Differentiable Manifolds & Riemannian Geometry

Lecture Notes for MATH 4033 and MATH 6250I

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Contents

Preface

This lecture note is written for courses MATH 4033 (Calculus on Manifolds) and MATH 6250I (Riemannian Geometry) taught by the author in the Hong Kong University of Science and Technology.

The main goal of these courses is to introduce advanced undergraduates and first-year graduates the basic language of differentiable manifolds, tensor calculus, and Riemannian geometry. It presents some of the most essential knowledge on differential geometry that is necessary for further studies or research in geometric analysis, general relativity, and related fields. Before reading the lecture notes, students are advised to have a solid *conceptual* background of linear algebra (MATH 2131) and multivariable calculus.

The course MATH 4033 covers Chapters 1 to 5 in this lecture note. These chapters are about the analytic, algebraic, and topological aspects of differentiable manifolds. Chapters 6 and 7 form a crush course on differential geometry of hypersurfaces in Euclidean spaces. The main purpose of these chapters is to give some motivations on why various abstract concepts in Riemannian geometry are introduced in the way they are. The remaining Chapters 8 to 11 form an introduction course to Riemannian geometry.

This version of lecture notes will be progressively updated to fix typographical errors and to add diagrams for the Riemannian geometry part.

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Part 1

Differentiable Manifolds

Chapter 1

Regular Surfaces

> "God made solids, but surfaces were the work of the devil."
>
> *Wolfgang Pauli*

A manifold is a space which locally resembles an Euclidean space. Before we learn about manifolds in the next chapter, we first introduce the notion of regular surfaces in $\mathbb{R}^3$ which motivates the definition of abstract manifolds and related concepts in the next chapter.

1.1. Local Parametrizations

In Multivariable Calculus, we expressed a surface in $\mathbb{R}^3$ in two ways, namely using a parametrization $F(u, v)$ or by a level set $f(x, y, z) = 0$. In this section, let us first focus on the former.

In MATH 2023, we used a parametrization $F(u, v)$ to describe a surface in $\mathbb{R}^3$ and to calculate various geometric and physical quantities such as surface areas, surface integrals and surface flux. To start the course, we first look into several technical and analytical aspects concerning $F(u, v)$, such as their domains and images, their differentiability, etc. In the past, we can usually cover (or almost cover) a surface by a single parametrization $F(u, v)$. Take the unit sphere as an example. We learned that it can be parametrized with the help of spherical coordinates:

$$F(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

where $0 < \theta < 2\pi$ and $0 < \varphi < \pi$. This parametrization covers almost every part of the sphere (except the north and south poles, and a half great circle connecting them). In order to cover the whole sphere, we need more parametrizations, such as $G(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with domain $-\pi < \theta < \pi$ and $0 < \varphi < \pi$.

Since the image of either F or G does not cover the whole sphere (although almost), from now on we call them *local parametrizations*.

Definition 1.1 (Local Parametrizations of Class C^k)**.** Consider a subset $M \subset \mathbb{R}^3$. A function $F(u,v) : \mathcal{U} \to \mathcal{O}$ from an open subset $\mathcal{U} \subset \mathbb{R}^2$ onto an open subset $\mathcal{O} \subset M$ is called a C^k *local parametrization* (or a C^k *local coordinate chart*) of M (where $k \geq 1$) if all of the following holds:

(1) $F : \mathcal{U} \to \mathbb{R}^3$ is C^k when the codomain is regarded as $\mathbb{R}^3$.

(2) $F : \mathcal{U} \to \mathcal{O}$ is a *homeomorphism*, meaning that $F : \mathcal{U} \to \mathcal{O}$ is bijective, and both F and F^{-1} are continuous.

(3) For all $(u,v) \in \mathcal{U}$, the cross product:

$$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0.$$

The coordinates (u,v) are called the *local coordinates* of M.

If $F : \mathcal{U} \to M$ is of class C^k for any integer k, then F is said to be a C^∞ (or smooth) local parametrization.

Definition 1.2 (Surfaces of Class C^k)**.** A subset $M \subset \mathbb{R}^3$ is called a C^k *surface*, where $k \in \mathbb{N} \cup \{\infty\}$, in $\mathbb{R}^3$ if at every point $p \in M$, there exists an open subset $\mathcal{U} \subset \mathbb{R}^2$, an open subset $\mathcal{O} \subset M$ containing p, and a C^k local parametrization $F : \mathcal{U} \to \mathcal{O}$ which satisfies all three conditions stated in Definition 1.1.

We say M is a *regular surface* in $\mathbb{R}^3$ if it is a C^∞ surface.

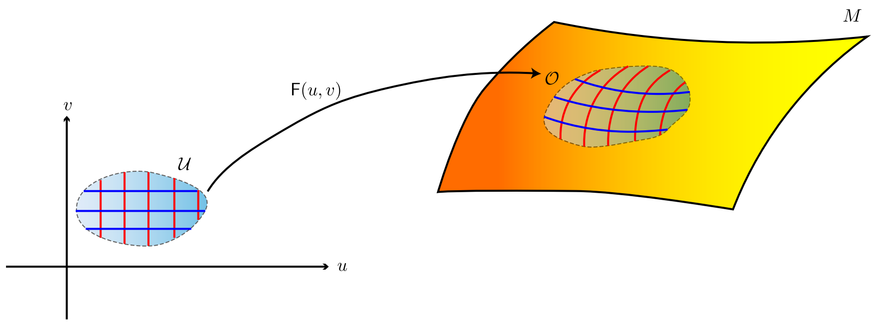
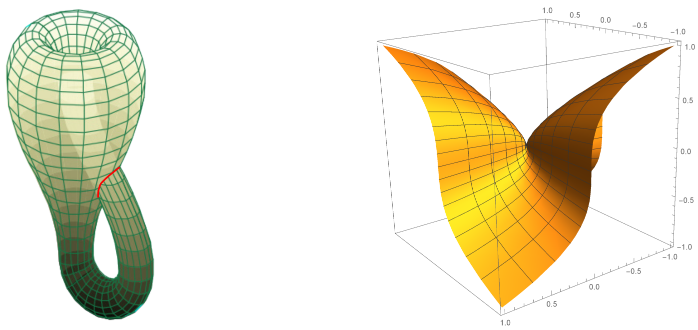


Figure 1.1. smooth local parametrization

To many students (myself included), the definition of regular surfaces looks obnoxious at the first glance. One way to make sense of it is to look at some examples and understand why each of the three conditions is needed in the definition.

The motivation behind condition (1) in the definition is that we are studying *differential* topology/geometry and so we want the parametrization to be differentiable as many times as we like. Condition (2) rules out surfaces that have self-intersection such as the Klein bottle (see Figure 1.2a). Finally, condition (3) guarantees the existence of a unique tangent plane at every point on M (see Figure 1.2b for a non-example).



(a) Klein Bottle has a self-intersection. **(b)** $F(u,v) = (u^3, v^3, uv)$ fails condition (3).

Figure 1.2. Examples of non-smooth parametrizations

Example 1.3 (Graph of a Function)**.** Consider a smooth function $f(u,v) : \mathcal{U} \to \mathbb{R}$ defined on an open subset $\mathcal{U} \subset \mathbb{R}^2$. The graph of f, denoted by Γ_f, is the subset $\{(u,v,f(u,v)) : (u,v) \in \mathcal{U}\}$ of $\mathbb{R}^3$. One can parametrize Γ_f by a global parametrization:

$$F(u,v) = (u,\ v,\ f(u,v)).$$

Condition (1) holds because f is given to be smooth. For condition (2), F is clearly one-to-one, and the image of F is the whole graph Γ_f. Regarding it as a map $F : \mathcal{U} \to \Gamma_f$, the inverse map

$$F^{-1}(x,y,z) = (x,y)$$

is clearly continuous. Therefore, $F : \mathcal{U} \to \Gamma_f$ is a homeomorphism. To verify condition (3), we compute the cross product:

$$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} = \left(-\frac{\partial f}{\partial u},\ -\frac{\partial f}{\partial v},\ 1\right) \neq 0$$

for all $(u,v) \in \mathcal{U}$. Therefore, F is a smooth local parametrization of Γ_f. Since the image of this single smooth local parametrization covers all of Γ_f, we have proved that Γ_f is a regular surface. □

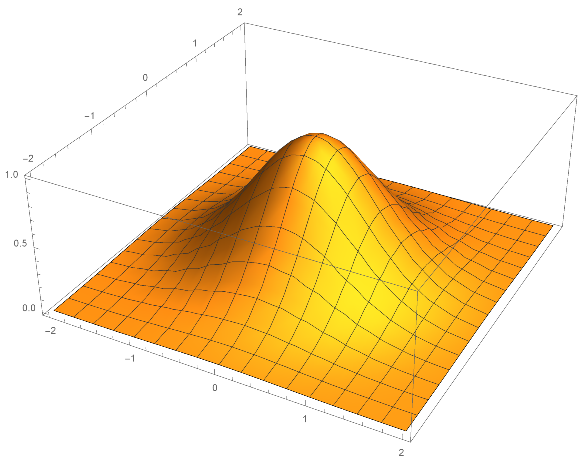


Figure 1.3. The graph of any smooth function is a regular surface.

Exercise 1.1. Show that $F(u, v) : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3$ defined by:

$$F(u, v) = (\sin u, \sin 2u, v)$$

satisfies conditions (1) and (3) in Definition 1.1, but not condition (2). [Hint: Try to show F^{-1} is not continuous by finding a diverging sequence $\{(u_n, v_n)\}$ such that $\{F(u_n, v_n)\}$ converges. See Figure 1.4 for reference.]

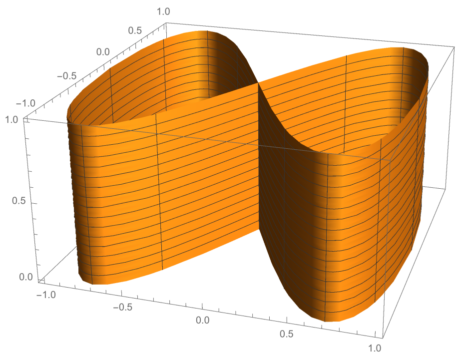


Figure 1.4. Plot of $F(u, v)$ in Exercise 1.1

In Figure 1.3, one can observe that there are two families of curves on the surface. These curves are obtained by varying one of the (u, v)-variables while keeping the other constant. Precisely, they are the curves represented by $F(u, v_0)$ and $F(u_0, v)$ where u_0 and v_0 are fixed. As such, the partial derivatives $\frac{\partial F}{\partial u}(p)$ and $\frac{\partial F}{\partial v}(p)$ give a pair of tangent vectors on the surface at point p. Therefore, their cross product $\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)$ is a normal vector to the surface at point p (see Figure 1.5). Here we have abused the notations for simplicity: $\frac{\partial F}{\partial u}(p)$ means $\frac{\partial F}{\partial u}$ evaluated at $(u, v) = F^{-1}(p)$. Similarly for $\frac{\partial F}{\partial v}(p)$.

Condition (3) requires that $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}$ is everywhere non-zero in the domain of F. An equivalent statement is that the vectors $\left\{\frac{\partial F}{\partial u}(p), \frac{\partial F}{\partial v}(p)\right\}$ are linearly independent for any $p \in F(\mathcal{U})$.

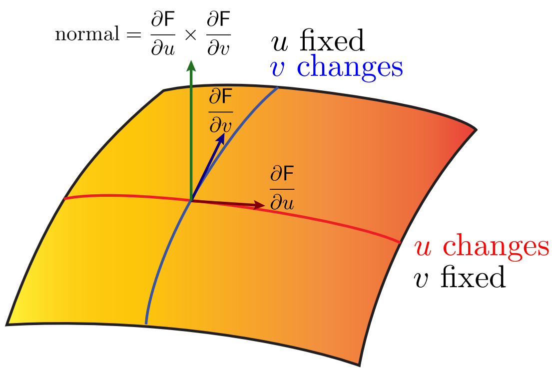


Figure 1.5. Tangent and normal vectors to a surface in $\mathbb{R}^3$

Example 1.4 (Sphere)**.** In $\mathbb{R}^3$, the unit sphere $\mathbb{S}^2$ centered at the origin can be represented by the equation $x^2 + y^2 + z^2 = 1$, or in other words, $z = \pm\sqrt{1 - x^2 - y^2}$. We can parametrize the upper and lower hemisphere by two separate local maps:

$$F_1(u,v) = (u,\ v,\ \sqrt{1-u^2-v^2}) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_+$$
$$F_2(u,v) = (u,\ v,\ -\sqrt{1-u^2-v^2}) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_-$$

where $B_1(0) = \{(u,v) : u^2+v^2 < 1\}$ is the *open* unit disk in $\mathbb{R}^2$ centered at the origin, and $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ are the upper and lower hemispheres of $\mathbb{S}^2$ respectively. Since $B_1(0)$ is open, the functions $\pm\sqrt{1-u^2-v^2}$ are smooth, and according to the previous example both F_1 and F_2 are smooth local parametrizations.

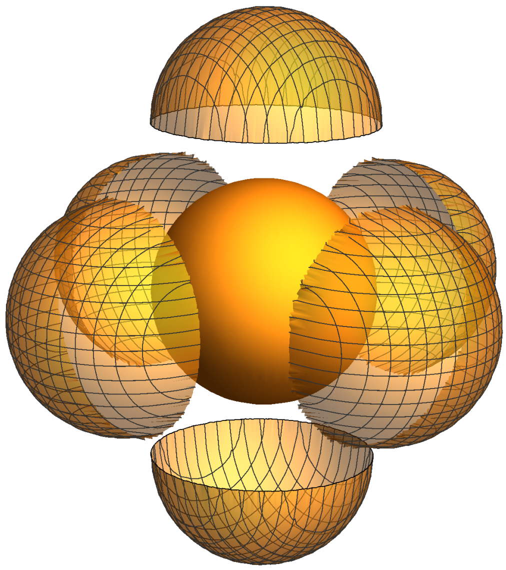


Figure 1.6. A unit sphere covered by six parametrization charts

However, not all points on the sphere are covered by $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$, since points on the equator are not. In order for show that $\mathbb{S}^2$ is a regular surface, we need to write down more smooth local parametrization(s) so that each point on the sphere can be covered by at least one smooth local parametrization chart. One can construct four more smooth local parametrizations (left, right, front and back) similar to F_1 and F_2 (see Figure 1.6). It is left as an exercise for readers to write down the other four parametrizations. These six parametrizations are all smooth and they cover the whole sphere. Therefore, it shows the sphere is a regular surface. □

Exercise 1.2. Write down the left, right, front and back parametrizations F_i's $(i = 3,4,5,6)$ of the sphere as shown in Figure 1.6. Indicate clearly the domain and range of each F_i.

Example 1.5 (Sphere: revisited)**.** We can in fact cover the sphere by two smooth local parametrization described below. Define $F_+(u,v) : \mathbb{R}^2 \to \mathbb{S}^2\backslash\{(0,0,1)\}$ where:

$$F_+(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

It is called the *stereographic parametrization* of the sphere (see Figure 1.7) , which assigns each point $(u,v,0)$ on the xy-plane of $\mathbb{R}^3$ to a point where the line segment joining $(u,v,0)$ and the north pole $(0,0,1)$ intersects the sphere. Clearly F_+ is a smooth function. We leave it as exercise for readers to verify that F_+ satisfies condition (3) and that $F_+^{-1} : \mathbb{S}^2\backslash\{(0,0,1)\} \to \mathbb{R}^2$ is given by:

$$F_+^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

As $z \neq 1$ for every (x,y,z) in the domain of F_+^{-1}, it is a continuous function. Therefore, F_+ is a smooth local parametrization. The inverse map F_+^{-1} is commonly called the stereographic projection of the sphere.

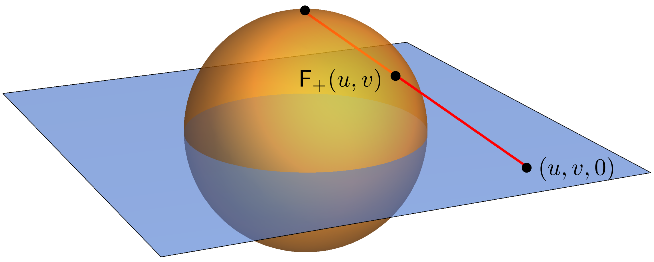


Figure 1.7. Stereographic parametrization of the sphere

Note that the range of F_+ does not include the point $(0,0,1)$. In order to show that the sphere is a regular surface, we need to cover it by another parametrization $F_- : \mathbb{R}^2 \to \mathbb{S}\backslash\{(0,0,-1)\}$ which assigns each point $(u,v,0)$ on the xy-plane to a point where the line segment joining $(u,v,0)$ and the south pole $(0,0,-1)$ intersects the sphere. It is an exercise for readers to write down the explicit parametrization F_-. $\square$

Exercise 1.3. Verify that F_+ in Example 1.4 satisfies condition (3) in Definition 1.1, and that the inverse map $F_+^{-1} : \mathbb{S}^2\backslash\{(0,0,1)\} \to \mathbb{R}^2$ is given as stated. [Hint: Write down $F_+(u,v) = (x,y,z)$ and solve (u,v) in terms of (x,y,z). Begin by finding u^2+v^2 in terms of z.]

Furthermore, write down explicitly the map F_- described in Example 1.4, and find its inverse map F_-^{-1}.

Exercise 1.4. Find smooth local parametrizations which together cover the whole ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b and c are positive constants.

Exercise 1.5. Let M be the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. The purpose of this exercise is to construct a smooth local parametrization analogous to the stereographic parametrization in Example 1.4:

Consider the unit circle $x^2 + y^2 = 1$ on the xy-plane. For each point $(u, 0)$ on the x-axis, we construct a straight-line joining the point $(0, 1)$ and $(u, 0)$. This line intersects the unit circle at a unique point p. Denote the xy-coordinates of p by $(x(u), y(u))$.

(a) Find the coordinates $(x(u), y(u))$ in terms of u.

(b) Define:
$$F_1(u, v) = (x(u), y(u), v)$$
with $\mathbb{R}^2$ as its domain. Describe the image of F_1.

(c) Denote $\mathcal{O}_1$ to be the image of F_1. Verify that $F_1 : \mathbb{R}^2 \to \mathcal{O}_1$ is smooth local parametrization of M.

(d) Construct another smooth local parametrization F_2 such that the images of F_1 and F_2 cover the whole surface M (hence establish that M is a regular surface).

Let's also look at a non-example of smooth local parametrizations. Consider the map:
$$G(u, v) = (u^3, v^3, 0), \quad (u, v) \in \mathbb{R} \times \mathbb{R}.$$
It is a smooth, injective map from $\mathbb{R}^2$ onto the xy-plane Π of $\mathbb{R}^3$, i.e. $G : \mathbb{R}^2 \to \Pi$. However, it can be computed that
$$\frac{\partial G}{\partial u}(0, 0) = \frac{\partial G}{\partial v}(0, 0) = 0$$
and so condition (3) in Definition 1.1 does not hold. The map G is not a smooth local parametrization of Π. However, note that Π is a regular surface because $F(u, v) = (u, v, 0)$ is a smooth global parametrization of Π, even though G is not a "good" parametrization.

In order to show M is a regular surface, what we need is to show at every point $p \in M$ there is *at least one* smooth local parametrization F near p. However, to show that M is not a regular surface, one then needs to come up with a point $p \in M$ such that there is *no* smooth local parametrization near that point p (which may not be easy).

1.2. Level Surfaces

Many surfaces are defined using an equation such as $x^2+y^2+z^2=1$, or $x^2+y^2=z^2+1$. They are level sets of a function $g(x,y,z)$. In this section, we are going to prove a theorem that allows us to show easily that some level sets $g^{-1}(c)$ are regular surfaces.

