 \quad \forall t$.

HW:

$X^k \nabla_k (T_{ij} V^i V^j) = 0$ at (t_1, p)

① $R_{ij} \ge \varepsilon R g_{ij} > 0$ (const. > 0) $\Rightarrow R_{ij}(t) \ge \varepsilon C g_{ij}(t)$

② $R_{ij} - \frac{R}{2} g_{ij} < 0$ $\Leftrightarrow$ sect > 0 $\qquad \frac{R}{2} g_{ij} - R_{ij}$ $\qquad R_{ijij} = \cdots$

① $T_{ij} := R_{ij} - \varepsilon R g_{ij}$ $\qquad \frac{\partial}{\partial t} T_{ij} = \Delta T_{ij} + \cdots \nabla_k T_{ij}$ ☺

$\tilde T_{ij} = \frac{R_{ij}}{R} - \varepsilon g_{ij}$

- $|\mathrm{Ric} - \frac{R}{3} g| \le C R^{1-\delta}$ ✓

Theorem 11.1

$\forall \eta > 0$, $\exists\, C = C(\eta) > 0$ (indep. of t) s.t. $|\nabla R|^2 \le \eta R^3 + C(\eta)$.

$f := \frac{|\nabla R|^2}{R} - \eta R^2 \quad \leftarrow \quad \frac{\partial f}{\partial t} \le \Delta f + \langle X, \nabla f \rangle + \text{neg}$.

$\frac{\partial}{\partial t} |\nabla R|^2 = \Delta |\nabla R|^2 + \cdots$ $\qquad \Rightarrow f \le C \quad \forall t, p$.

$\Rightarrow |\nabla R|^2 \le C \eta R^3 + C R$.

$g^{ij} \nabla_i R \nabla_j R$ $\qquad \le\ ?$ $\qquad \le C^{\frac{3}{2}} + R^3 \left(\frac{3}{2}\right)$

$\frac{\partial R}{\partial x_i} \quad \frac{\partial g_{ij}}{\partial x_j}$

$\# @ !!$

$\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R} - \eta R^2\right) \le \Delta\left(\frac{|\nabla R|^2}{R} - \eta R^2\right) + 16 |\nabla R|^2 - \frac{4}{3} \eta R^3$.

WANT: ≤ 0.