Bonnet–Myers' Theorem

(M,g) complete Riem. mfd. and $\mathrm{Ric} \geq (n-1)k\,g$, k constant >0.

$\Rightarrow \mathrm{diam}(M,g) \leq \frac{\pi}{\sqrt{k}}$.

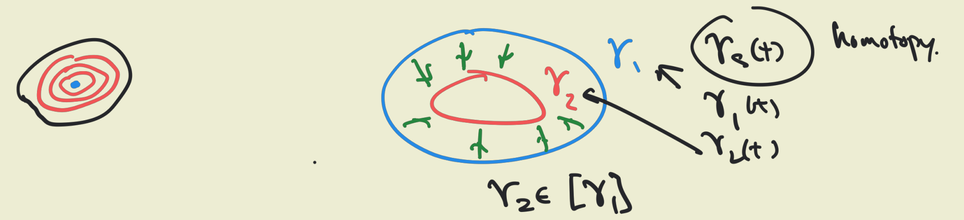
Equality hold. Shiu-Yuen Cheng

furthermore, if $\mathrm{diam}(M,g) = \frac{\pi}{\sqrt{k}}$, then $(M,g) \overset{\text{iso}}{\cong} (S^n, g_{round})$

Synge's Theorem

Let (M,g) compact Riem. mfd., sect > 0.

- If $\dim M$ is even, and M orientable then M is simply-connected.
- If $\dim M$ is odd, then M is orientable.



Lemma In every homotopy class $[\gamma]$, $\exists$ geodesic $\gamma' \in [\gamma]$ such that $L(\gamma') \leq L(\tilde\gamma)$ $\forall \tilde\gamma \in [\gamma]$.

① Suppose otherwise $\dim M$ even, M orientable but M is not simply-connected.

$\exists [\gamma] \neq 0$.

lemma $\Rightarrow \exists \gamma' \in [\gamma]$ geodesic s.t. and $L(\gamma') \leq L(\tilde\gamma)$ $\forall \tilde\gamma \in [\gamma]$.

$P_{\gamma'} : T_pM \to T_pM$ $\quad T = \dot\gamma'$

$\det P_{\gamma'} = 1$. $\quad P_{\gamma'}(T) = T$.

$E(t) := \mathrm{span}\{T(t)\}^\perp$

$P_{\gamma'} : \underset{\text{odd}}{E} \to \underset{\text{odd}}{E}$. $\quad \det(P_{\gamma'}|_E) = 1$.

$P_{\gamma'}|_E$: $\underbrace{\lambda_1, \dots, \lambda_r}_{\mathbb{R}}$, $\underbrace{z_1, \bar z_1, \dots, z_k, \bar z_k}_{\notin \mathbb{R}}$ $\quad \underbrace{\lambda_1 \cdots \lambda_r}_{\text{odd}} > 0$, ± 1

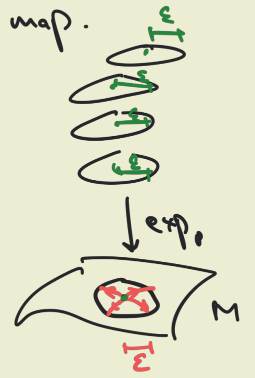
$\Rightarrow \exists \lambda_i = 1$.

$P_{\gamma'}(X_0) = X_0 \in E$. $\quad \nabla_{\dot\gamma'} X(t) = 0$, $X(0) = X_0$

$$I(X,X) = \int_a^b \underbrace{|\nabla_{\dot\gamma'} X|^2}_{=0} - \underbrace{\mathrm{Rm}(X,T,T,X)}_{+}\, dt < 0.$$

Cartan–Hadamard's Theorem. $\quad M^n \cong \mathbb{R}^n / \pi_1(M)$.

(M,g) complete Riem. manifold. sect ≤ 0. then $\exp_p : T_pM \to M$ is a covering map.



Cor: If $\pi_1(M) = 0$, then $M \overset{\text{diffeo}}{\cong} \mathbb{R}^n$.

① $V(p) = 0$.

$|V(t)| \geq at$. $\quad \forall t > 0$.

$\Rightarrow V(t) \neq 0$ $\quad \forall t > 0$.

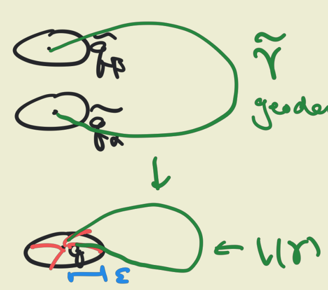
$\Rightarrow (\exp_p)_*$ is invertible everywhere. $\quad \Leftrightarrow (\exp_p)_{*q}$ is singular.

$(\exp_p)_* : T(T_pM) \to T_pM$. invertible

IFT $\Rightarrow \exp_p : T_pM \to M$ is a local diffeo.