Theorem 1.6. *Let $g(x,y,z):\mathbb{R}^3\to\mathbb{R}$ be a smooth function of three variables. Consider a non-empty level set $g^{-1}(c)$ where c is a constant. If $\nabla g(x_0,y_0,z_0)\neq 0$ at all points $(x_0,y_0,z_0)\in g^{-1}(c)$, then the level set $g^{-1}(c)$ is a regular surface.*

Proof. The key idea of the proof is to use the Implicit Function Theorem. Given any point $p=(x_0,y_0,z_0)\in g^{-1}(c)$, since $\nabla g(x_0,y_0,z_0)\neq(0,0,0)$, at least one of the first partials:

$$\frac{\partial g}{\partial x}(p),\ \frac{\partial g}{\partial y}(p),\ \frac{\partial g}{\partial z}(p)$$

is non-zero. Without loss of generality, assume $\frac{\partial g}{\partial z}(p)\neq 0$, then the Implicit Function Theorem shows that locally around the point p, the level set $g^{-1}(c)$ can be regarded as a graph $z=f(x,y)$ of some smooth function f of (x,y). To be precise, there exists an open set $\mathcal{O}$ of $g^{-1}(c)$ containing p such that there is a smooth function $f(x,y):\mathcal{U}\subset\mathbb{R}^2\to\mathbb{R}$ from an open set $\mathcal{U}$ such that $(x,y,f(x,y))\in\mathcal{O}\subset g^{-1}(c)$ for any $(x,y)\in\mathcal{U}$. As such, the smooth local parametrization $F:\mathcal{U}\to\mathcal{O}$ defined by:

$$F(u,v)=(u,v,f(u,v))$$

is a smooth local parametrization of $g^{-1}(c)$.

In the case where $\frac{\partial g}{\partial y}(p)\neq 0$, the above argument is similar as locally around p one can regard $g^{-1}(c)$ as a graph $y=h(x,z)$ for some smooth function h. Similar in the case $\frac{\partial g}{\partial x}(p)\neq 0$.

Since every point p can be covered by the image of a smooth local parametrization, the level set $g^{-1}(c)$ is a regular surface. □

Example 1.7. The unit sphere $x^2+y^2+z^2=1$ is a level surface $g^{-1}(1)$ where $g(x,y,z):=x^2+y^2+z^2$. The gradient vector $\nabla g=(2x,\ 2y,\ 2z)$ is zero only when $(x,y,z)=(0,0,0)$. Since the origin is not on the unit sphere, we have $\nabla g(x_0,y_0,z_0)\neq(0,0,0)$ for any $(x_0,y_0,z_0)\in g^{-1}(1)$. Therefore, the unit sphere is a regular surface.

Similarly, one can also check that the surface $x^2+y^2=z^2+1$ is a regular surface. It is a level set $h^{-1}(1)$ where $h(x,y,z)=x^2+y^2-z^2$. Since $\nabla h=(2x,\ 2y,\ -2z)$, the origin is the only point p at which $\nabla h(p)=(0,0,0)$ and it is not on the level set $h^{-1}(1)$. Therefore, $h^{-1}(1)$ is a regular surface. □

However, the cone $x^2+y^2=z^2$ cannot be shown to be a regular surface using Theorem 1.6. It is a level surface $h^{-1}(0)$ where $h(x,y,z):=x^2+y^2-z^2$. The origin $(0,0,0)$ is on the cone and $\nabla h(0,0,0)=(0,0,0)$. Theorem 1.6 fails to give any conclusion.

The converse of Theorem 1.6 is not true. Consider $g(x,y,z)=z^2$, then $g^{-1}(0)$ is the xy-plane which is clearly a regular surface. However, $\nabla g=(0,0,2z)$ is zero at the origin which is contained in the xy-plane.

Exercise 1.6. [**dC76**, P.66] Let $f(x,y,z) = (x+y+z-1)^2$. For what values of c is the set $f^{-1}(c)$ a regular surface?

Exercise 1.7. A torus is defined by the equation:

$$z^2 = R^2 - \left(\sqrt{x^2+y^2} - r\right)^2$$

where $R > r > 0$ are constants. Show that it is a regular surface.

The proof of Theorem 1.6 makes use of the Implicit Function Theorem which is an existence result. It shows a certain level set is a regular surface, but it fails to give an explicit smooth local parametrization around each point.

There is one practical use of Theorem 1.6 though. Suppose we are given $F(u,v)$ which satisfies conditions (1) and (3) in Definition 1.1 and that F is continuous and F^{-1} exists. In order to verify that it is a smooth local parametrization, we need to prove continuity of F^{-1}, which is sometimes difficult. Here is one example:

$$F(u,v) = (\sin u \, \cos v, \; \sin u \, \sin v, \; \cos u), \quad 0 < u < \pi, \; 0 < v < 2\pi$$

is a smooth local parametrization of a unit sphere. It is clearly a smooth map from $(0,\pi) \times (0,2\pi) \subset \mathbb{R}^2$ to $\mathbb{R}^3$, and it is quite straight-forward to verify condition (3) in Definition 1.1 and that F is one-to-one. However, it is rather difficult to write down an explicit F^{-1}, let alone to show it is continuous.

The following result tells us that if the surface is given by a level set satisfying conditions stated in Theorem 1.6, and F satisfies conditions (1) and (3), then F^{-1} is *automatically* continuous. Precisely, we have the following:

Proposition 1.8. *Assume all given conditions stated in Theorem 1.6. Furthermore, suppose $F(u,v)$ is a bijective map from an open set $\mathcal{U} \subset \mathbb{R}^2$ to an open set $\mathcal{O} \subset M := g^{-1}(c)$ which satisfies conditions (1) and (3) in Definition 1.1. Then, F satisfies condition (2) as well and hence is a smooth local parametrization of $g^{-1}(c)$.*

Proof. Given any point $p \in g^{-1}(c)$, we can assume without loss of generality that $\frac{\partial g}{\partial z}(p) \neq 0$. Recall from Multivariable Calculus that $\nabla g(p)$ is a normal vector to the level surface $g^{-1}(c)$ at point p. Furthermore, if $F(u,v)$ is a map satisfying conditions (1) and (3) of Definition 1.1, then $\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)$ is also a normal vector to $g^{-1}(c)$ at p.

Now that the $\hat{k}$-component of $\nabla g(p)$ is non-zero since $\frac{\partial g}{\partial z}(p) \neq 0$, so the $\hat{k}$-component of the cross product $\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)$ is also non-zero. If we express $F(u,v)$ as:

$$F(u,v) = (x(u,v), y(u,v), z(u,v)),$$

then the $\hat{k}$-component of $\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)$ is given by:

$$\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)(p) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}(p).$$

Define $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ by $\pi(x,y,z) = (x,y)$. The above shows that the composition $\pi \circ F$ given by

$$(\pi \circ F)(u,v) = (x(u,v), y(u,v))$$

has non-zero Jacobian determinant at p. By the Inverse Function Theorem, $\pi \circ F$ has a smooth local inverse near p. In particular, $(\pi \circ F)^{-1}$ is continuous near p.

Finally, by the fact that $(\pi \circ F) \circ F^{-1} = \pi$ and that $(\pi \circ F)^{-1}$ exists and is continuous locally around p, we can argue that $F^{-1} = (\pi \circ F)^{-1} \circ \pi$ is also continuous near p. It completes the proof. $\square$

Exercise 1.8. Rewrite the proof of Proposition 1.8 by assuming $\frac{\partial g}{\partial y}(p) \neq 0$ instead.

Example 1.9. We have already shown that the unit sphere $x^2 + y^2 + z^2 = 1$ is a regular surface using Theorem 1.6 by regarding it is the level set $g^{-1}(1)$ where $g(x, y, z) = x^2 + y^2 + z^2$. We also discussed that

$$F(u, v) = (\sin u\, \cos v,\ \sin u\, \sin v,\ \cos u), \quad 0 < u < \pi,\ 0 < v < 2\pi$$

is a *possible* smooth local parametrization. It is clearly smooth, and by direct computation, one can show

$$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} = \sin u\, (\sin u \cos v,\ \sin u \sin v,\ \cos u)$$

and so $\left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right| = \sin u \neq 0$ for any (u, v) in the domain $(0, \pi) \times (0, 2\pi)$. We leave it as an exercise for readers to verify that F is one-to-one (and so bijective when its codomain is taken to be its image).

Condition (2) is not easy to verify because it is difficult to write down the inverse map F^{-1} explicitly. However, thanks for Proposition 1.8, F is a smooth local parametrization since it satisfies conditions (1) and (3), and it is one-to-one. □

Exercise 1.9. Consider that the *Mercator projection* of the unit sphere:

$$F(u, v) = \left(\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u}\right)$$

where $\sinh u := \frac{1}{2}(e^u - e^{-u})$ and $\cosh u := \frac{1}{2}(e^u + e^{-u})$.

(a) What are the domain and range of F?

(b) Show that F is a smooth local parametrization.

Exercise 1.10. Consider the following parametrization of a torus $\mathbb{T}^2$:

$$F(u, v) = ((r \cos u + R) \cos v,\ (r \cos u + R) \sin v,\ r \sin u)$$

where $(u, v) \in (0, 2\pi) \times (0, 2\pi)$, and $R > r > 0$ are constants. Show that F is a smooth local parametrization.

1.3. Transition Maps

Let $M \subset \mathbb{R}^3$ be a regular surface, and $F_\alpha(u_1, u_2) : \mathcal{U}_\alpha \to M$ and $F_\beta(v_1, v_2) : \mathcal{U}_\beta \to M$ be two smooth local parametrizations of M with overlapping images, i.e. $\mathcal{W} := F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta) \neq \emptyset$. Under this set-up, it makes sense to define the maps $F_\beta^{-1} \circ F_\alpha$ and $F_\alpha^{-1} \circ F_\beta$. However, we need to shrink their domains so as to guarantee they are well-defined. Precisely:

$$(F_\beta^{-1} \circ F_\alpha) : F_\alpha^{-1}(\mathcal{W}) \to F_\beta^{-1}(\mathcal{W})$$
$$(F_\alpha^{-1} \circ F_\beta) : F_\beta^{-1}(\mathcal{W}) \to F_\alpha^{-1}(\mathcal{W})$$

Note that $F_\alpha^{-1}(\mathcal{W})$ and $F_\beta^{-1}(\mathcal{W})$ are open subsets of $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ respectively. The map $F_\beta^{-1} \circ F_\alpha$ describes a relation between two sets of coordinates (u_1, u_2) and (v_1, v_2) of M. In other words, one can regard $F_\beta^{-1} \circ F_\alpha$ as a *change-of-coordinates*, or *transition* map and we can write:

$$F_\beta^{-1} \circ F_\alpha(u_1, u_2) = (v_1(u_1, u_2),\, v_2(u_1, u_2)).$$

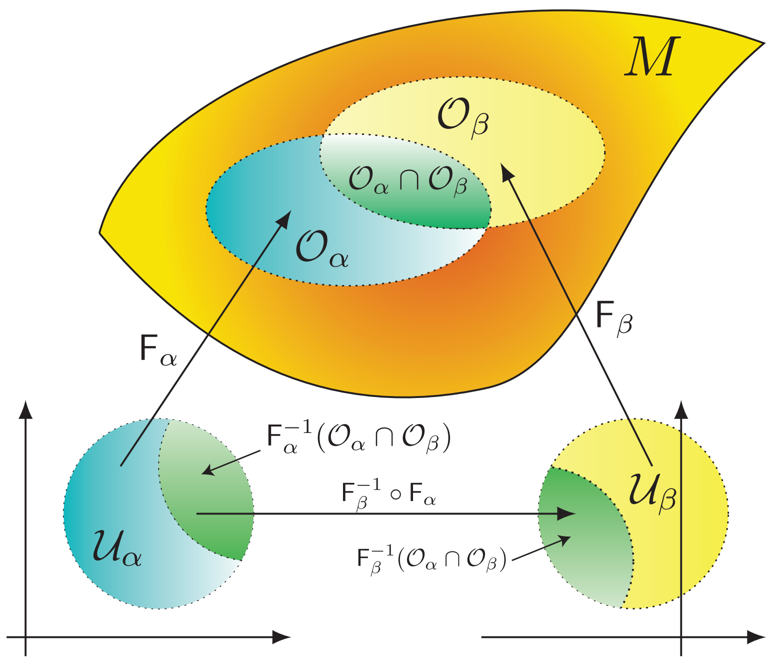


Figure 1.8. Transition maps

One goal of this section is to show that this transition map $F_\beta^{-1} \circ F_\alpha$ is smooth provided that F_α and F_β are two overlapping smooth local parametrizations. Before we present the proof, let us look at some examples of transition maps.

Example 1.10. The xy-plane Π in $\mathbb{R}^3$ is a regular surface which admits a global smooth parametrization $F_\alpha(x,y) = (x,y,0) : \mathbb{R}^2 \to \Pi$. Another way to locally parametrize Π is by polar coordinates $F_\beta : (0,\infty) \times (0,2\pi) \to \Pi$

$$F_\beta(r,\theta) = (r\cos\theta, r\sin\theta, 0)$$

Readers should verify that they are smooth local parametrizations. The image of F_α is the entire xy-plane Π, whereas the image of F_β is the xy-plane with the origin and positive x-axis removed. The transition map $F_\alpha^{-1} \circ F_\beta$ is given by:

$$\begin{aligned} F_\alpha^{-1} \circ F_\beta : (0,\infty) \times (0,2\pi) &\to \mathbb{R}^2 \backslash \{(x,0) : x \geq 0\} \\ (r,\theta) &\mapsto (r\cos\theta, r\sin\theta) \end{aligned}$$

To put it in a simpler form, we can say $(x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$. □

Exercise 1.11. Consider the stereographic parametrizations F_+ and F_- in Example 1.5. Compute the transition maps $F_+^{-1} \circ F_-$ and $F_-^{-1} \circ F_+$. State the maximum possible domain for each map. Are they smooth on their domains?

Exercise 1.12. The unit cylinder Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$\begin{aligned} F &: (0,2\pi) \times \mathbb{R} \to \Sigma^2 & \widetilde{F} &: (-\pi,\pi) \times \mathbb{R} \to \Sigma^2 \\ F(\theta,z) &:= (\cos\theta, \sin\theta, z) & \widetilde{F}(\widetilde{\theta},\widetilde{z}) &:= (\cos\widetilde{\theta}, \sin\widetilde{\theta}, \widetilde{z}) \end{aligned}$$

Compute the transition maps $F^{-1} \circ \widetilde{F}$ and $\widetilde{F}^{-1} \circ F$. State their maximum possible domains. Are they smooth on their domains?

Exercise 1.13. The Möbius strip Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$F : (-1,1) \times (0,2\pi) \to \Sigma^2 \qquad \widetilde{F} : (-1,1) \times (-\pi,\pi) \to \Sigma^2$$

$$F(u,\theta) = \begin{bmatrix} \left(3 + u\cos\frac{\theta}{2}\right)\cos\theta \\ \left(3 + u\cos\frac{\theta}{2}\right)\sin\theta \\ u\sin\frac{\theta}{2} \end{bmatrix} \qquad \widetilde{F}(\widetilde{u},\widetilde{\theta}) = \begin{bmatrix} \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \\ \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \\ \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \end{bmatrix}$$

Compute the transition maps, state their maximum possible domains and verify that they are smooth.

The proposition below shows that the transition maps between any pair of smooth local parametrizations are smooth:

Proposition 1.11. *Let $M \subset \mathbb{R}^3$ be a regular surface, and $F_\alpha(u_1,u_2) : \mathcal{U}_\alpha \to M$ and $F_\beta(v_1,v_2) : \mathcal{U}_\beta \to M$ be two smooth local parametrizations of M with overlapping images, i.e. $\mathcal{W} := F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta) \neq \emptyset$. Then, the transition maps defined below are also smooth maps:*

$$\begin{aligned} (F_\beta^{-1} \circ F_\alpha) &: F_\alpha^{-1}(\mathcal{W}) \to F_\beta^{-1}(\mathcal{W}) \\ (F_\alpha^{-1} \circ F_\beta) &: F_\beta^{-1}(\mathcal{W}) \to F_\alpha^{-1}(\mathcal{W}) \end{aligned}$$

Proof. It suffices to show $F_\beta^{-1} \circ F_\alpha$ is smooth as the other one $F_\alpha^{-1} \circ F_\beta$ can be shown by symmetry. Furthermore, since differentiability is a local property, we may fix a point $p \in \mathcal{W} \subset M$ and show that $F_\beta^{-1} \circ F_\alpha$ is smooth at the point $F_\alpha^{-1}(p)$.

By condition of (3) of smooth local parametrizations, we have:

$$\frac{\partial F_\alpha}{\partial u_1}(p) \times \frac{\partial F_\alpha}{\partial u_2}(p) \neq 0$$

By straight-forward computations, one can show that this cross product is given by:

$$\frac{\partial F_\alpha}{\partial u_1} \times \frac{\partial F_\alpha}{\partial u_2} = \left(\det \frac{\partial(y,z)}{\partial(u_1,u_2)}(p),\ \det \frac{\partial(z,x)}{\partial(u_1,u_2)}(p),\ \det \frac{\partial(x,y)}{\partial(u_1,u_2)}(p)\right).$$

Hence, at least one of the determinants is non-zero. Without loss of generality, assume that:

$$\det \frac{\partial(x,y)}{\partial(u_1,u_2)}(p) \neq 0.$$

Both $\frac{\partial F_\alpha}{\partial u_1}(p) \times \frac{\partial F_\alpha}{\partial u_2}(p)$ and $\frac{\partial F_\beta}{\partial v_1}(p) \times \frac{\partial F_\beta}{\partial v_2}(p)$ are normal vectors to the surface at p. Given that the former has non-zero $\hat{k}$-component, then so does the latter. Therefore, we have:

$$\det \frac{\partial(x,y)}{\partial(v_1,v_2)}(p) \neq 0.$$

Then we proceed as in the proof of Proposition 1.8. Define $\pi(x,y,z) = (x,y)$, then

$$\begin{aligned} \pi \circ F_\beta : \mathcal{U}_\beta &\to \mathbb{R}^2 \\ (v_1,v_2) &\mapsto (x(v_1,v_2), y(v_1,v_2)) \end{aligned}$$

has non-zero Jacobian determinant $\det \dfrac{\partial(x,y)}{\partial(v_1,v_2)}$ at p. Therefore, by the Inverse Function Theorem, $(\pi \circ F_\beta)^{-1}$ exists and is smooth near p. Since $F_\beta^{-1} \circ F_\alpha = (\pi \circ F_\beta)^{-1} \circ (\pi \circ F_\alpha)$, and all of $(\pi \circ F_\beta)^{-1}$, π and F_α are smooth maps, their composition is also a smooth map. We have proved $F_\beta^{-1} \circ F_\alpha$ is smooth near p. Since p is arbitrary, $F_\beta^{-1} \circ F_\alpha$ is in fact smooth on the domain $F_\alpha^{-1}(\mathcal{W})$. $\square$

Exercise 1.14. Rewrite the proof of Proposition 1.11, *mutatis mutandis*, by assuming $\det \dfrac{\partial(y,z)}{\partial(u_1,u_2)}(p) \neq 0$ instead.

1.4. Maps and Functions from Surfaces

Let M be a regular surface in $\mathbb{R}^3$ with a smooth local parametrization $F(u_1, u_2) : \mathcal{U} \to M$. Then, for any $p \in F(\mathcal{U})$, one can define the partial derivatives for a function $f : M \to \mathbb{R}$ at p as follows. The subtle issue is that the domain of f is the surface M, but by pre-composing f with F, i.e. $f \circ F$, one can regard it as a map from $\mathcal{U} \subset \mathbb{R}^2$ to $\mathbb{R}$. With a little abuse of notations, we denote:

$$\frac{\partial f}{\partial u_j}(p) := \frac{\partial (f \circ F)}{\partial u_j}(u_1, u_2)$$

where (u_1, u_2) is the point corresponding to p, i.e. $F(u_1, u_2) = p$.

Remark 1.12. Note that $\frac{\partial f}{\partial u_j}(p)$ is defined locally on $F(\mathcal{U})$, and depends on the choice of local parametrization F near p. □

Definition 1.13 (Functions of Class C^k)**.** Let M be a regular surface in $\mathbb{R}^3$, and $f : M \to \mathbb{R}$ be a function defined on M. We say f is C^k at $p \in M$ if for any smooth local parametrization $F : \mathcal{U} \to M$ with $p \in F(\mathcal{U})$, the composition $f \circ F$ is C^k at (u_1, u_2) corresponding to p.

If f is C^k at p for any $p \in M$, then we say that f is a C^k function on M. Here k can be taken to be ∞, and in such case we call f to be a C^∞ (or smooth) function.

Remark 1.14. Although we require $f \circ F$ to be C^k at $p \in M$ for *any* local parametrization F in order to say that f is C^k, by Proposition 1.11 it suffices to show that $f \circ F$ is C^k at p for *at least one* F near p. It is because

$$f \circ \widetilde{F} = (f \circ F) \circ (F^{-1} \circ \widetilde{F})$$

and compositions of C^k maps (between Euclidean spaces) are C^k. □

Example 1.15. Let M be a regular surface in $\mathbb{R}^3$, then each of the x, y and z coordinates in $\mathbb{R}^3$ can be regarded as a function from M to $\mathbb{R}$. For any smooth local parametrization $F : \mathcal{U} \to M$ around p given by

$$F(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)),$$

we have $x \circ F(u_1, u_2) = x(u_1, u_2)$. Since F is C^∞, we get $x \circ F$ is C^∞ as well. Therefore, the coordinate functions x, y and z for any regular surface is smooth. □

Example 1.16. Let $f : M \to \mathbb{R}$ be the function from a regular surface M in $\mathbb{R}^3$ defined by:

$$f(p) := |p - p_0|^2$$

where $p_0 = (x_0, y_0, z_0)$ is a fixed point of $\mathbb{R}^3$. Suppose $F(u, v)$ is a local parametrization of M. We want to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

Write $(x, y, z) = F(u, v)$ so that x, y and z are functions of (u, v). Then

$$\begin{aligned}
\frac{\partial f}{\partial u} &:= \frac{\partial}{\partial u}(f \circ F) \\
&= \frac{\partial}{\partial u} f(x(u, v), y(u, v), z(u, v)) \\
&= \frac{\partial}{\partial u}\left((x(u, v) - x_0)^2 + (y(u, v) - y_0)^2 + (z(u, v) - z_0)^2\right) \\
&= 2(x - x_0)\frac{\partial x}{\partial u} + 2(y - y_0)\frac{\partial y}{\partial u} + 2(z - z_0)\frac{\partial z}{\partial u}
\end{aligned}$$

Note that we can differentiate x, y and z by u because $F(u,v)$ is smooth. Similarly, we have:

$$\frac{\partial f}{\partial v} = 2(x-x_0)\frac{\partial x}{\partial v} + 2(y-y_0)\frac{\partial y}{\partial v} + 2(z-z_0)\frac{\partial z}{\partial v}.$$

Again since $F(u,v)$ (and hence x, y and z) is a smooth function of (u,v), we can differentiate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ as many times as we wish. This concludes that f is a smooth funciton. $\square$

Exercise 1.15. Let $p_0(x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ and let $f(p) = |p - p_0|$ be the Euclidean distance between p and p_0 in $\mathbb{R}^3$. Suppose M is a regular surface in $\mathbb{R}^3$, one can then restrict the domain of f to M and consider it as a function:

$$\begin{aligned} f : M &\to \mathbb{R} \\ p &\mapsto |p - p_0| \end{aligned}$$

Under what condition is the function $f : M \to \mathbb{R}$ smooth?

Now let M and N be two regular surfaces in $\mathbb{R}^3$. Then, one can also talk about mappings $\Phi : M \to N$ between them. In this section, we will define the notion of smooth maps between two surfaces.

Suppose $F : \mathcal{U}_M \to M$ and $G : \mathcal{U}_N \to N$ are two smooth local parametrizations of M and N respectively. One can then consider the composition $G^{-1} \circ \Phi \circ F$ after shrinking the domain. It is then a map between open subsets of $\mathbb{R}^2$.

However, in order for this composition to be well-defined, we require the image of $\Phi \circ F$ to be contained in the image of G, which is not always guaranteed. Let $\mathcal{W} := \Phi(\mathcal{O}_M) \cap \mathcal{O}_N$ be the overlapping region on N of these two images. Then, provided that $\mathcal{W} \neq \emptyset$, the composition $G^{-1} \circ \Phi \circ F$ becomes well-defined as a map on:

$$G^{-1} \circ \Phi \circ F : (\Phi \circ F)^{-1}(\mathcal{W}) \to \mathcal{U}_N.$$

From now on, whenever we talk about this composition $G^{-1} \circ \Phi \circ F$, we always implicitly assume that $\mathcal{W} \neq \emptyset$ and its domain is $(\Phi \circ F)^{-1}(\mathcal{W})$.

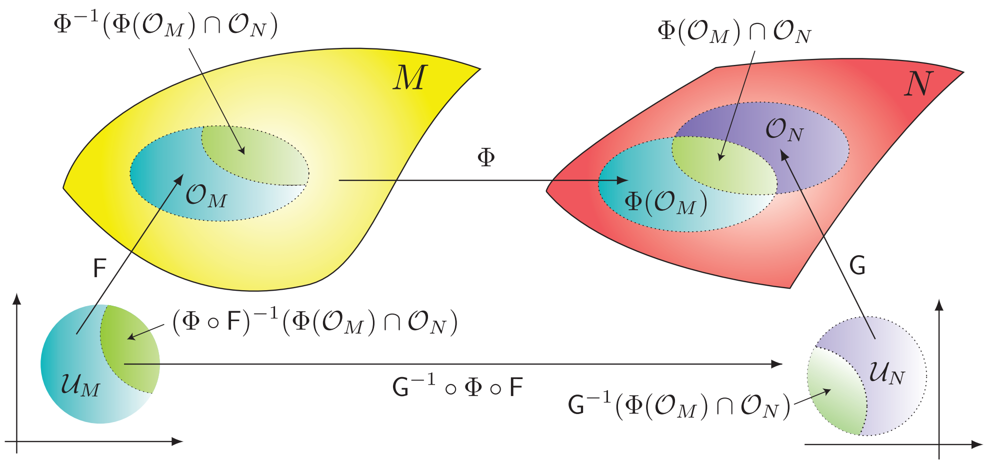


Figure 1.9. maps between regular surfaces

Definition 1.17 (Maps of Class C^k)**.** Let M and N be two regular surfaces in $\mathbb{R}^3$, and $\Phi : M \to N$ be a map between them. We say Φ is C^k at $p \in M$ if for any smooth local parametrization $F : \mathcal{U}_M \to M$ with $p \in F(\mathcal{U}_M)$, and $G : \mathcal{U}_N \to N$ with $\Phi(p) \in G(\mathcal{U}_N)$, the composition $G^{-1} \circ \Phi \circ F$ is C^k at $F^{-1}(p)$ as a map between subsets of $\mathbb{R}^2$.

If Φ is C^k at p for any $p \in M$, then we say that Φ is C^k on M. Here k can be taken to be ∞, and in such case we call Φ to be C^∞ (or smooth) on M.

Remark 1.18. Although we require $G^{-1} \circ \Phi \circ F$ to be C^k at $p \in M$ for *any* local parametrizations F and G in order to say that Φ is C^k, by Proposition 1.11 it suffices to show that $G^{-1} \circ \Phi \circ F$ is C^k at p for *at least one* pair of F and G. It is because

$$\widetilde{G}^{-1} \circ \Phi \circ \widetilde{F} = (\widetilde{G}^{-1} \circ G) \circ (G^{-1} \circ \Phi \circ F) \circ (F^{-1} \circ \widetilde{F})$$

and compositions of C^k maps (between Euclidean spaces) are C^k. □

Example 1.19. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Consider the antipodal map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ taking P to $-P$. In Example 1.4, two of the local parametrizations are given by:

$$F_1(u_1, u_2) = (u_1,\ u_2,\ \sqrt{1 - u_1^2 - u_2^2}) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_+$$
$$F_2(v_1, v_2) = (v_1,\ v_2,\ -\sqrt{1 - v_1^2 - v_2^2}) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_-$$

where $B_1(0)$ is the open unit disk in $\mathbb{R}^2$ centered at the origin, and $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ are the upper and lower hemispheres of $\mathbb{S}^2$ respectively. One can compute that:

$$\begin{aligned} F_2^{-1} \circ \Phi \circ F_1(u_1, u_2) &= F_2^{-1} \circ \Phi \left(u_1,\ u_2,\ \sqrt{1 - u_1^2 - u_2^2}\right) \\ &= F_2^{-2} \left(-u_1,\ -u_2,\ -\sqrt{1 - u_1^2 - u_2^2}\right) \\ &= (-u_1,\ -u_2) \end{aligned}$$

Clearly, the map $(u_1, u_2) \mapsto (-u_1, -u_2)$ is C^∞. It shows the antipodal map Φ is C^∞ at every point in $F_1(B_1(0))$. One can show in similar way using other local parametrizations that Φ is C^∞ at points on $\mathbb{S}^2$ not covered by F_1.

Note that, for instance, the images of $\Phi \circ F_1$ and F_1 are disjoint, and so $F_1^{-1} \circ \Phi \circ F_1$ is not well-defined. We don't need to verify whether it is smooth. □

Exercise 1.16. Let Φ be the antipodal map considered in Example 1.19, and F_+ and F_- be the two stereographic parametrizations of $\mathbb{S}^2$ defined in Example 1.5. Compute the maps $F_+^{-1} \circ \Phi \circ F_+$, $F_-^{-1} \circ \Phi \circ F_+$, $F_+^{-1} \circ \Phi \circ F_-$ and $F_-^{-1} \circ \Phi \circ F_-$. State their domains, and verify that they are smooth on their domains.

Exercise 1.17. Denote $\mathbb{S}^2$ to be the unit sphere $x^2 + y^2 + z^2 = 1$. Let $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ be the rotation map about the z-axis defined by:

$$\Phi(x, y, z) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha, z)$$

where α is a fixed angle. Show that Φ is smooth.

Let M and N be two regular surfaces. If a map $\Phi : M \to N$ is C^∞, invertible, with C^∞ inverse map $\Phi^{-1} : N \to M$, then we say:

Definition 1.20 (Diffeomorphisms)**.** A map $\Phi : M \to N$ between two regular surfaces M and N in $\mathbb{R}^3$ is said to be a *diffeomorphism* if Φ is C^∞ and invertible, and also the inverse map Φ^{-1} is C^∞. If such a map Φ exists between M and N, then we say the surfaces M and N are *diffeomorphic*.

Example 1.21. The antipodal map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ described in Example 1.19 is a diffeomorphism between $\mathbb{S}^2$ and itself. □

Example 1.22. The sphere $x^2 + y^2 + z^2 = 1$ and the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ are diffeomorphic, under the map $\Phi(x, y, z) = (ax, by, cz)$ restricted on $\mathbb{S}^2$. □

Exercise 1.18. Given any pair of C^∞ functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, show that the graphs Γ_f and Γ_g are diffeomorphic.

Exercise 1.19. Show that $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ defined in Exercise 1.17 is a diffeomorphism.

1.5. Tangent Planes and Tangent Maps

1.5.1. Tangent Planes of Regular Surfaces. The tangent plane is an important geometric object associated to a regular surface. Condition (3) of a smooth local parametrization $F(u, v)$ requires that the cross-product $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}$ is non-zero for any (u, v) in the domain, or equivalently, both tangent vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ must be non-zero vectors and they are non-parallel to each other.

Therefore, the two vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ span a two-dimensional subspace in $\mathbb{R}^3$. We call this subspace the *tangent plane*, which is defined rigorously as follows:

Definition 1.23 (Tangent Plane). Let M be a regular surface in $\mathbb{R}^3$ and p be a point on M. Suppose $F(u, v) : \mathcal{U} \subset \mathbb{R}^2 \to M$ is a smooth local parametrization around p, then the *tangent plane* at p, denoted by T_pM, is defined as follows:

$$T_pM := \operatorname{span}\left\{\frac{\partial F}{\partial u}(p), \frac{\partial F}{\partial v}(p)\right\} = \left\{a\frac{\partial F}{\partial u}(p) + b\frac{\partial F}{\partial v}(p) : a, b \in \mathbb{R}\right\}.$$

Here we have abused the notations for simplicity: $\frac{\partial F}{\partial u}(p)$ means $\frac{\partial F}{\partial u}$ evaluated at $(u, v) = F^{-1}(p)$. Similarly for $\frac{\partial F}{\partial v}(p)$.

Rigorously, T_pM is a plane passing through the *origin* while $p + T_pM$ is the plane tangent to the surface at p (see Figure 1.10). The difference between T_pM and $p + T_pM$ is very subtle, and we will almost neglect this difference.

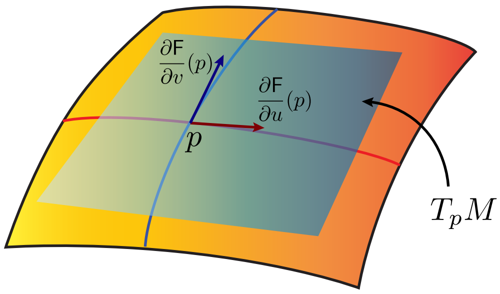


Figure 1.10. Tangent plane $p + T_pM$ at $p \in M$

Exercise 1.20. Show that the equation of the tangent plane $p + T_pM$ of the graph of a smooth function $f(x, y)$ at $p = (x_0, y_0, f(x_0, y_0))$ is given by:

$$z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

Exercise 1.21. [**dC76**, P.88] Consider the surface M given by $z = xf(y/x)$, where $x \neq 0$ and f is a smooth function. Show that the tangent planes $p + T_pM$ must pass through the origin $(0, 0, 0)$.

1.5.2. Tangent Maps between Regular Surfaces. Given a smooth map $\Phi : M \to N$ between two regular surfaces M and N, there is a naturally defined map called the *tangent map*, denoted by Φ_* in this course, between the tangent planes T_pM and $T_{\Phi(p)}N$.

Let us consider a smooth local parametrization $F(u_1, u_2) : \mathcal{U}_M \to M$. The composition $\Phi \circ F$ can be regarded as a map from $\mathcal{U}_M$ to $\mathbb{R}^3$, so one can talk about its partial derivatives $\frac{\partial(\Phi\circ F)}{\partial u_i}$:

$$\frac{\partial \Phi}{\partial u_i}(\Phi(p)) := \left.\frac{\partial(\Phi\circ F)}{\partial u_i}\right|_{(u_1,u_2)} = \left.\frac{d}{dt}\right|_{t=0} \Phi\circ F((u_1,u_2)+t\hat{e}_i)$$

where (u_1,u_2) is a point in $\mathcal{U}_M$ such that $F(u_1,u_2)=p$. The curve $F((u_1,u_2)+t\hat{e}_i)$ is a curve on M with parameter t along the u_i-direction. The curve $\Phi\circ F((u_1,u_2)+t\hat{e}_i)$ is then the image of the u_i-curve of M under the map Φ (see Figure 1.11). It is a curve on N so $\frac{\partial \Phi}{\partial u_i}$ which is a tangent vector to the surface N.

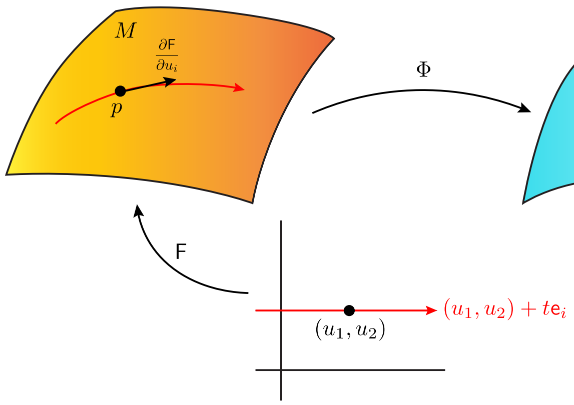


Figure 1.11. Partial derivative of the map $\Phi : M \to N$

Exercise 1.22. Denote $\mathbb{S}^2$ to be the unit sphere $x^2+y^2+z^2=1$. Let $\Phi:\mathbb{S}^2\to\mathbb{S}^2$ be the rotation map about the z-axis defined by:

$$\Phi(x,y,z) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha, z)$$

where α is a fixed angle. Calculate the following partial derivatives under the given local parametrizations:

(a) $\dfrac{\partial\Phi}{\partial\theta}$ and $\dfrac{\partial\Phi}{\partial\varphi}$ under $F(\theta,\varphi)=(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi)$;

(b) $\dfrac{\partial\Phi}{\partial u}$ and $\dfrac{\partial\Phi}{\partial v}$ under F_2 in Example 1.4;

(c) $\dfrac{\partial\Phi}{\partial u}$ and $\dfrac{\partial\Phi}{\partial v}$ under F_+ in Example 1.5.

Next, we write the partial derivative $\frac{\partial\Phi}{\partial u_i}$ in a *fancy* way. Define:

$$\Phi_*\left(\frac{\partial F}{\partial u_i}\right) := \frac{\partial \Phi}{\partial u_i}.$$

Then, one can regard Φ_* as a map that takes the tangent vector $\frac{\partial F}{\partial u_i}$ in T_pM to another vector $\frac{\partial\Phi}{\partial u_j}$ in $T_{\Phi(p)}N$. Since $\left\{\frac{\partial F}{\partial u_i}(p)\right\}$ is a basis of T_pM, one can then extend Φ_* linearly and define it as the *tangent map* of Φ. Precisely, we have:

Definition 1.24 (Tangent Maps)**.** Let $\Phi : M \to N$ be a smooth map between two regular surfaces M and N in $\mathbb{R}^3$. Let $F : \mathcal{U}_M \to M$ and $G : \mathcal{U}_N \to N$ be two smooth local parametrizations covering p and $\Phi(p)$ respectively. Then, the *tangent map* of Φ at $p \in M$ is denoted by $(\Phi_*)_p$ and is defined as:

$$(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$$
$$(\Phi_*)_p \left(\sum_{i=1}^{2} a_i \frac{\partial F}{\partial u_i}(p) \right) = \sum_{i=1}^{2} a_i \frac{\partial \Phi}{\partial u_i}(\Phi(p))$$

If the point p is clear from the context, $(\Phi_*)_p$ can be simply denoted by Φ_*.