$\exp_p : T_pM \to M$ is a covering map.

* Justify they are disjoint.



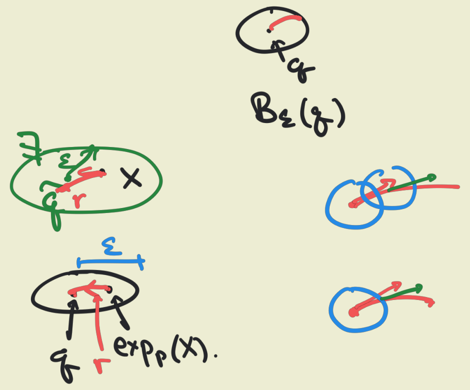
$(\exp_p)^{-1}(B_\varepsilon(q)) = \bigsqcup_\alpha \tilde B_{\tilde q_\alpha}$

$\supset$ easy.

$\subset$?

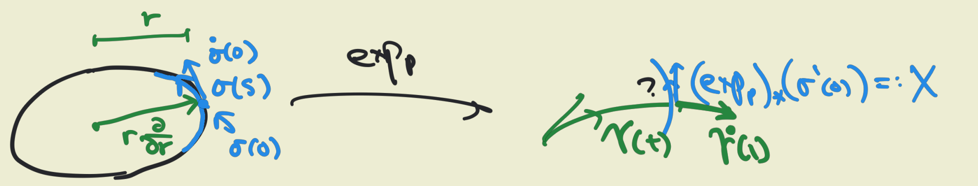
$\forall x \in (\exp_p)^{-1}(B_\varepsilon(q))$

$\Rightarrow (\exp_p)(x) \in B_\varepsilon(q)$



Gauss lemma

Proof



$\gamma_s(t) := \exp_p(t\sigma(s))$

$\left.\frac{\partial\gamma}{\partial s}\right|_{(s,t)=(0,0)} = 0 \qquad \left.\frac{\partial\gamma}{\partial t}\right|_{(s,t)=(0,0)} = \exp_{p*0}(\sigma(0)) = \sigma(0)$

$\left.\frac{\partial\gamma}{\partial s}\right|_{(s,t)=(0,1)} = \left.\frac{d}{ds}\right|_{s=0} \exp_p(\sigma(s)) = (\exp_p)_*(\sigma'(0)) =: X$

$\left.\frac{\partial\gamma}{\partial t}\right|_{(s,t)=(0,1)} = \left.\frac{d}{dt}\right|_{t=1} \exp_p(t\sigma(0)) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(\sigma(0)+t\sigma(0))$

$= (\exp_p)_{*\sigma(0)}(\sigma(0)) = \dot\gamma(1)$

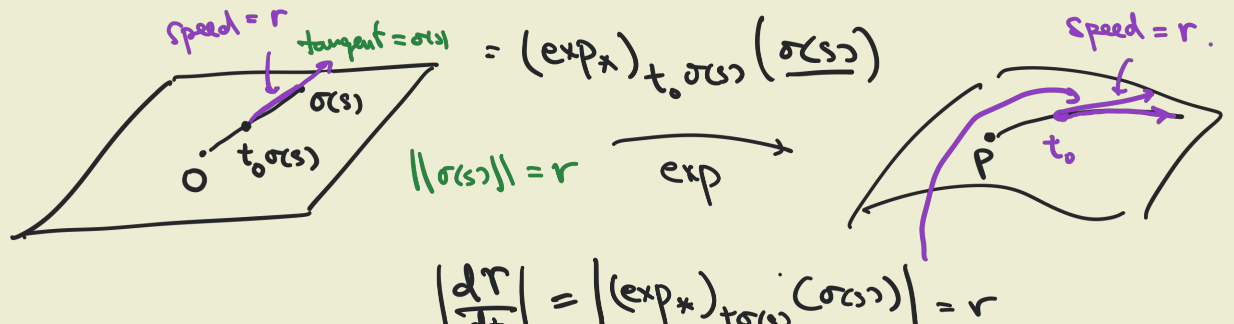
$$\frac{d}{dt} g\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right)\Big|_{s=0} = g\left(\nabla_{\dot\gamma}\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right) + g\Big(\frac{\partial\gamma}{\partial s}, \underbrace{\cancel{\nabla_{\dot\gamma}\frac{\partial\gamma}{\partial t}}}_{0 \text{ geodesic}}\Big)\Big|_{s=0}$$

$$= g\left(\nabla_{\frac{\partial\gamma}{\partial s}}\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial t}\right)$$

$$= \frac{1}{2}\frac{d}{ds}\left|\frac{\partial\gamma}{\partial t}\right|_g^2 \quad \text{see below. } 0$$