Remark 1.25. Some textbooks may use $d\Phi_p$ to denote the tangent map of Φ at p. □

Example 1.26. Consider the unit sphere $\mathbb{S}^2$ locally parametrized by

$$F(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

and the rotation map:

$$\Phi(x, y, z) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha, z)$$

From Exercise 1.22, one should have figured out that:

$$\frac{\partial \Phi}{\partial \theta} = (-\sin\varphi\sin(\theta+\alpha), \sin\varphi\cos(\theta+\alpha), 0)$$
$$\frac{\partial \Phi}{\partial \varphi} = (\cos\varphi\cos(\theta+\alpha), \cos\varphi\sin(\theta+\alpha), -\sin\varphi)$$

Next we want to write them in terms of the basis $\left\{ \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \varphi} \right\}$. However, we should be careful about the base points of these vectors. Consider a point $p \in \mathbb{S}^2$ with local coordinates (θ, φ), the vectors $\frac{\partial \Phi}{\partial \theta}$ and $\frac{\partial \Phi}{\partial \varphi}$ computed above are based at the point $\Phi(p)$ with local coordinates $(\theta + \alpha, \varphi)$. Therefore, we should express them in terms of the basis $\left\{ \frac{\partial F}{\partial \theta}(\Phi(p)), \frac{\partial F}{\partial \varphi}(\Phi(p)) \right\}$, not $\left\{ \frac{\partial F}{\partial \theta}(p), \frac{\partial F}{\partial \varphi}(p) \right\}$!

At $\Phi(p)$, we have:

$$\frac{\partial F}{\partial \theta}(\Phi(p)) = (-\sin\varphi\sin(\theta+\alpha), \sin\varphi\cos(\theta+\alpha), 0) = \frac{\partial \Phi}{\partial \theta}(\Phi(p))$$
$$\frac{\partial F}{\partial \varphi}(\Phi(p)) = (\cos\varphi\cos(\theta+\alpha), \cos\varphi\sin(\theta+\alpha), -\sin\varphi) = \frac{\partial \Phi}{\partial \varphi}(\Phi(p))$$

Therefore, the tangent map $(\Phi_*)_p$ acts on the basis vectors by:

$$(\Phi_*)_p \left(\frac{\partial F}{\partial \theta}(p) \right) = \frac{\partial F}{\partial \theta}(\Phi(p))$$
$$(\Phi_*)_p \left(\frac{\partial F}{\partial \varphi}(p) \right) = \frac{\partial F}{\partial \varphi}(\Phi(p))$$

In other words, the matrix representation $[(\Phi_*)_p]$ with respect to the bases

$$\left\{ \frac{\partial F}{\partial \theta}(p), \frac{\partial F}{\partial \varphi}(p) \right\} \text{ for } T_p\mathbb{S}^2 \qquad \left\{ \frac{\partial F}{\partial \theta}(\Phi(p)), \frac{\partial F}{\partial \varphi}(\Phi(p)) \right\} \text{ for } T_{\Phi(p)}\mathbb{S}^2$$

is the identity matrix. However, it is not perfectly correct to say $(\Phi_*)_p$ is an identity map, since the domain and co-domain are different tangent planes. □

Exercise 1.23. Let Φ be as in Example 1.26. Consider the stereographic parametrization $F_+(u,v)$ defined in Example 1.5. Suppose $p \in \mathbb{S}^2$, express the matrix representation $[(\Phi_*)_p]$ with respect to the bases $\left\{\frac{\partial F_+}{\partial u}, \frac{\partial F_+}{\partial v}\right\}_p$ and $\left\{\frac{\partial F_+}{\partial u}, \frac{\partial F_+}{\partial v}\right\}_{\Phi(p)}$

1.5.3. Tangent Maps and Jacobian Matrices. Let $\Phi : M \to N$ be a smooth map between two regular surfaces. Instead of computing the matrix representation of the tangent map Φ_* directly by taking partial derivatives (c.f. Example 1.26), one can also find it out by computing a Jacobian matrix.

Suppose $F(u_1,u_2) : \mathcal{U}_M \to M$ and $G(v_1,v_2) : \mathcal{U}_N \to N$ are local parametrizations of M and N. The composition $G^{-1} \circ \Phi \circ F$ can be regarded as a map between the u_1u_2-plane to the v_1v_2-plane. As such, one can write

$$G^{-1} \circ \Phi \circ F(u_1,u_2) = (v_1(u_1,u_2), v_2(u_1,u_2)).$$

By considering $\Phi \circ F(u_1,u_2) = G(v_1(u_1,u_2), v_2(u_1,u_2))$, one can differentiate both sides with respect to u_i:

$$\frac{\partial}{\partial u_i}(\Phi \circ F) = \frac{\partial}{\partial u_i} G(v_1(u_1,u_2), v_2(u_1,u_2)) = \sum_{k=1}^{2} \frac{\partial G}{\partial v_k}\frac{\partial v_k}{\partial u_i}. \tag{1.1}$$

Here we used the chain rule. Note that $\left\{\frac{\partial G}{\partial v_k}\right\}$ is a basis for $T_{\Phi(p)}N$.

Using (1.1), one can see:

$$\Phi_*\left(\frac{\partial F}{\partial u_1}\right) := \frac{\partial \Phi}{\partial u_1} = \frac{\partial}{\partial u_1}(\Phi \circ F) = \frac{\partial v_1}{\partial u_1}\frac{\partial G}{\partial v_1} + \frac{\partial v_2}{\partial u_1}\frac{\partial G}{\partial v_2}$$
$$\Phi_*\left(\frac{\partial F}{\partial u_2}\right) := \frac{\partial \Phi}{\partial u_2} = \frac{\partial}{\partial u_2}(\Phi \circ F) = \frac{\partial v_1}{\partial u_2}\frac{\partial G}{\partial v_1} + \frac{\partial v_2}{\partial u_2}\frac{\partial G}{\partial v_2}$$

Hence the matrix representation of $(\Phi_*)_p$ with respect to the bases $\left\{\frac{\partial F}{\partial u_i}(p)\right\}$ and $\left\{\frac{\partial G}{\partial v_i}(\Phi(p))\right\}$ is the Jacobian matrix:

$$\left.\frac{\partial(v_1,v_2)}{\partial(u_1,u_2)}\right|_{F^{-1}(p)} = \begin{bmatrix} \frac{\partial v_1}{\partial u_1} & \frac{\partial v_1}{\partial u_2} \\ \frac{\partial v_2}{\partial u_1} & \frac{\partial v_2}{\partial u_2} \end{bmatrix}_{F^{-1}(p)}$$

Example 1.27. Let $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ be the rotation map as in Example 1.26. Consider again the local parametrization:

$$F(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

By standard trigonometry, one can find out that $\Phi(F(\theta,\varphi)) = F(\theta+\alpha,\varphi)$. Equivalently, the map $F^{-1} \circ \Phi \circ F$ (in a suitable domain) is the map:

$$(\theta,\varphi) \mapsto (\theta+\alpha,\varphi).$$

As α is a constant, the Jacobian matrix of $F^{-1} \circ \Phi \circ F$ is the identity matrix, and so the matrix $[(\Phi_*)_p]$ with respect to the bases $\left\{\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \varphi}\right\}_p$ and $\left\{\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \varphi}\right\}_{\Phi(p)}$ is the identity matrix (which was also obtained by somewhat tedious computations in Example 1.26). □

Exercise 1.24. Do Exercise 1.23 by considering Jacobian matrices.

Chapter 2

Abstract Manifolds

> "Manifolds are a bit like pornography: hard to define, but you know one when you see one."
>
> *Shmuel Weinberger*

2.1. Smooth Manifolds

Intuitively, a manifold is a space which locally resembles an Euclidean space. Regular surfaces are examples of manifolds. Being locally Euclidean, a manifold is equipped with a local coordinate system around every point so that many concepts in Calculus on Euclidean spaces can carry over to manifolds.

Unlike regular surfaces, we do *not* require a manifold to be a subset of $\mathbb{R}^n$. A manifold can just stand alone by itself like the Universe is regarded as a curved space-time sheet with nothing "outside" in General Relativity. However, we do require that a manifold satisfies certain *topological* conditions.

2.1.1. Point-Set Topology. In order to state the formal definition of a manifold, there are some topological terms (such as Hausdorff, second countable, etc.) we will briefly introduce. However, we will not take a long detour to go through every single topological concept, otherwise we will not have time to cover the more interesting material about smooth manifolds. Moreover, these topological conditions are very common as long as the space we are looking at is not "strange".

A *topological space* X is a set equipped with a collection $\mathcal{T}$ of subsets of X such that:

(a) $\emptyset, X \in \mathcal{T}$; and

(b) for any arbitrary sub-collection $\{U_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{T}$, we have $\displaystyle\bigcup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{T}$; and

(c) for any finite sub-collection $\{U_1, \ldots, U_N\} \subset \mathcal{T}$, we have $\displaystyle\bigcap_{i=1}^{N} U_\alpha \in \mathcal{T}$.

If $\mathcal{T}$ is such a collection, we call $\mathcal{T}$ a *topology* of X. Elements in $\mathcal{T}$ are called *open sets* of X.

Example 2.1. The Euclidean space $\mathbb{R}^n$ equipped with the collection

$$\mathcal{T} = \text{collection of all open sets (in usual sense) in } \mathbb{R}^n$$

is an example of a topological space. The collection $\mathcal{T}$ is called the *usual topology* of $\mathbb{R}^n$. □

Example 2.2. Any subset $S \subset \mathbb{R}^n$, equipped with the collection

$$\mathcal{T}_S = \{S \cap U : U \text{ is an open set (in usual sense) in } \mathbb{R}^n\}$$

is an example of a topological space. The collection $\mathcal{T}_S$ is called the *subspace topology*. □

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, one can talk about functions or mapping between them. A map $\Phi : X \to Y$ is said to be *continuous* with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ if for any $U \in \mathcal{T}_Y$, we have $\Phi^{-1}(U) \in \mathcal{T}_X$. This definition is a generalization of continuous functions between Euclidean spaces equipped with the usual topologies. If the map $\Phi : X \to Y$ is one-to-one and onto, and both Φ and Φ^{-1} are continuous, then we say Φ is a *homeomorphism* and the spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *homeomorphic*.

A topological space $(X, \mathcal{T})$ is said to be *Hausdorff* if for any pair of distinct points $p, q \in X$, we have $U_1, U_2 \in \mathcal{T}$ such that $p \in U_1$, $q \in U_2$ and $U_1 \cap U_2 = \emptyset$. In other words, points of a Hausdorff space can be *separated* by open sets. It is intuitive that $\mathbb{R}^n$ with the usual topology is a Hausdorff space. Any subset $S \subset \mathbb{R}^n$ with subspace topology is also a Hausdorff space.

A topological space $(X, \mathcal{T}_X)$ is said to be *second countable* if there is a countable sub-collection $\{U_i\}_{i=1}^\infty \subset \mathcal{T}$ such that any set $U \in \mathcal{T}$ can be expressed as a union of some of these U_i's. For instance, $\mathbb{R}^n$ with usual topology is second countable since by density of rational numbers, any open set can be expressed as a countable union of open balls with rational radii and centers.

This introduction to point-set topology is intended to be short. It may not make sense to everybody, but it doesn't hurt! Point-set topology is not the main dish of the course. Many spaces we will look at are either Euclidean spaces, their subsets or sets *derived* from Euclidean spaces. Most of them are Hausdorff and second countable. Readers who want to learn more about point-set topology may consider taking MATH 4225. For more thorough treatment on point-set topology, please consult [**Mun00**]. Meanwhile, the *take-home message* of this introduction is that we don't have to worry much about point-set topology in this course!

2.1.2. Definitions and Examples.

Now we are ready to learn what a manifold is. We will first introduce topological manifolds, which are objects that locally look like Euclidean space in certain *continuous* sense:

Definition 2.3 (Topological Manifolds)**.** A Hausdorff, second countable topological space M is said to be an *n-dimensional topological manifold*, or in short a *topological n-manifold*, if for any point $p \in M$, there exists a homeomorphism $F : \mathcal{U} \to \mathcal{O}$ between a non-empty open subset $\mathcal{U} \subset \mathbb{R}^n$ and an open subset $\mathcal{O} \subset M$ containing p. This homeomorphism F is called a *local parametrization* (or *local coordinate chart*) around p.

Example 2.4. Any regular surface is a topological manifold since its local parametrizations are all homeomorphisms. Therefore, spheres, cylinders, torus, etc. are all topological manifolds.

However, a double cone (see Figure 2.1) is not a topological manifold since the vertex is a "bad" point. Any open set containing the vertex cannot be homeomorphic to any open set in Euclidean space.

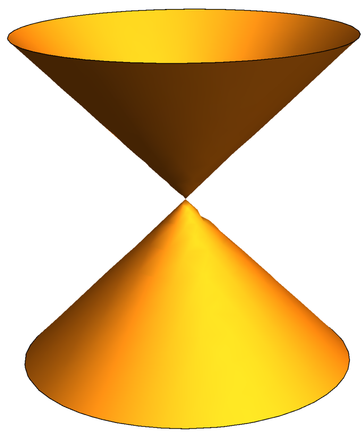


Figure 2.1. Double cone is not locally Euclidean near its vertex.

Remark 2.5. Note that around every p there may be more than one local parametrizations. If $F_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha$ and $F_\beta : \mathcal{U}_\beta \to \mathcal{O}_\beta$ are two local parametrizations around p, then the composition:

$$(F_\beta^{-1} \circ F_\alpha) : F_\alpha^{-1}(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to F_\beta^{-1}(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$$
$$(F_\alpha^{-1} \circ F_\beta) : F_\beta^{-1}(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to F_\alpha^{-1}(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$$

are often called the *transition maps* between these local parametrizations. We need to restrict their domains to smaller sets so as to guarantee the transition maps are well-defined (c.f. Section 1.3). □

On a topological manifold, there is a coordinate system around every point. However, many concepts in Calculus involve taking *derivatives*. In order to carry out differentiations on manifolds, it is not sufficient to be merely locally *homeomorphic* to Euclidean spaces. We need the local parametrization F to be *differentiable* in a certain sense.

For regular surfaces in $\mathbb{R}^3$, the local parametrization $F : \mathcal{U} \to \mathbb{R}^3$ are maps between Euclidean spaces, so it makes sense to take derivatives of F. However, an abstract manifold may not be sitting in $\mathbb{R}^3$ or $\mathbb{R}^N$, and therefore it is difficult to make of sense of differentiability of $F : \mathcal{U} \to \mathcal{O}$. To get around this issue, we will *not* talk about the differentiability of a local parametrization F, but instead talk about the differentiability of *transition maps*.

In Proposition 1.11 of Chapter 1 we showed that any two overlapping local parametrizations F_α and F_β of a regular surface M have smooth transition maps $F_\beta^{-1} \circ F_\alpha$ and $F_\alpha^{-1} \circ F_\beta$. Now consider an abstract topological manifold. Although the local parametrizations F_α and F_β may not have a codomain sitting in Euclidean spaces, the transition maps $F_\beta^{-1} \circ F_\alpha$ and $F_\alpha^{-1} \circ F_\beta$ are indeed maps between open subsets of Euclidean spaces!

While we cannot differentiate local parametrizations $F : \mathcal{U} \to \mathcal{O} \subset M$ for abstract manifolds, we can do so for the transition maps $F_\beta^{-1} \circ F_\alpha$ and $F_\alpha^{-1} \circ F_\beta$. This motivates the definition of a smooth manifold:

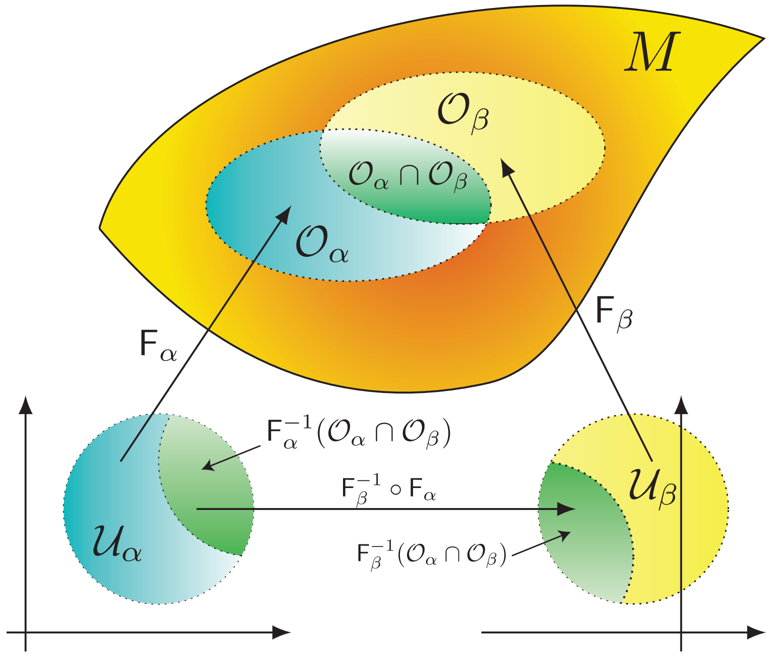


Figure 2.2. transition maps of a manifold

Definition 2.6 (Smooth Manifolds). A n-dimensional topological manifold M is said to be an *n-dimensional smooth manifold*, or in short a *smooth n-manifold*, if there is a collection $\mathcal{A}$ of local parametrizations $F_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha$ such that

(1) $\bigcup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha = M$, i.e. these local parametrizations cover all of M; and

(2) all transition maps $F_\alpha^{-1} \circ F_\beta$ are smooth (i.e. C^∞) on their domains.

Remark 2.7. Two local parametrizations F_α and F_β with smooth transition maps $F_\alpha^{-1} \circ F_\beta$ and $F_\beta^{-1} \circ F_\alpha$ are said to be *compatible*. □

Remark 2.8. We often use the superscript n, i.e. M^n, to mean that the manifold M is n-dimensional. □

Remark 2.9. A 2-dimensional manifold is sometimes called a *surface*. In this course, we will use the term *regular surfaces* for those surfaces in $\mathbb{R}^3$ discussed in Chapter 1, while we will use the term *smooth surfaces* to describe 2-dimensional smooth manifolds in the sense of Definition 2.6. □

Example 2.10. Any topological manifold which can be covered by one global parametrization (i.e. image of F is all of M) is a smooth manifold. Examples of which include $\mathbb{R}^n$ which can be covered by one parametrization $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$. The graph of Γ_f of any continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is also a smooth manifold covered by one parametrization $F(x) = (x, f(x)) : \mathbb{R}^n \to \Gamma_f$. Any regular curve $\gamma(t)$ is a smooth manifold of dimension 1. □

Example 2.11. All regular surfaces in $\mathbb{R}^3$ are smooth manifolds by Proposition 1.11 (which we showed their transition maps are smooth). Therefore, spheres, cylinders, tori, etc. are all smooth manifolds. □

Example 2.12 (Extended complex plane). Define $M = \mathbb{C} \cup \{\infty\}$. One can show (omitted here) that it is a Hausdroff, second countable topological space. Furthermore, one can cover M by two local parametrizations:

$$\begin{aligned} F_1 : \mathbb{R}^2 &\to \mathbb{C} \subset M \\ (x, y) &\mapsto x + yi \end{aligned} \qquad \begin{aligned} F_2 : \mathbb{R}^2 &\to (\mathbb{C}\backslash\{0\}) \cup \{\infty\} \subset M \\ (x, y) &\mapsto \frac{1}{x + yi} \end{aligned}$$

The overlap part on M is given by $\mathbb{C}\backslash\{0\}$, corresponding to $\mathbb{R}^2\backslash\{(0,0)\}$ in $\mathbb{R}^2$ under the parametrizations F_1 and F_2. One can compute that the transition maps are given by:

$$F_2^{-1} \circ F_1(x, y) = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$
$$F_1^{-1} \circ F_2(x, y) = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$

Both are smooth maps on $\mathbb{R}^2\backslash\{(0,0)\}$. Therefore, $\mathbb{C} \cup \{\infty\}$ is a smooth manifold. □

Exercise 2.1. Show that the n-dimensional sphere

$$\mathbb{S}^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_{n+1}^2 = 1\}$$

is a smooth n-manifold. [Hint: Generalize the stereographic projection to higher dimensions]

Exercise 2.2. Discuss: According to Example 2.10, the graph of any *continuous* function $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth manifold as there is no transition map. However, wouldn't it imply the single cone:

$$\left\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}\right\}$$

is a smooth manifold? It appears to have a "corner" point at the vertex, isn't it?

2.1.3. Product and Quotient Manifolds. Given two smooth manifolds M^m and N^n, one can form an $(m+n)$-dimensional manifold $M^m \times N^n$, which is defined by:

$$M^m \times N^n := \{(x, y) : x \in M^m \text{ and } y \in N^n\}.$$

Given a local parametrization $F : \mathcal{U}_M \to \mathcal{O}_M$ for M^m, and a local parametrizaiton $G : \mathcal{U}_N \to \mathcal{O}_N$ for N^n, one can define a local parametrization:

$$\begin{aligned} F \times G : \mathcal{U}_M \times \mathcal{U}_N &\to \mathcal{O}_M \times \mathcal{O}_N \subset M^m \times N^n \\ (u, \mathsf{v}) &\mapsto (F(u), G(\mathsf{v})) \end{aligned}$$

If $\{F_\alpha\}$ is a collection of local parametrizations of M^m with smooth transition maps, and $\{G_\beta\}$ is that of N^n with smooth transition maps, then one can form a collection of local parametrizations $F_\alpha \times G_\beta$ of the product $M^m \times N^n$. It can be shown that these local parametrizations of $M^m \times N^n$ also have smooth transition maps between open subsets of $\mathbb{R}^{m+n}$ (see Exercise 2.3).

Exercise 2.3. Show that if F_α and $F_{\widetilde{\alpha}}$ are local parametrizations of M^m with smooth transition maps, and similarly for G_β and $G_{\widetilde{\beta}}$ for N^n, then $F_\alpha \times G_\beta$ and $F_{\widetilde{\alpha}} \times G_{\widetilde{\beta}}$ have smooth transition maps.

The result from Exercise 2.3 showed that the product $M^m \times N^n$ of two smooth manifolds M^m and N^n is a smooth manifold with dimension $m+n$. Inductively, the product $M_1^{m_1} \times \ldots M_k^{m_k}$ of k smooth manifolds $M_1^{m_1}, \ldots, M_k^{m_k}$ is a smooth manifold with dimension $m_1 + \ldots + m_k$.

Example 2.13. The cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ can be regarded as $\mathbb{R} \times \mathbb{S}^1$. The torus can be regarded as $\mathbb{S}^1 \times \mathbb{S}^1$. They are both smooth manifolds. By taking products of known smooth manifolds, one can generate a great deal of new smooth manifolds. The n-dimensional cylinder can be easily seen to be a smooth manifold by regarding it as $\mathbb{R} \times \mathbb{S}^{n-1}$. The n-dimensional torus $\underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n \text{ times}}$ is also a smooth manifold. □

Another common way to produce a new manifold from an old one is to take quotients. Take $\mathbb{R}$ as an example. Let us define an equivalence relation $\sim$ by declaring that $x \sim y$ if and only if $x - y$ is an integer. For instance, we have $3 \sim 5$ while $4 \not\sim \frac{9}{2}$. Then, we can talk about *equivalence classes* $[x]$ which is the following set:

$$[x] := \{y \in \mathbb{R} : y \sim x\}.$$

For instance, we have $5 \in [2]$ as $5 \sim 2$. Likewise $-3 \in [2]$ as $-3 \sim 2$ too. The set $[2]$ is the set of all integers. Similarly, one can also argue $[-1] = [0] = [1] = [2] = [3] = \ldots$ are all equal to the set of all integers.

On the contrary, $1 \notin [0.2]$ as $1 \not\sim 0.2$. Yet $-1.8, -0.8, 0.2, \ldots$ are all in the set $[0.2]$. The set $[0.2]$ is simply the set of all numbers in the form of $0.2 + N$ where N is any integer. One can also see that $[-1.8] = [-0.8] = [0.2] = [1.2] = \ldots$.

Under such notations, we see that $[1] = [2]$ while $[1] \neq [0.2]$. The notion of equivalence classes provides us with a way to "decree" what elements in the "mother" set ($\mathbb{R}$ in this case) are regarded as equal. This is how topologists and geometers interpret *gluing*. In this example, we can think of $1, 2, 3$, etc. are glued together, and also $-1.8, -0.8, 0.2$, etc. are glued together. Formally, we denote

$$\mathbb{R}/\sim := \{[x] : x \in \mathbb{R}\}$$

which is the set of all equivalence classes under the relation $\sim$. This new set $\mathbb{R}/\sim$ is called a *quotient set* of $\mathbb{R}$ by the equivalence relation $\sim$. By sketching the set, we can see $\mathbb{R}/\sim$ is topologically a circle $\mathbb{S}^1$ (see Figure 2.3):

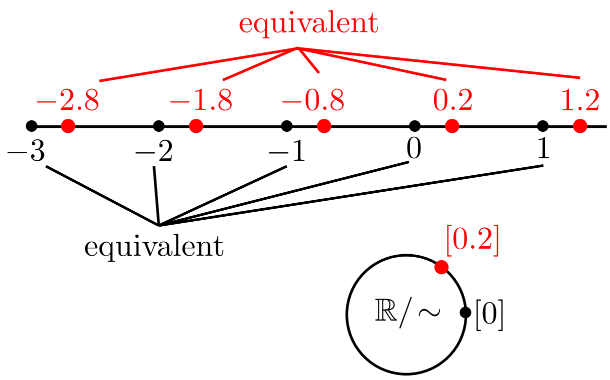


Figure 2.3. Quotient set $\mathbb{R}/\sim$

Exercise 2.4. Describe the set $\mathbb{R}^2/\sim$ where we declare $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$.

Example 2.14 (Real Projective Space)**.** The real projective space $\mathbb{RP}^n$ is the quotient set of $\mathbb{R}^{n+1}\backslash\{0\}$ under the equivalence relation: $(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$ if and only if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $(x_0, x_1, \ldots, x_n) = (\lambda y_0, \lambda y_1, \ldots, \lambda y_n)$. Each equivalence class is commonly denoted by:

$$[x_0 : x_1 : \cdots : x_n]$$

For instance, we have $[0 : 1 : -1] = [0 : -\pi : \pi]$. Under this notation, we can write:

$$\mathbb{RP}^n := \left\{[x_0 : x_1 : \cdots : x_n] : (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}\backslash\{0\}\right\}$$

It is important to note that $[0 : 0 : \cdots : 0] \notin \mathbb{RP}^n$.

We are going to show that $\mathbb{RP}^n$ is an n-dimensional smooth manifold. For each $i = 0, 1, \ldots, n$, we denote:

$$\mathcal{O}_i := \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\} \subset \mathbb{RP}^n.$$

Define $F_i : \mathbb{R}^n \to \mathcal{O}_i$ by:

$$F_i(x_1, \ldots, x_n) = [x_1 : \cdots : \underbrace{1}_{i} : \ldots x_n].$$

For instance, $F_0(x_1, \ldots, x_n) = [1 : x_1 : \cdots : x_n]$, $F_1(x_1, \ldots, x_n) = [x_1 : 1 : x_2 : \cdots : x_n]$ and $F_n(x_1, \ldots, x_n) = [x_1 : \cdots : x_n : 1]$.

The overlap between images of F_0 and F_1, for instance, is given by:

$$\mathcal{O}_0 \cap \mathcal{O}_1 = \{[x_0 : x_1 : x_2 : \cdots : x_n] : x_0, x_1 \neq 0\}$$
$$F_0^{-1}(\mathcal{O}_0 \cap \mathcal{O}_1) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \neq 0\}$$

One can compute that the transition map $F_1^{-1} \circ F_0$ is given by:

$$F_1^{-1} \circ F_0(x_1, \ldots, x_n) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$$

which is smooth on the domain $F_0^{-1}(\mathcal{O}_0 \cap \mathcal{O}_1)$. The smoothness of transition maps between any other pairs can be verified in a similar way. □

Exercise 2.5. Express the transition map $F_3^{-1} \circ F_1$ of $\mathbb{RP}^5$ and verify that it is smooth on its domain.

Example 2.15 (Complex Projective Space)**.** The complex projective space $\mathbb{CP}^n$ is an important manifold in Complex Geometry (one of my research interests) and Algebraic Geometry. It is defined similarly as $\mathbb{RP}^n$, with all $\mathbb{R}$'s replaced by $\mathbb{C}$'s. Precisely, we declare for any two elements in $(z_0, \ldots, z_n), (w_0, \ldots, w_n) \in \mathbb{C}^{n+1}\backslash\{(0, \ldots, 0)\}$, we have $(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n)$ if and only if there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that $z_i = \lambda w_i$ for any $i = 0, \ldots n$. Under this equivalence relation, the equivalence classes denoted by $[z_0 : z_1 : \cdots : z_n]$ constitute the complex projective space:

$$\mathbb{CP}^n := \{[z_0 : z_1 : \cdots : z_n] : z_i \text{ not all zero }\}.$$

It can be shown to be a smooth $2n$-manifold in exactly the same way as in $\mathbb{RP}^n$. □

Exercise 2.6. Show that $\mathbb{CP}^n$ is an $2n$-dimensional smooth manifold by constructing local parametrizations in a similar way as for $\mathbb{RP}^n$. For the transition maps, express one or two of them explicitly and verify that they are smooth. What's more can you say about the transition maps apart from being smooth?

Exercise 2.7. Consider the equivalence relation of $\mathbb{R}^n$ defined as follows:

$$x \sim y \text{ if and only if } x - y \in \mathbb{Z}^n$$

Show that $\mathbb{R}^n/\sim$ is a smooth n-manifold.

Example 2.16. The Klein Bottle K (see Figure 1.2a) cannot be put inside $\mathbb{R}^3$ without self-intersection, but it can be done in $\mathbb{R}^4$. It is covered by two local parametrizations given below:

$$F_1 : \mathcal{U}_1 \to \mathbb{R}^4 \quad \text{where} \quad \mathcal{U}_1 := \{(u_1, v_1) : 0 < u_1 < 2\pi \text{ and } 0 < v_1 < 2\pi\}$$

$$F_1(u_1, v_1) = \begin{bmatrix} (\cos v_1 + 2)\cos u_1 \\ (\cos v_1 + 2)\sin u_1 \\ \sin v_1 \cos \frac{u_1}{2} \\ \sin v_1 \sin \frac{u_1}{2} \end{bmatrix}.$$

$$F_2 : \mathcal{U}_2 \to \mathbb{R}^4 \quad \text{where} \quad \mathcal{U}_2 := \{(u_2, v_2) : 0 < u_2 < 2\pi \text{ and } 0 < v_2 < 2\pi\}$$

$$F_2(u_2, v_2) = \begin{bmatrix} -(\cos v_2 + 2)\cos u_2 \\ (\cos v_2 + 2)\sin u_2 \\ \sin v_2 \cos\left(\frac{u_2}{2} + \frac{\pi}{4}\right) \\ \sin v_2 \sin\left(\frac{u_2}{2} + \frac{\pi}{4}\right) \end{bmatrix}.$$

Geometrically speaking, the Klein bottle is generated by rotating the unit circle by two independent rotations, one parallel to the xy-plane, another parallel to the zw-plane. For geometric explanations for these parametrizations, see [**dC94**, P.36].

We leave it to readers to check that F_1 and F_2 are both injective and compatible with each other. It will show that K is a 2-manifold. □

Exercise 2.8. Consider the Klein bottle K given in Example 2.16.

(a) Show that both F_1 and F_2 are injective.

(b) Let $\mathcal{W} = F_1(\mathcal{U}_1) \cap F_2(\mathcal{U}_2)$. Find $F_1^{-1}(\mathcal{W})$ and $F_2^{-1}(\mathcal{W})$.

(c) Compute the transition maps $F_2^{-1} \circ F_1$ and $F_1^{-1} \circ F_2$ defined on the overlaps.

2.1.4. Differential Structures. A smooth manifold M^n is equipped with a collection smooth local parametrizations $F_\alpha : \mathcal{U}_\alpha \subset \mathbb{R}^n \to \mathcal{O}_\alpha \subset M^n$ such that the images of these F_α's cover the entire manifold, i.e.

$$M = \bigcup_{\text{all } \alpha\text{'s}} \mathcal{O}_\alpha = \bigcup_{\text{all } \alpha\text{'s}} F_\alpha(\mathcal{U}_\alpha).$$

These local parametrizations need to be compatible with each other in a sense that any overlapping parametrizations F_α and F_β must have smooth transition maps $F_\alpha^{-1} \circ F_\beta$ and $F_\beta^{-1} \circ F_\alpha$. Such a collection of local parametrizations $\mathcal{A} = \{F_\alpha, \mathcal{U}_\alpha, \mathcal{O}_\alpha\}_\alpha$ is called a *smooth atlas* of M.

Given a smooth atlas $\mathcal{A}$ of M, we can enlarge the atlas by including more local parametrizations $F_{\text{new}} : \mathcal{U}_{\text{new}} \to \mathcal{O}_{\text{new}}$ that are compatible to all local parametrizations in $\mathcal{A}$. The *differential structure* generated by an atlas $\mathcal{A}$ is a *gigantic* atlas that contains all local parametrizations which are compatible with every local parametrizations in $\mathcal{A}$ (for more formal definition, please read [**Lee09**, Section 1.3]).

Let's take the plane $\mathbb{R}^2$ as an example. It can be parametrized by at least three different ways:

- the identity map $F_1 := \text{id} : \mathbb{R}^2 \to \mathbb{R}^2$.

- the map $F_2 : \mathbb{R}^2 \to (0, \infty) \times (0, \infty) \subset \mathbb{R}^2$, defined as:
$$F_2(u, v) := (e^u, e^v).$$
- and pathologically, by $F_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:
$$F_3(u, v) = (u, v + |u|).$$

It is clear that $F_1^{-1} \circ F_2(u, v) = (e^u, e^v)$ and $F_2^{-1} \circ F_1(u, v) = (\log u, \log v)$ are smooth on the domains at which they are defined. Therefore, we say that F_1 and F_2 are compatible, and the differential structure generated by F_1 will contain F_2.

On the other hand, $F_1^{-1} \circ F_3(u, v) = (u, v + |u|)$ is not smooth, and so F_1 and F_3 are not compatible. Likewise, $F_2^{-1} \circ F_3(u, v) = (\log u, \log(v + |u|))$ is not smooth either. Therefore, F_3 does *not* belong to the differential structure generated by F_1 and F_2.

As we can see from above, a manifold M can have many distinct differential structures. In this course, when we talk about manifolds, we usually only consider *one* differential structure of the manifold, and very often we will only deal with the most "natural" differential structure such as the one generated by F_1 or F_2 above for $\mathbb{R}^3$, but not like the pathological one such as F_3. Therefore, we usually will not specify the differential structure when we talk about a manifold, unless it is necessary in some rare occasions.

Exercise 2.9. Show that any smooth manifold has uncountably many distinct differential structures. [Hint: Let $\mathbb{B}(1) := \{x \in \mathbb{R}^n : |x| < 1\}$, consider maps $\Psi_s : \mathbb{B}(1) \to \mathbb{B}(1)$ defined by $\Psi_s(x) = |x|^s x$ where $s > 0$.]

2.2. Functions and Maps on Manifolds

2.2.1. Definitions and Examples. Let's first review how smooth functions $f : M \to \mathbb{R}$ and smooth maps $\Phi : M \to N$ are defined for regular surfaces (Definitions 1.13 and 1.17). Given two and $G : \mathcal{U}_N \to \mathcal{O}_N \subset N$, the compositions $f \circ F$ and $G^{-1} \circ \Phi \circ F$ are functions or maps between Euclidean spaces. We say the function f is smooth if $f \circ F$ is smooth for any local parametrizations $F : \mathcal{U}_M \to \mathcal{O}_M \subset M$. We say Φ is smooth if $G^{-1} \circ \Phi \circ F$ is smooth.

The definitions of differentiable functions and maps for regular surfaces carry over to abstract manifolds in a natural way:

Definition 2.17 (Functions and Maps of Class C^k)**.** Let M^m and N^n be two smooth manifolds of dimensions m and n respectively. Then:

A scalar-valued function $f : M \to \mathbb{R}$ is said to be C^k *at* $p \in M$ if for any smooth local parametrization $F : \mathcal{U} \to M$ with $p \in F(\mathcal{U})$, the composition $f \circ F$ is C^k at the point $F^{-1}(p) \in \mathcal{U}$ as a function from subset from $\mathbb{R}^m$ to $\mathbb{R}$. Furthermore, if $f : M \to \mathbb{R}$ is C^k at every $p \in M$, then we say f is C^k *on* M.

A map $\Phi : M \to N$ is said to be C^k *at* $p \in M$ if for any smooth local parametrization $F : \mathcal{U}_M \to \mathcal{O}_M \subset M$ with $p \in F(\mathcal{U}_M)$, and $G : \mathcal{U}_N \to \mathcal{O}_N \subset N$ with $\Phi(p) \in G(\mathcal{U}_N)$, the composition $G^{-1} \circ \Phi \circ F$ is C^k at $F^{-1}(p)$ as a map between subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$. Furthermore, if $\Phi : M \to N$ is C^k at every $p \in M$, then Φ is said to be C^k *on* M.

When k is ∞, we can also say that the function or map is *smooth*.

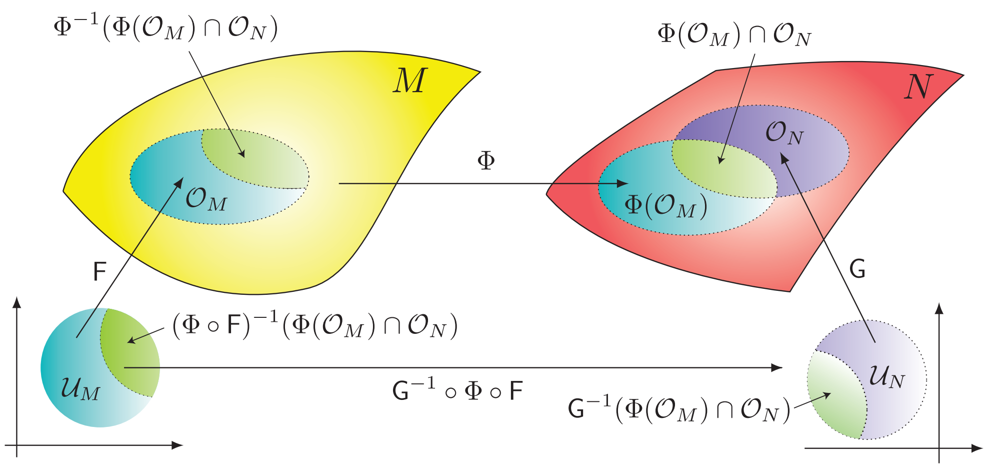


Figure 2.4. maps between two manifolds

Remark 2.18. By the definition of a smooth manifold (see condition (2) in Definition 2.6), transition maps are always smooth. Therefore, although we require $f \circ F$ and $G^{-1} \circ \Phi \circ F$ to be smooth for *any* local parametrizations around p, it suffices to show that they are smooth for at least *one* F covering p and at least *one* G covering $\Phi(p)$.

Exercise 2.10. Suppose $\Phi : M \to N$ and $\Psi : N \to P$ are C^k maps between smooth manifolds M, N and P. Show that the composition $\Psi \circ \Phi$ is also C^k.