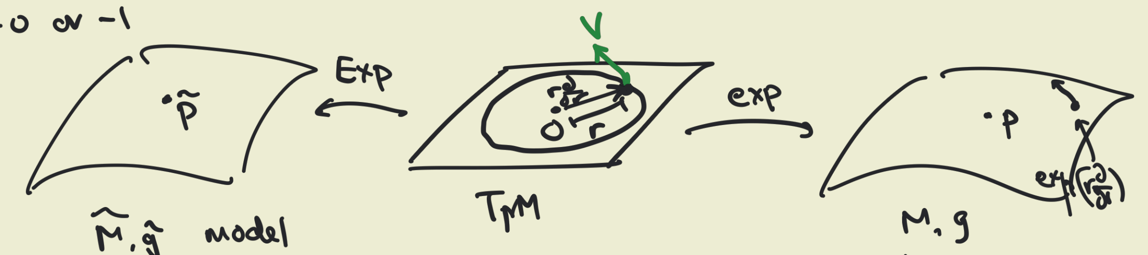
$$\left.\frac{\partial\gamma}{\partial t}\right|_{t=t_0} = \frac{d}{dt}\exp(t\sigma(s))\Big|_{t=t_0} = \frac{d}{dt}\Big|_{t=0}\exp(t_0\sigma(s)+t\sigma(s))$$

$= (\exp_*)_{t_0\sigma(s)}(\sigma(s))$



$$\left|\frac{d\gamma}{dt}\right|_g = \left|(\exp_*)_{t_0\sigma(s)}(\sigma(s))\right|_g = r$$

$C = 0$ or -1



if $\tilde g((\mathrm{Exp})_*V, (\mathrm{Exp})_*V) = g(\exp_*V, \exp_*V)$ then done. $\quad (\mathrm{Exp})^*\tilde g = (\exp)^* g$.

suffices to consider V tangent to

$$J(t) := \frac{\partial}{\partial s}\Big|_{s=0} \exp\left(t\left(r\frac{\partial}{\partial r} + s\frac{V}{r}\right)\right)$$

$$= (\exp)_{*t r\frac{\partial}{\partial r}}\left(t\frac{V}{r}\right)$$

Recall $J(t) = u(t)E(t)$ where $u(t) = \begin{cases} u'(0)t & \text{if } C=0 \\ \frac{u'(0)}{\sqrt{-C}}\sinh(\sqrt{-C}\,t) & \text{if } C<0 \end{cases}$

$J(r) = (\exp_*)_{r\frac{\partial}{\partial r}}(V)$.

$|J(r)| = u(r) = \begin{cases} u'(0)\,r & \text{if } C=0 \\ \frac{u'(0)}{\sqrt{-C}}\sinh(\sqrt{-C}\,r) & \text{if } C<0 \end{cases}$

$J(t) = u(t)E(t)$

$\nabla_{\dot\gamma} J(0) = \dot u(0)E(0)$.

$\quad \| \quad \frac{V}{r}$

$\Rightarrow |\dot u(0)| = \left|\frac{V}{r}\right|$

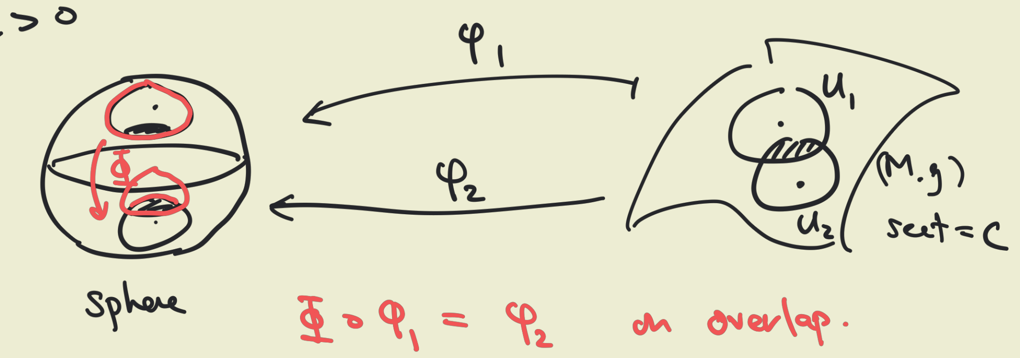
HW3: $\quad W = W^i e_i$

$J(t) = \frac{\partial}{\partial s}\Big|_{s=0}\exp\left(t\left(\frac{\partial}{\partial r} + sW\right)\right)$

in geodesic normal coord. $= tW^i\frac{\partial}{\partial x_i}(t)$

$\Rightarrow \nabla_{\dot\gamma} J(0) = W$

$C > 0$



$\Phi \circ \varphi_1 = \varphi_2$ on overlap.

replace φ_1 by $\Phi \circ \varphi_1$, and so on

$\Rightarrow \exists\, \varphi : M \to S^n$ isometry.