Example 2.19. Consider the 3-dimensional sphere

$$\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \in \mathbb{R}\}$$

and the complex projective plane

$$\mathbb{CP}^1 = \{[z : w] : z \neq 0 \text{ or } w \neq 0\}.$$

Define a map $\Phi : \mathbb{S}^3 \to \mathbb{CP}^1$ by:

$$\Phi(x_1, x_2, x_3, x_4) = [x_1 + ix_2, x_3 + ix_4].$$

Locally parametrize $\mathbb{S}^3$ by stereographic projection:

$$\begin{aligned} F &: \mathbb{R}^3 \to \mathbb{S}^3 \\ F(u_1, u_2, u_3) &= \left(\frac{2u_1}{1 + \sum_k u_k^2}, \frac{2u_2}{1 + \sum_k u_k^2}, \frac{2u_3}{1 + \sum_k u_k^2}, \frac{-1 + \sum_k u_k^2}{1 + \sum_k u_k^2} \right) \end{aligned}$$

The image of F is $\mathbb{S}^3 \backslash \{(0,0,0,1)\}$. As usual, we locally parametrize $\mathbb{CP}^1$ by:

$$\begin{aligned} G &: \mathbb{R}^2 \to \mathbb{CP}^1 \\ G(v_1, v_2) &= [1 : v_1 + iv_2] \end{aligned}$$

The domain of $G^{-1} \circ \Phi \circ F$ is $(\Phi \circ F)^{-1} \left(\Phi \circ F(\mathbb{R}^3) \cap G(\mathbb{R}^2) \right)$, and the map is explicitly given by:

$$\begin{aligned} & G^{-1} \circ \Phi \circ F(u_1, u_2, u_3) \\ &= G^{-1} \circ \Phi \left(\frac{2u_1}{1 + \sum_k u_k^2}, \frac{2u_2}{1 + \sum_k u_k^2}, \frac{2u_3}{1 + \sum_k u_k^2}, \frac{-1 + \sum_k u_k^2}{1 + \sum_k u_k^2} \right) \\ &= G^{-1} \left[\frac{2u_1}{1 + \sum_k u_k^2} + i \frac{2u_2}{1 + \sum_k u_k^2} : \frac{2u_3}{1 + \sum_k u_k^2} + i \frac{-1 + \sum_k u_k^2}{1 + \sum_k u_k^2} \right] \\ &= G^{-1} \left[1 : \frac{2u_3 + i \left(-1 + \sum_k u_k^2 \right)}{2u_1 + 2iu_2} \right] \\ &= G^{-1} \left[1 : \frac{2u_1 u_3 + u_2(-1 + \sum_k u_k^2)}{2(u_1^2 + u_2^2)} + i \frac{-2u_2 u_3 + u_1(-1 + \sum_k u_k^2)}{2(u_1^2 + u_2^2)} \right] \\ &= \left(\frac{2u_1 u_3 + u_2(-1 + \sum_k u_k^2)}{2(u_1^2 + u_2^2)}, \frac{-2u_2 u_3 + u_1(-1 + \sum_k u_k^2)}{2(u_1^2 + u_2^2)} \right). \end{aligned}$$

For any (u_1, u_2, u_3) in the domain of $G^{-1} \circ \Phi \circ F$, which is $(\Phi \circ F)^{-1} \left(\Phi \circ F(\mathbb{R}^3) \cap G(\mathbb{R}^2) \right)$, we have in particular $\Phi \circ F(u_1, u_2, u_3) \in G(\mathbb{R}^2)$, and so

$$\frac{2u_1}{1 + \sum_k u_k^2} + i \frac{2u_2}{1 + \sum_k u_k^2} \neq 0.$$

Therefore, $(u_1, u_2) \neq (0,0)$ whenever (u_1, u_2, u_3) is in the domain of $G^{-1} \circ \Phi \circ F$. From the above computations, $G^{-1} \circ \Phi \circ F$ is smooth.

One can also check similarly that $\widetilde{G}^{-1} \circ \Phi \circ \widetilde{F}$ is smooth for other combinations of local parametrizations, concluding Φ is a smooth map.

□

Example 2.20. Let $M \times N$ be the product of two smooth manifolds M and N. Then, the projection map π_M and π_N defined by:

$$\begin{aligned} \pi_M &: M \times N \to M \\ & (p, q) \mapsto p \\ \pi_N &: M \times N \to N \\ & (p, q) \mapsto q \end{aligned}$$

are both smooth manifolds. It can be shown by considering local parametrizations $F : \mathcal{U}_M \to \mathcal{O}_M$ of M, and $G : \mathcal{U}_N \to \mathcal{O}_N$ of N. Then $F \times G : \mathcal{U}_M \times \mathcal{U}_N \to \mathcal{O}_M \times \mathcal{O}_N$ is a local parametrization of $M \times N$. To show that π_M is smooth, we compute:

$$\begin{aligned} F^{-1} \circ \pi_M \circ (F \times G)(u, \mathsf{v}) &= F^{-1} \circ \pi_M (F(u), G(\mathsf{v})) \\ &= F^{-1}(F(u)) \\ &= u \end{aligned}$$

The map $(u, \mathsf{v}) \mapsto u$ is clearly a smooth map between Euclidean spaces. Therefore, π_M is a smooth map between $M \times N$ and M. Similarly, π_N is also a smooth map between $M \times N$ and N. $\square$

Exercise 2.11. Suppose $\Phi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{m+1}\backslash\{0\}$ is a smooth map which satisfies

$$\Phi(cx_0, cx_1, \ldots, cx_n) = c^d \Phi(x_0, x_1, \ldots, x_n)$$

for any $c \in \mathbb{R}\backslash\{0\}$ and $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}\backslash\{0\}$. Show that the induced map $\widetilde{\Phi} : \mathbb{RP}^n \to \mathbb{RP}^m$ defined by:

$$\widetilde{\Phi}([x_0 : x_1 : \cdots : x_n]) = \Phi(x_0, x_1, \ldots, x_n)$$

is well-defined and smooth. [Hint: To check $\widetilde{\Phi}$ is well-defined means to verify that two equivalent inputs $[x_0 : x_1 : \cdots : x_n] = [y_0 : y_1 : \cdots : y_n]$ will give the same outputs $\Phi(x_0, x_1, \ldots, x_n)$ and $\Phi(y_0, y_1, \ldots, y_n)$.]

Exercise 2.12. Let $M = \{(w, z) \in \mathbb{C}^2 : |w|^2 + |z|^2 = 1\}$.

(a) Show that M is a 3-dimensional manifold.

(b) Define
$$\Phi(w, z) := \left(z\bar{w} + w\bar{z}, i(w\bar{z} - z\bar{w}), |z|^2 - |w|^2\right)$$
for any $(w, z) \in M$. Show that $\Phi(w, z) \in \mathbb{R}^3$ and it lies on the unit sphere $\mathbb{S}^2$, and then verify that $\Phi : M \to \mathbb{S}^2$ is a smooth map.

2.2.2. Diffeomorphisms. Two smooth manifolds M and N are said to be *diffeomorphic* if they are in one-to-one correspondence with each other in smooth sense. Here is the rigorous definition:

Definition 2.21 (Diffeomorphisms)**.** A smooth map $\Phi : M \to N$ between two smooth manifolds M and N is said to be a *diffeomorphism* if Φ is a one-to-one and onto (i.e. bijective), and that the inverse map $\Phi^{-1} : N \to M$ is also smooth.

If such a map exists between M and N, the two manifolds M and N are said to be *diffeomorphic*.

Example 2.22. Given a smooth function $f : \mathcal{U} \to \mathbb{R}$ from an open subset $\mathcal{U} \subset \mathbb{R}^n$. The graph Γ_f defined as:

$$\Gamma_f := \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathcal{U}\}$$

is a smooth manifold by Example 2.10. We claim that the projection map:

$$\begin{aligned} \pi : \Gamma_f &\to \mathcal{U} \\ (x, f(x)) &\mapsto x \end{aligned}$$

is a diffeomorphism. Both Γ_f and $\mathcal{U}$ are covered by one global parametrization. The parametrization of $\mathcal{U}$ is simply the identity map $\mathrm{id}_{\mathcal{U}}$ on $\mathcal{U}$. The parametrization of Γ_f is given by:

$$\begin{aligned} F : \mathcal{U} &\to \Gamma_f \\ x &\mapsto (x, f(x)) \end{aligned}$$

To show that π is smooth, we consider $\mathrm{id}_{\mathcal{U}}^{-1} \circ \pi \circ F$, which is given by:

$$\begin{aligned} \mathrm{id}_{\mathcal{U}}^{-1} \circ \pi \circ F(x) &= \mathrm{id}_{\mathcal{U}}^{-1} \circ \pi(x, f(x)) \\ &= \mathrm{id}_{\mathcal{U}}^{-1}(x) \\ &= x. \end{aligned}$$

Therefore, the composite $\mathrm{id}_{\mathcal{U}}^{-1} \circ \pi \circ F$ is simply the identity map on $\mathcal{U}$, which is clearly smooth.

π is one-to-one and onto with inverse map π^{-1} given by:

$$\begin{aligned} \pi^{-1} : \mathcal{U} &\to \Gamma_f \\ x &\mapsto (x, f(x)) \end{aligned}$$

To show π^{-1} is smooth, we consider the composite $F^{-1} \circ \pi^{-1} \circ \mathrm{id}_{\mathcal{U}}$:

$$\begin{aligned} F^{-1} \circ \pi^{-1} \circ \mathrm{id}_{\mathcal{U}}(x) &= F^{-1} \circ \pi^{-1}(x) \\ &= F^{-1}(x, f(x)) \\ &= x. \end{aligned}$$

Therefore, the composite $F^{-1} \circ \pi^{-1} \circ \mathrm{id}_{\mathcal{U}}$ is also the identity map on $\mathcal{U}$, which is again smooth. □

Example 2.23. Let M be the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. We are going to show that M is diffeomorphic to $\mathbb{R}^2 \backslash \{(0,0)\}$ via the diffeomorphism:

$$\begin{aligned} \Phi : M &\to \mathbb{R}^2 \backslash \{(0,0)\} \\ (x, y, z) &\mapsto e^z(x, y) \end{aligned}$$

We leave it for readers to verify that Φ is one-to-one and onto, and hence Φ^{-1} exists. To show it is a diffeomorphism, we first parametrize M by two local coordinate charts:

$$\begin{aligned} F_1 : (0, 2\pi) \times \mathbb{R} &\to M & \qquad F_2 : (-\pi, \pi) \times \mathbb{R} &\to M \\ F_1(\theta, z) &= (\cos\theta, \sin\theta, z) & \qquad F_2(\widetilde{\theta}, \widetilde{z}) &= (\cos\widetilde{\theta}, \sin\widetilde{\theta}, \widetilde{z}) \end{aligned}$$

The target space $\mathbb{R}^2 \backslash \{(0,0)\}$ is an open set of $\mathbb{R}^2$, and hence can be globally parametrized by $\mathrm{id} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\}$.

We need to show $\Phi \circ F_i$ and $F_i^{-1} \circ \Phi^{-1}$ are smooth for any $i = 1, 2$. As an example, we verify one of them only:

$$\begin{aligned} \Phi \circ F_1(\theta, z) &= \Phi(\cos\theta, \sin\theta, z) \\ &= (e^z \cos\theta, e^z \sin\theta). \end{aligned}$$

To show $F_1^{-1} \circ \Phi^{-1} = (\Phi \circ F_1)^{-1}$ is smooth, we use Inverse Function Theorem. The Jacobian of $\Phi \circ F_1$ is given by:

$$D(\Phi \circ F_1) = \det \begin{bmatrix} -e^z \sin\theta & e^z \cos\theta \\ e^z \cos\theta & e^z \sin\theta \end{bmatrix} = -e^{2z} \neq 0.$$

Therefore, $\Phi \circ F_1$ has a C^∞ local inverse around every point in the domain. Since $\Phi \circ F_1$ is one-to-one and onto, such a local inverse is a global inverse.

Similarly, one can show $\Phi \circ F_2$ and $F_2^{-1} \circ \Phi^{-1}$ are smooth. All these show Φ and Φ^{-1} are smooth maps between M and $\mathbb{R}^2 \backslash \{(0,0)\}$, and hence are diffeomorphisms. $\square$

Exercise 2.13. Show that the open square $(-1,1) \times (-1,1) \subset \mathbb{R}^2$ is diffeomorphic to $\mathbb{R}^2$. [Hint: consider the trig functions $\tan$ or $\tan^{-1}$.]

Exercise 2.14. Consider the map $\Phi : B_1(0) \to \mathbb{R}^n$ defined by:

$$\Phi(x) = \frac{|x|}{\sqrt{1 - |x|^2}}$$

where $B_1(0)$ is the open unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$. Show that Φ is a diffeomorphism.

Exercise 2.15. Let $M = \mathbb{R}^2/\sim$ where $\sim$ is the equivalence relation:

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Z}^2.$$

From Exercise 2.7, we have already showed that M is a smooth manifold. Show that M is diffeomorphic to a $\mathbb{S}^1 \times \mathbb{S}^1$.

2.3. Tangent Spaces and Tangent Maps

At a point p on a regular surface $M \subset \mathbb{R}^3$, the tangent plane T_pM at p is spanned by the basis $\left\{\frac{\partial F}{\partial u_i}\right\}_{i=1,2}$ where F is a local parametrization around p. The basis $\frac{\partial F}{\partial u_i}$ are vectors in $\mathbb{R}^3$ since F has an image in $\mathbb{R}^3$. However, this definition of tangent plane can hardly be generalized to abstract manifolds, as an abstract manifold M may not sit inside any Euclidean space. Instead, we define the tangent space at a point p on a smooth manifold as the vector space spanned by partial differential operators $\left\{\frac{\partial}{\partial u_i}\right\}_{i=1}^n$. Heuristically, we generalize the concept of tangent planes of regular surfaces in $\mathbb{R}^3$ by "removing" the label F from the *geometric* vector $\frac{\partial F}{\partial u_i}$, so that it becomes an *abstract* vector $\frac{\partial}{\partial u_i}$. For this generalization, we first need to define partial derivatives on abstract manifolds.

2.3.1. Partial Derivatives and Tangent Vectors. Let M^n be a smooth manifold and $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{O} \subset M^n$ be a smooth local parametrization. Then similar to regular surfaces, for any $p \in \mathcal{O}$, it makes sense to define partial derivative for a function $f : M \to \mathbb{R}$ at p by pre-composing f with F, i.e. $f \circ F$, which is a map from $\mathcal{U} \subset \mathbb{R}^n$ to $\mathbb{R}$. Let $(u_1, \ldots, u_n)$ be the coordinates of $\mathcal{U} \subset \mathbb{R}^n$, then with a little abuse of notations, we denote:

$$\frac{\partial f}{\partial u_j}(p) := \frac{\partial (f \circ F)}{\partial u_j}(u)$$

where u is the point in $\mathcal{U}$ corresponding to p, i.e. $F(u) = p$.

Remark 2.24. Note that $\frac{\partial f}{\partial u_j}(p)$ is defined locally on $\mathcal{O}$, and depends on the choice of local parametrization F near p. □

The partial derivative $\frac{\partial}{\partial u_j}(p)$ can be thought as an operator:

$$\begin{aligned} \frac{\partial}{\partial u_j}(p) : C^1(M, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial u_j}(p). \end{aligned}$$

Here $C^1(M, \mathbb{R})$ denotes the set of all C^1 functions from M to $\mathbb{R}$.

On regular surfaces $\frac{\partial F}{\partial u_j}(p)$ is a tangent vector at p. On an abstract manifold, $\frac{\partial F}{\partial u_j}(p)$ cannot be defined since F may not be in an Euclidean space. Instead, the partial differential operator $\frac{\partial}{\partial u_j}(p)$ plays the role of $\frac{\partial F}{\partial u_j}(p)$, and we will call the operator $\frac{\partial}{\partial u_j}(p)$ a *tangent vector* for an abstract manifold. It does sound less concrete and more abstract than a geometric tangent vector, but one good quote by John von Neumann is:

> "In mathematics you don't understand things. You just get used to them."

Example 2.25. Let $F(x, y, z) = (x, y, z)$ be the identity parametrization of $\mathbb{R}^3$, and $G(\rho, \theta, \varphi) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$ be local parametrization of $\mathbb{R}^3$ by spherical coordinates.

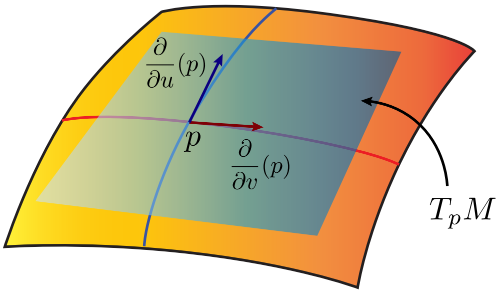


Figure 2.5. $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are used in place of $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ on abstract manifolds.

Then at any point $p \in \mathbb{R}^3$, the vectors $\dfrac{\partial}{\partial x}(p), \dfrac{\partial}{\partial y}(p), \dfrac{\partial}{\partial z}(p)$ are regarded as the *abstract form* of the *geometric* vectors $\dfrac{\partial F}{\partial x}(p), \dfrac{\partial F}{\partial y}(p), \dfrac{\partial F}{\partial z}(p)$, which are respectively $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ in standard notations.

Also, the vectors $\dfrac{\partial}{\partial \rho}(p), \dfrac{\partial}{\partial \theta}(p), \dfrac{\partial}{\partial \varphi}(p)$ are regarded as the *abstract form* of the *geometric* vectors $\dfrac{\partial G}{\partial \rho}(p), \dfrac{\partial G}{\partial \theta}(p), \dfrac{\partial G}{\partial \varphi}(p)$, which are respectively the vectors at p tangent to the ρ-, θ- and φ-directions on the sphere. $\square$

Example 2.26. Take another example:

$$\mathbb{RP}^2 = \{[x_0 : x_1 : x_2] : \text{at least one } x_i \neq 0\}.$$

According to Example 2.14, one of its local parametrizations is given by:

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{RP}^2 \\ (x_1, x_2) &\mapsto [1 : x_1 : x_2] \end{aligned}$$

Such a manifold is not assumed to be in $\mathbb{R}^N$, so we can't define $\dfrac{\partial F}{\partial x_1}, \dfrac{\partial F}{\partial x_2}$ as *geometric vectors* in $\mathbb{R}^N$. However, as a substitute, we will regard the operators $\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}$ as *abstract* tangent vectors along the directions of x_1 and x_2 respectively. $\square$

2.3.2. Tangent Spaces. Having generalized the concept of partial derivatives to abstract manifolds, we now ready to state the definition of tangent vectors for abstract manifolds.

Definition 2.27 (Tangent Spaces)**.** Let M be a smooth n-manifold, $p \in M$ and $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{O} \subset M$ be a smooth local parametrization around p. The *tangent space at p of M*, denoted by T_pM, is defined as:

$$T_pM = \text{span}\left\{\frac{\partial}{\partial u_1}(p), \ldots, \frac{\partial}{\partial u_n}(p)\right\},$$

where $\dfrac{\partial}{\partial u_i}$'s are partial differential operators with respect to the local parametrization $F(u_1, \ldots, u_n)$.

It *seems* that the definition of T_pM depends on the choice of local parametrization F. However, we can show that it does not. We first show that $\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^{n}$ are linearly independent, then we have $\dim T_pM = n = \dim M$.

Given a local parametrization $F : \mathcal{U} \to \mathcal{O} \subset M$ with local coordinates denoted by $(u_1, \ldots, u_n)$, then each coordinate u_i can be regarded as a locally defined function $u_i : \mathcal{O} \to \mathbb{R}$. Then we have:

$$\frac{\partial u_k}{\partial u_j}(p) = \delta_{kj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Next we want to show $\left\{\frac{\partial}{\partial u_i}\right\}_{i=1}^{n}$ are linearly independent. Suppose a_i's are real numbers such that

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial u_i} = 0,$$

meaning that $\sum_{i=1}^{n} a_i \frac{\partial f}{\partial u_i} = 0$ for any differential function f (including the coordinate functions u_k's). Therefore, we have:

$$0 = \sum_{i=1}^{n} a_i \frac{\partial u_k}{\partial u_i} = \sum_{i=1}^{n} a_i \delta_{ki} = a_k$$

for any k. This shows $\left\{\frac{\partial}{\partial u_i}\right\}_{i=1}^{n}$ are linearly independent, show that $\dim T_pM = \dim M$.

Now we show T_pM does not depend on the choice of local coordinates. Suppose $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{O} \subset M$ and $\widetilde{F} : \widetilde{\mathcal{U}} \subset \mathbb{R}^n \to \widetilde{\mathcal{O}}$ be two local parametrizations. We use $(u_1, \ldots, u_n)$ to denote the Euclidean coordinates on $\mathcal{U}$, and use $(v_1, \ldots, v_n)$ for $\widetilde{\mathcal{U}}$.

The partial derivatives $\frac{\partial f}{\partial u_j}$ are $\frac{\partial f}{\partial v_i}$ are different. Via the transition map $\widetilde{F}^{-1} \circ F$, $(v_1, \ldots, v_n)$ can be regarded as functions of $(u_1, \ldots, u_n)$, and therefore it makes sense of defining $\frac{\partial v_j}{\partial u_i}$.

Given a smooth function $f : M \to \mathbb{R}$, by the chain rule, one can write the partial derivative $\frac{\partial f}{\partial u_j}$ in terms of $\frac{\partial f}{\partial v_i}$ as follows:

$$\begin{aligned} \frac{\partial f}{\partial u_i}(p) &:= \left.\frac{\partial (f \circ F)}{\partial u_i}\right|_{F^{-1}(p)} \\ &= \left.\frac{\partial}{\partial u_i}\right|_{F^{-1}(p)} (f \circ \widetilde{F}) \circ (\widetilde{F}^{-1} \circ F) \\ &= \sum_{j=1}^{n} \left.\frac{\partial (f \circ \widetilde{F})}{\partial v_j}\right|_{\widetilde{\mathsf{F}}^{-1}(\mathsf{p})} \left.\frac{\partial v_j}{\partial u_i}\right|_{F^{-1}(p)} \\ &= \sum_{j=1}^{n} \left.\frac{\partial v_j}{\partial u_i}\right|_{F^{-1}(p)} \frac{\partial f}{\partial v_j}(p) \end{aligned} \tag{2.1}$$

In short, we can write:

$$\frac{\partial}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial u_i} \frac{\partial}{\partial v_j}.$$

In other words, $\dfrac{\partial}{\partial u_i}$ can be expressed as a linear combination of $\dfrac{\partial}{\partial v_j}'s$.

Therefore, span $\left\{\dfrac{\partial}{\partial u_i}(p)\right\}_{i=1}^{n} \subset$ span $\left\{\dfrac{\partial}{\partial v_i}(p)\right\}_{i=1}^{n}$. Since both spans of vectors have equal dimension, their span must be equal. This shows T_pM is independent of choice of local parametrizations. However, it is important to note that each individual basis vector $\dfrac{\partial}{\partial u_i}(p)$ *does* depend on local parametrizations.

Example 2.28. Consider again the real projective plane:

$$\mathbb{RP}^2 = \{[x_0 : x_1 : x_2] : \text{at least one } x_i \neq 0\}.$$

Consider the two local parametrizations:

$$\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{RP}^2 & G &: \mathbb{R}^2 \to \mathbb{RP}^2 \\ F(x_1, x_2) &= [1 : x_1 : x_2] & G(y_0, y_2) &= [y_0 : 1 : y_2] \end{aligned}$$

Then, (y_0, y_2) can be regarded as a function of (x_1, x_2) via the transition map $G^{-1} \circ F$, which is explicitly given by:

$$\begin{aligned} (y_0, y_2) &= G^{-1} \circ F(x_1, x_2) = G^{-1}([1 : x_1 : x_2]) \\ &= G^{-1}([x_1^{-1} : 1 : x_1^{-1}x_2]) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right). \end{aligned}$$

Using the chain rule, we can then express $\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}$ in terms of $\dfrac{\partial}{\partial y_0}, \dfrac{\partial}{\partial y_2}$:

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{\partial y_0}{\partial x_1}\frac{\partial}{\partial y_0} + \frac{\partial y_2}{\partial x_1}\frac{\partial}{\partial y_2} \\ &= -\frac{1}{x_1^2}\frac{\partial}{\partial y_0} - \frac{x_2}{x_1^2}\frac{\partial}{\partial y_2} \\ &= -y_0^2\frac{\partial}{\partial y_0} - y_0 y_2\frac{\partial}{\partial y_2}. \end{aligned}$$

We leave $\dfrac{\partial}{\partial x_2}$ as an exercise. □

Exercise 2.16. Express $\dfrac{\partial}{\partial x_2}$ as a linear combination $\dfrac{\partial}{\partial y_0}, \dfrac{\partial}{\partial y_2}$ in Example 2.28. Leave the final answer in terms of y_0 and y_2 only.

Exercise 2.17. Consider the extended complex plane $M := \mathbb{C} \cup \{\infty\}$ (discussed in Example 2.12) with local parametrizations:

$$\begin{aligned} F_1 &: \mathbb{R}^2 \to \mathbb{C} \subset M & F_2 &: \mathbb{R}^2 \to (\mathbb{C}\backslash\{0\}) \cup \{\infty\} \subset M \\ (x_1, x_2) &\mapsto x_1 + x_2 i & (y_1, y_2) &\mapsto \frac{1}{y_1 + y_2 i} \end{aligned}$$

Express the tangent space basis $\left\{\dfrac{\partial}{\partial x_i}\right\}$ in terms of the basis $\left\{\dfrac{\partial}{\partial y_j}\right\}$.

Exercise 2.18. Given two smooth manifolds M^m and N^n, and a point $(p, q) \in M \times N$, show that the tangent plane $T_{(p,q)}(M \times N)$ is isomorphic to $T_pM \oplus T_qN$. Recall that $V \oplus W$ is the *direct sum* of two vector spaces V and W, defined as:

$$V \oplus W = \{(v, w) : v \in V \text{ and } w \in W\}.$$

2.3.3. Tangent Maps. Given a smooth map Φ between two regular surfaces in $\mathbb{R}^3$, we discussed in Section 1.5.2 on how to define its partial derivatives using local parametrizations. To recap, suppose $\Phi : M \to N$ and $F(u_1, u_2) : \mathcal{U}_M \to \mathcal{O}_M \subset M$ and $G(v_1, v_2) : \mathcal{U}_N \to \mathcal{O}_N \subset N$ are local parametrizations of M and N respectively. Via Φ, the local coordinates (v_1, v_2) of N can be regarded as functions of (u_1, u_2), i.e.

$$(v_1, v_2) = G^{-1} \circ \Phi \circ F(u_1, u_2).$$

Then, according to (1.1), the partial derivative $\dfrac{\partial \Phi}{\partial u_i}$ of the map Φ is given by:

$$\begin{aligned}
\frac{\partial \Phi}{\partial u_i}(\Phi(p)) := \left.\frac{\partial(\Phi \circ F)}{\partial u_i}\right|_{F^{-1}(p)} &= \left.\frac{\partial(G \circ (G^{-1} \circ \Phi \circ F))}{\partial u_i}\right|_{F^{-1}(p)} \\
&= \sum_{j=1}^{2} \left.\frac{\partial v_j}{\partial u_i}\right|_{F^{-1}(p)} \frac{\partial G}{\partial v_j}(\Phi(p))
\end{aligned}$$

which is a vector on the tangent plane $T_{\Phi(p)}N$.

Now, if we are given a smooth map $\Phi : M^m \to N^n$ between two smooth abstract manifolds M^m and N^n with local parametrizations $F(u_1, \ldots, u_m) : \mathcal{U}_M \subset \mathbb{R}^m \to \mathcal{O}_M \subset M^m$ around $p \in M$, and $G(v_1, \ldots, v_n) : \mathcal{U}_N \subset \mathbb{R}^n \to \mathcal{O}_N \subset N^n$ around $\Phi(p) \in N$, then the tangent space $T_{\Phi(p)}N$ is spanned by $\left\{ \dfrac{\partial}{\partial v_j}(\Phi(p)) \right\}_{j=1}^{n}$. In view of (1.1), a natural generalization of partial derivatives $\dfrac{\partial \Phi}{\partial u_i}(p)$ to smooth maps between manifolds is:

Definition 2.29 (Partial Derivatives of Maps between Manifolds)**.** Let $\Phi : M^m \to N^n$ be a smooth map between two smooth manifolds M and N. Let $F(u_1, \ldots, u_m) : \mathcal{U}_M \to \mathcal{O}_M \subset M$ be a smooth local parametrization around p and $G(v_1, \ldots, v_n) : \mathcal{U}_N \to \mathcal{O}_N \subset N$ be a smooth local parametrization around $\Phi(p)$. Then, the *partial derivative of* Φ *with respect to* u_i *at* p is defined to be:

$$\frac{\partial \Phi}{\partial u_i}(p) := \sum_{j=1}^{n} \left.\frac{\partial v_j}{\partial u_i}\right|_{F^{-1}(p)} \frac{\partial}{\partial v_j}(\Phi(p)). \tag{2.2}$$

Here $(v_1, \ldots, v_n)$ are regarded as functions of $(u_1, \ldots, u_m)$ in a sense that:

$$(v_1, \ldots, v_n) = G^{-1} \circ \Phi \circ F(u_1, \ldots, u_m).$$

Note that the partial derivative $\dfrac{\partial \Phi}{\partial u_i}$ defined in (2.2) depends on the local parametrization F. However, one can show that it does not depend on the choice of the local parametrization G in the target space.

Suppose $\widetilde{G}(w_1, \ldots, w_n)$ is another local parametrization around $\Phi(p)$. Then by the chain rule:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\partial w_j}{\partial u_i} \frac{\partial}{\partial w_j} &= \sum_{j=1}^{n} \frac{\partial w_j}{\partial u_i} \left(\sum_{k=1}^{n} \frac{\partial v_k}{\partial w_j} \frac{\partial}{\partial v_k} \right) \\
&= \sum_{j,k=1}^{n} \frac{\partial w_j}{\partial u_i} \frac{\partial v_k}{\partial w_j} \frac{\partial}{\partial v_k} \\
&= \sum_{k=1}^{n} \frac{\partial v_k}{\partial u_i} \frac{\partial}{\partial v_k}
\end{aligned}$$

Therefore, the way to define $\dfrac{\partial\Phi}{\partial u_i}$ in (2.2) is independent of choice of local parametrization G for the target manifold N.

Example 2.30. Consider the map $\Phi : \mathbb{RP}^1 \times \mathbb{RP}^2 \to \mathbb{RP}^5$ defined by:

$$\Phi([x_0 : x_1], [y_0 : y_1 : y_2]) = [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2].$$

Under the standard local parametrizations $F(u) = [1 : u]$ for $\mathbb{RP}^1$, $G(v_1, v_2) = [1 : v_1 : v_2]$ for $\mathbb{RP}^2$, and $H(w_1, \ldots, w_5) = [1 : w_1 : \cdots : w_5]$ for $\mathbb{RP}^5$, the local expression of Φ is given by:

$$\begin{aligned}
&H^{-1} \circ \Phi \circ (F \times G)(u, v_1, v_2) \\
&= H^{-1} \circ \Phi([1 : u], [1 : v_1 : v_2]) \\
&= H^{-1}\left([1 : v_1 : v_2 : u : uv_1 : uv_2]\right) \\
&= (v_1, v_2, u, uv_1, uv_2).
\end{aligned}$$

Via the map Φ, we can regard $(w_1, w_2, w_3, w_4, w_5) = (v_1, v_2, u, uv_1, uv_2)$, and the partial derivatives of Φ are given by:

$$\begin{aligned}
\frac{\partial\Phi}{\partial u} &= \frac{\partial w_1}{\partial u}\frac{\partial}{\partial w_1} + \ldots + \frac{\partial w_5}{\partial u}\frac{\partial}{\partial w_5} = \frac{\partial}{\partial w_3} + v_1\frac{\partial}{\partial w_4} + v_2\frac{\partial}{\partial w_5} \\
\frac{\partial\Phi}{\partial v_1} &= \frac{\partial w_1}{\partial v_1}\frac{\partial}{\partial w_1} + \ldots + \frac{\partial w_5}{\partial v_1}\frac{\partial}{\partial w_5} = \frac{\partial}{\partial w_1} + u\frac{\partial}{\partial w_4} \\
\frac{\partial\Phi}{\partial v_2} &= \frac{\partial w_1}{\partial v_2}\frac{\partial}{\partial w_1} + \ldots + \frac{\partial w_5}{\partial v_2}\frac{\partial}{\partial w_5} = \frac{\partial}{\partial w_2} + u\frac{\partial}{\partial w_5}
\end{aligned}$$

□

Similar to tangent maps between regular surfaces in $\mathbb{R}^3$, we define:

$$(\Phi_*)_p\left(\frac{\partial}{\partial u_i}(p)\right) := \frac{\partial\Phi}{\partial u_i}(p)$$

and extend the map linearly to all vectors in T_pM. This is then a linear map between T_pM and $T_{\Phi(p)}N$, and we call this map the *tangent map*.

Definition 2.31 (Tangent Maps)**.** Under the same assumption stated in Definition 2.29, the *tangent map* of Φ at $p \in M$ denoted by $(\Phi_*)_p$ is defined as:

$$(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$$

$$(\Phi_*)_p\left(\sum_{i=1}^n a_i\frac{\partial}{\partial u_i}(p)\right) = \sum_{i=1}^n a_i\frac{\partial\Phi}{\partial u_i}(p)$$

If the point p is clear from the context, $(\Phi_*)_p$ can be simply denoted by Φ_*.

For brevity, we will from now on say "$(u_1, \ldots, u_m)$ are local coordinates of M around p" instead of saying in a clumsy way that "$F : \mathcal{U} \to M$ is a local parametrization of M around p and that $(u_1, \ldots, u_m)$ are coordinates on $\mathcal{U}$".

Given a local coordinates $(u_1, \ldots, u_m)$ around p, and local coordinates $(v_1, \ldots, v_n)$ around $\Phi(p)$, then from (2.2), the matrix representation of $(\Phi_*)_p$ with respect to bases $\left\{\dfrac{\partial}{\partial u_i}(p)\right\}_{i=1}^m$ and $\left\{\dfrac{\partial}{\partial v_j}(\Phi(p))\right\}_{j=1}^n$ is given by $\left[\dfrac{\partial v_j}{\partial u_i}\right]_{i=1,\ldots,m}^{j=1,\ldots,n}$ where i stands for the column, and j stands for the row. The matrix is nothing but the Jacobian matrix:

$$[(\Phi_*)_p] = \left[\frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_m)}\right]_{F^{-1}(p)} = \left[D(G^{-1} \circ \Phi \circ F)\right]_{F^{-1}(p)}. \tag{2.3}$$

Example 2.32. Consider again the map $\Phi : \mathbb{RP}^1 \times \mathbb{RP}^2 \to \mathbb{RP}^5$ in Example 2.30. Under the local parametrizations considered in that example, we then have (for instance):

$$\Phi_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial w_3} + v_1 \frac{\partial}{\partial w_4} + v_2 \frac{\partial}{\partial w_5}.$$

Using the results computed in Example 2.30, the matrix representation of Φ_* is given by:

$$[\Phi_*] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ v_1 & u & 0 \\ v_2 & 0 & u \end{bmatrix}$$

Hence, Φ_* is injective. Remark: To be rigorous, we have only shown $(\Phi_*)_p$ is injective at any p covered by the local coordinate charts we picked. The matrix $[\Phi_*]$ using other local coordinate charts can be computed in a similar way (left as an exercise). □

Exercise 2.19. Consider the map $\Phi : \mathbb{RP}^1 \times \mathbb{RP}^2 \to \mathbb{RP}^5$ defined as in Example 2.30. This time, we use the local parametrizations

$$F(u) = [u : 1]$$
$$G(v_0, v_2) = [v_0 : 1 : v_2]$$
$$H(w_0, w_1, w_3, w_4, w_5) = [w_0 : w_1 : 1 : w_3 : w_4 : w_5]$$

for $\mathbb{RP}^1$, $\mathbb{RP}^2$ and $\mathbb{RP}^5$ respectively. Compute matrix representation of Φ_* using these local parametrizations.

Exercise 2.20. Note that in Definition 2.31 we defined Φ_* using local coordinates. Show that Φ_* is independent of local coordinates. Precisely, show that if:

$$\sum_i a_i \frac{\partial}{\partial u_i} = \sum_i b_i \frac{\partial}{\partial w_i}$$

where $\{u_i\}$ and $\{w_i\}$ are two local coordinates of M, then we have:

$$\sum_i a_i \frac{\partial \Phi}{\partial u_i} = \sum_i b_i \frac{\partial \Phi}{\partial w_i}, \quad \text{which implies} \quad \Phi_*\left(\sum_i a_i \frac{\partial}{\partial u_i}\right) = \Phi_*\left(\sum_i b_i \frac{\partial}{\partial w_i}\right).$$

Exercise 2.21. The identity map id_M of a smooth manifolds M takes any point $p \in M$ to itself, i.e. $\mathrm{id}_M(p) = p$. Show that its tangent map $(\mathrm{id}_M)_*$ at p is the identity map on the tangent space T_pM.

Exercise 2.22. Consider two smooth manifolds M^m and N^n, and their product $M^m \times N^n$. Find the tangent maps $(\pi_M)_*$ and $(\pi_N)_*$ of projection maps:

$$\pi_M : M \times N \to M$$
$$(x, y) \mapsto x$$
$$\pi_N : M \times N \to N$$
$$(x, y) \mapsto y$$

2.4. Inverse Function Theorem

2.4.1. Chain Rule. Consider a smooth function $\Psi(v_1, \ldots, v_k) : \mathbb{R}^k \to \mathbb{R}^m$, another smooth function $\Phi(u_1, \ldots, u_n) : \mathbb{R}^n \to \mathbb{R}^k$ and the composition $\Psi \circ \Phi$. Under this composition, $(v_1, \ldots, v_k)$ can be regarded as a function of $(u_1, \ldots, u_n)$, and the output $(w_1, \ldots, w_m) = \Psi(v_1, \ldots, v_k)$ is ultimately a function of $(u_1, \ldots, u_n)$. In Multivariable Calculus, the chain rule is usually stated as:

$$\frac{\partial w_j}{\partial u_i} = \sum_l \frac{\partial w_j}{\partial v_l} \frac{\partial v_l}{\partial u_i}$$

or equivalently in an elegant way using Jacobian matrices:

$$\frac{\partial(w_1, \ldots, w_m)}{\partial(u_1, \ldots, u_n)} = \frac{\partial(w_1, \ldots, w_m)}{\partial(v_1, \ldots, v_k)} \frac{\partial(v_1, \ldots, v_k)}{\partial(u_1, \ldots, u_n)}.$$

Our goal here is to show that the chain rule can be generalized to maps between smooth manifolds, and can be rewritten using tangent maps:

Theorem 2.33 (Chain Rule: smooth manifolds)**.** *Let $\Phi : M^m \to N^n$ and $\Psi : N^n \to P^k$ be two smooth maps between smooth manifolds M, N and P, then we have:*

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$$

Proof. Suppose $F(u_1, \ldots, u_m)$ is a smooth local parametrization of M, $G(v_1, \ldots, v_n)$ is a smooth local parametrization of N and $H(w_1, \ldots, w_k)$ is a smooth parametrization of P. Locally, $(w_1, \ldots, w_k)$ are then functions of $(v_1, \ldots, v_n)$ via Ψ; and $(v_1, \ldots, v_n)$ are functions of $(u_1, \ldots, u_m)$ via Φ, i.e.

$$\begin{aligned}(w_1, \ldots, w_k) &= H^{-1} \circ \Psi \circ G(v_1, \ldots, v_n)\\ (v_1, \ldots, v_n) &= G^{-1} \circ \Phi \circ F(u_1, \ldots, u_m)\end{aligned}$$

Ultimately, we can regard $(w_1, \ldots, w_k)$ as functions of $(u_1, \ldots, u_m)$ via the composition $\Psi \circ \Phi$:

$$\begin{aligned}(w_1, \ldots, w_k) &= (H^{-1} \circ \Psi \circ G) \circ (G^{-1} \circ \Phi \circ F)(u_1, \ldots, u_m)\\ &= H^{-1} \circ (\Psi \circ \Phi) \circ F(u_1, \ldots, u_m)\end{aligned}$$

To find the tangent map $(\Psi \circ \Phi)_*$, we need to figure out how it acts on the basis vectors $\dfrac{\partial}{\partial u_i}$, and recall that it is defined (see (2.2)) as follows:

$$(\Psi \circ \Phi)_* \left(\frac{\partial}{\partial u_i}\right) = \frac{\partial(\Psi \circ \Phi)}{\partial u_i} = \sum_{j=1}^{k} \frac{\partial w_j}{\partial u_i} \frac{\partial}{\partial w_j}.$$

Next, we use the (standard) chain rule for maps between Euclidean spaces:

$$\sum_{j=1}^{k} \frac{\partial w_j}{\partial u_i} \frac{\partial}{\partial w_j} = \sum_{j=1}^{k} \sum_{l=1}^{n} \frac{\partial w_j}{\partial v_l} \frac{\partial v_l}{\partial u_i} \frac{\partial}{\partial w_j}.$$

Therefore, we get:

$$(\Psi \circ \Phi)_* \left(\frac{\partial}{\partial u_i}\right) = \sum_{j=1}^{k} \sum_{l=1}^{n} \frac{\partial w_j}{\partial v_l} \frac{\partial v_l}{\partial u_i} \frac{\partial}{\partial w_j}.$$

Next, we verify that $\Psi_* \circ \Phi_* \left(\dfrac{\partial}{\partial u_i}\right)$ will give the same output:

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial \Phi}{\partial u_i} = \sum_{l=1}^{n} \frac{\partial v_l}{\partial u_i}\frac{\partial}{\partial v_l}$$

$$\begin{aligned}\Psi_* \circ \Phi_*\left(\frac{\partial}{\partial u_i}\right) &= \Psi_*\left(\sum_{l=1}^{n} \frac{\partial v_l}{\partial u_i}\frac{\partial}{\partial v_l}\right) = \sum_{l=1}^{n} \frac{\partial v_l}{\partial u_i}\Psi_*\left(\frac{\partial}{\partial v_l}\right) \\ &= \sum_{l=1}^{n} \frac{\partial v_l}{\partial u_i}\frac{\partial \Psi}{\partial v_l} = \sum_{l=1}^{n} \frac{\partial v_l}{\partial u_i}\left(\sum_{j=1}^{k} \frac{\partial w_j}{\partial v_l}\frac{\partial}{\partial w_j}\right) \\ &= \sum_{j=1}^{k}\sum_{l=1}^{n} \frac{\partial w_j}{\partial v_l}\frac{\partial v_l}{\partial u_i}\frac{\partial}{\partial w_j}.\end{aligned}$$

Therefore, we have:

$$(\Psi \circ \Phi)_*\left(\frac{\partial}{\partial u_i}\right) = \Psi_* \circ \Phi_*\left(\frac{\partial}{\partial u_i}\right)$$

for any i, and hence $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$. □

Here is one immediate corollary of the chain rule:

Corollary 2.34. *If $\Phi : M \to N$ is a diffeomorphism between two smooth manifolds M and N, then at each point $p \in M$ the tangent map $\Phi_* : T_pM \to T_{\Phi(p)}N$ is invertible.*

Proof. Given that Φ is a diffeomorphism, the inverse map $\Phi^{-1} : N \to M$ exists. Since $\Phi^{-1} \circ \Phi = \mathrm{id}_M$, using the chain rule and Exercise 2.21, we get:

$$\mathrm{id}_{TM} = (\mathrm{id}_M)_* = (\Phi^{-1} \circ \Phi)_* = (\Phi^{-1})_* \circ \Phi_*.$$

Similarly, one can also show $\Phi_* \circ \Phi_*^{-1} = \mathrm{id}_{TN}$. Therefore, Φ_*, and $(\Phi^{-1})_*$ as well, are invertible. □

Exercise 2.23. Given two diffeomorphic smooth manifolds M and N, what can you say about $\dim M$ and $\dim N$?

Exercise 2.24. Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Consider the maps $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$ defined by

$$\pi(x, y, z) := [x : y : z]$$

and $\Phi : \mathbb{RP}^2 \to \mathbb{R}^4$ defined by:

$$\Phi([x : y : z]) = (x^2 - y^2, xy, xz, yz).$$

Locally parametrize $\mathbb{S}^2$ stereographically:

$$F(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)$$

and $\mathbb{RP}^2$ by a standard parametrization:

$$G(w_1, w_2) = [1 : w_1 : w_2].$$

Compute $[\Phi_*]$, $[\pi_*]$ and $[(\Phi \circ \pi)_*]$ directly, and verify that $[(\Phi \circ \pi)_*] = [\Phi_*][\pi_*]$.

2.4.2. Inverse Function Theorem. Given a diffeomorphism $\Phi : M \to N$, it is necessary that Φ_* is invertible. One natural question to ask is that if we are given Φ_* is invertible, can we conclude that Φ is a diffeomorphism?

Unfortunately, it is too good to be true. One easy counter-example is the map $\Phi : \mathbb{R} \to \mathbb{S}^1$, defined as:

$$\Phi(t) = (\cos t, \sin t).$$

As both $\mathbb{R}$ and $\mathbb{S}^1$ are one dimensional manifolds, to show that Φ_* is invertible it suffices to show that $\Phi_* \neq 0$, which can be verified by considering:

$$\Phi_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial \Phi}{\partial t} = (-\sin t, \cos t) \neq 0.$$

However, it is clear that Φ is not even one-to-one, and hence Φ^{-1} does not exist.

Fortunately, the Inverse Function Theorem tells us that Φ is *locally* invertible near p whenever $(\Phi_*)_p$ is invertible. In Multivariable Calculus/Analysis, the Inverse Function Theorem asserts that if the Jacobian matrix of a smooth map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ at p is invertible, then there exists an open set $\mathcal{U} \subset \mathbb{R}^n$ containing p, and an open set $\mathcal{V} \subset \mathbb{R}^n$ containing $\Phi(p)$ such that $\Phi|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism.

Now suppose $\Phi : M \to N$ is a smooth map between two smooth manifolds M and N. According to (2.2), the matrix representation of the tangent map Φ_* is a Jacobian matrix. Therefore, one can generalize the Inverse Function Theorem to smooth manifolds. To start, we first define:

Definition 2.35 (Local Diffeomorphisms)**.** Let $\Phi : M \to N$ be a smooth map between two smooth manifolds M and N. We say Φ is a *local diffeomorphism* near p if there exists an open set $\mathcal{O}_M \subset M$ containing p, and an open set $\mathcal{O}_N \subset N$ containing $\Phi(p)$ such that $\Phi|_{\mathcal{O}_M} : \mathcal{O}_M \to \mathcal{O}_N$ is a diffeomorphism.

If such a smooth map exists, we say M is *locally diffeomorphic to N near p*, or equivalently, *N is locally diffeomorphic to M near $\Phi(p)$*. If M is locally diffeomorphic to N near every point $p \in M$, then we say M is *locally diffeomorphic* to N.

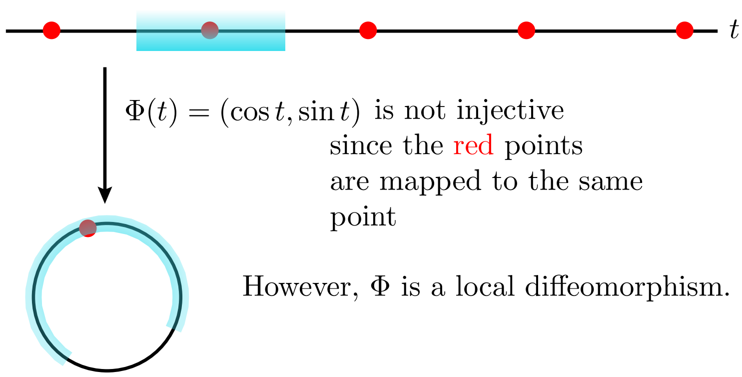


Figure 2.6. A local diffeomorphism which is not injective.

Theorem 2.36 (Inverse Function Theorem)**.** *Let $\Phi : M \to N$ be a smooth map between two smooth manifolds M and N. If $(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$ is invertible, then M is locally diffeomorphic to N near p.*

Proof. The proof to be presented uses the Inverse Function Theorem for Euclidean spaces and then extends it to smooth manifolds. For the proof of the Euclidean case, readers may consult the lecture notes of MATH 3033/3043.

Let F be a local parametrization of M near p, and G be a local parametrization of N near $\Phi(p)$. Given that $(\Phi_*)_p$ is invertible, by (2.3) we know that the following Jacobian matrix $D(G^{-1} \circ \Phi \circ F)$ is invertible at $F^{-1}(p)$. By Inverse Function Theorem for Euclidean spaces, there exist an open set $\mathcal{U}_M \subset \mathbb{R}^{\dim M}$ containing $F^{-1}(p)$, and an open set $\mathcal{U}_N \subset \mathbb{R}^{\dim N}$ containing $G^{-1}(\Phi(p))$ such that:

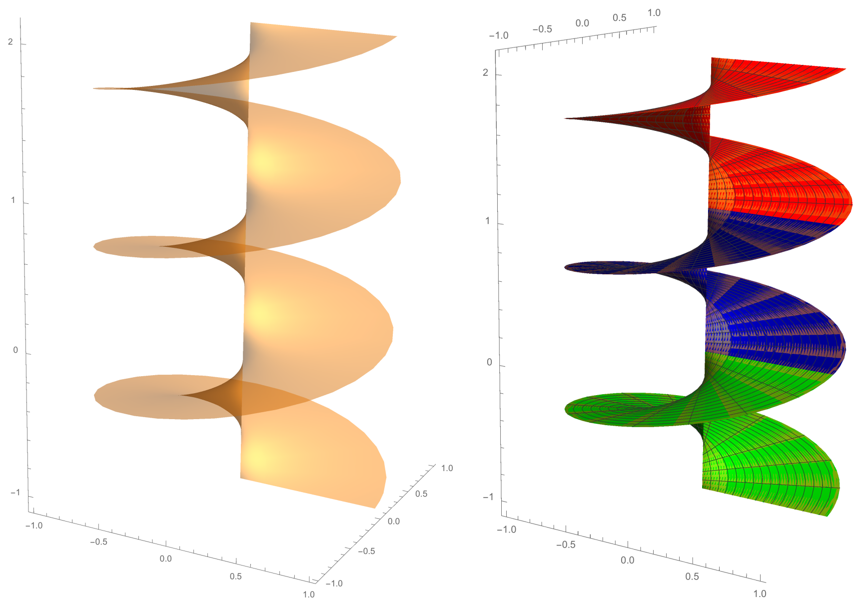
$$G^{-1} \circ \Phi \circ F\big|_{\mathcal{U}_M} : \mathcal{U}_M \to \mathcal{U}_N$$

is a diffeomorphism, i.e. the inverse $F^{-1} \circ \Phi^{-1} \circ G$ exists when restricted to $\mathcal{U}_N$ and is smooth.

Denote $\mathcal{O}_M = F(\mathcal{U}_M)$ and $\mathcal{O}_N = G(\mathcal{U}_N)$. By the definition of smooth maps, this shows $\Phi|_{\mathcal{O}_M}$ and $\Phi^{-1}\big|_{\mathcal{O}_N}$ are smooth. Hence $\Phi|_{\mathcal{O}_M}$ is a local diffeomorphism near p. $\square$

Example 2.37. The helicoid Σ is defined to be the following surface in $\mathbb{R}^3$:

$$\Sigma := \{(r\cos\theta, r\sin\theta, \theta) \in \mathbb{R}^3 : r > 0 \text{ and } \theta \in \mathbb{R}\}.$$



(a) A helicoid **(b)** Locally diffeomorphic to $\mathbb{R}^2\backslash\{0\}$

Figure 2.7. a helicoid is not globally diffeomorphic to $\mathbb{R}^2\backslash\{0\}$, but is locally diffeomorphic to $\mathbb{R}^2\backslash\{0\}$.

It can be parametrized by:

$$F : (0, \infty) \times \mathbb{R} \to \Sigma$$
$$F(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$$

Consider the map $\Phi : \Sigma \to \mathbb{R}^2\backslash\{0\}$ defined as:

$$\Phi(r\cos\theta, r\sin\theta, \theta) = (r\cos\theta, r\sin\theta).$$

It is clear that Φ is not injective: for instance, $\Phi(\cos 2\pi, \sin 2\pi, 2\pi) = \Phi(\cos 0, \sin 0, 0)$. However, we can show that $(\Phi_*)_p$ is injective at each point $p \in \Sigma$.

The set $\mathbb{R}^2 \backslash \{0\}$ is open in $\mathbb{R}^2$. The matrix $[\Phi_*]$ is the Jacobian matrix of $\Phi \circ F$:

$$\begin{aligned}
\Phi \circ F(r, \theta) &= \Phi(r \cos\theta, r\sin\theta, \theta) \\
&= (r\cos\theta, r\sin\theta)
\end{aligned}$$

$$[\Phi_*] = D(\Phi \circ F) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

As $\det[\Phi_*] = r \neq 0$, the linear map $[\Phi_*]$ is invertible. By Inverse Function Theorem, Φ is a local diffeomorphism.

Exercise 2.25. Show that $\mathbb{S}^n$ and $\mathbb{RP}^n$ are locally diffeomorphic via the map:

$$\Phi(x_0, \ldots, x_n) = [x_0 : \cdots : x_n].$$

2.5. Immersions and Submersions

2.5.1. Review of Linear Algebra: injectivity and surjectivity.

Given a linear map $T : V \to W$ between two finite dimensional vector spaces V and W, the following are equivalent:

(a) T is injective;

(b) $\ker T = \{0\}$;

(c) The row reduced echelon form (RREF) of the matrix of T has no *free column*.

In each RREF of a matrix, we call the first non-zero entry (if exists) of each row to be a pivot. A free column of an RREF is a column which does *not* have a pivot. For instance, the following RREF:

$$R = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has three pivots, and two free columns (namely the second and fourth columns). Any map with a matrix which can be row reduced to this R is not injective.

Surjectivity of a linear map $T : V \to W$ can also be stated in several equivalent ways:

(a) T is surjective;

(b) rank$(T) = \dim W$;

(c) All rows in the RREF of the matrix of T are non-zero.

For instance, all rows of the matrix R above are non-zero. Hence any map with a matrix which can be row reduced to R is surjective.

Exercise 2.26. Let $T : V \to W$ be a linear map between two finite dimensional vector spaces V and W. Given that T is injective, what can you say about $\dim V$ and $\dim W$? Explain. Now given that T is surjective, what can you say about $\dim V$ and $\dim W$? Explain.

2.5.2. Immersions.

Loosely speaking, an immersion from one smooth manifold to another is a map that is "locally injective". Here is the rigorous definition:

Definition 2.38 (Immersions). Let $\Phi : M \to N$ be a smooth map between two smooth manifolds M and N. We say Φ is an *immersion at* $p \in M$ if the tangent map $(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$ is injective. If Φ is an immersion at every point on M, then we simply say Φ is an *immersion*.

Remark 2.39. As a linear map $T : V \to W$ between any two finite dimensional vector spaces cannot be injective if $\dim V > \dim W$, an immersion $\Phi : M \to N$ can only exist when $\dim M \leq \dim N$. □

Example 2.40. The map $\Phi : \mathbb{R} \to \mathbb{S}^1$ defined by:

$$\Phi(t) = (\cos t, \sin t)$$

is an immersion. The tangent space of $\mathbb{R}$ at any point t_0 is simply span $\left\{ \left.\frac{\partial}{\partial t}\right|_{t=t_0} \right\}$. The tangent map $(\Phi_*)_{t_0}$ is given by:

$$(\Phi_*)_{t_0}\left(\frac{\partial}{\partial t}\right) = \left.\frac{\partial \Phi}{\partial t}\right|_{t=t_0} = (-\sin t_0, \cos t_0) \neq 0.$$

Therefore, the "matrix" of Φ_* is a one-by-one matrix with a non-zero entry. Clearly, there is no free column and so Φ_* is injective at every $t_0 \in \mathbb{R}$. This shows Φ is an immersion.

This example tells us that an immersion Φ is not necessary injective. □

Example 2.41. Let M^2 be a regular surface in $\mathbb{R}^3$, then the inclusion map $\iota : M^2 \to \mathbb{R}^3$, defined as $\iota(p) = p \in \mathbb{R}^3$, is a smooth map, since for any local parametrization $F(u_1, u_2)$ of M^2, we have $\iota \circ F = F$, which is smooth by definition (see p.4). We now show that ι is an immersion:

$$(\iota_*)_p \left(\frac{\partial F}{\partial u_i}\right) = \left.\frac{\partial(\iota \circ F)}{\partial u_i}\right|_{F^{-1}(p)} = \left.\frac{\partial F}{\partial u_i}\right|_{F^{-1}(p)}.$$

Let $F(u_1, u_2) = (x_1(u_1, u_2),\ x_2(u_1, u_2),\ x_3(u_1, u_2))$, then

$$\frac{\partial F}{\partial u_i} = \sum_{j=1}^{3} \frac{\partial x_j}{\partial u_i} \hat{e}_i$$

where $\{\hat{e}_i\}$ is the standard basis of $\mathbb{R}^3$. Therefore, the matrix of ι_* is given by:

$$[\iota_*] = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} \end{bmatrix}.$$

By the condition $0 \neq \dfrac{\partial F}{\partial u_1} \times \dfrac{\partial F}{\partial u_2} = \dfrac{\partial(x_2, x_3)}{\partial(u_1, u_2)}\hat{e}_1 + \dfrac{\partial(x_3, x_1)}{\partial(u_1, u_2)}\hat{e}_2 + \dfrac{\partial(x_1, x_2)}{\partial(u_1, u_2)}\hat{e}_3$, at each $p \in M$ one least one of the following is invertible:

$$\frac{\partial(x_2, x_3)}{\partial(u_1, u_2)}, \quad \frac{\partial(x_3, x_1)}{\partial(u_1, u_2)}, \quad \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)}.$$

and hence has the 2×2 identity as its RREF. Using this fact, one can row reduce $[\iota_*]$ so that it becomes:

$$[\iota_*] \to \ldots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has no free column. Therefore, $[\iota_*]$ is an injective linear map at every $p \in M$. This shows ι is an immersion. □

Exercise 2.27. Define a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^4$ by:

$$\Phi(x, y) = (x^3, x^2y, xy^2, y^3).$$

Show that Φ is an immersion at any $(x, y) \neq (0, 0)$.

Exercise 2.28. Given two immersions $\Phi : M \to N$ and $\Psi : N \to P$ between smooth manifolds M, N and P, show that $\Psi \circ \Phi : M \to P$ is also an immersion.

Exercise 2.29. Consider two smooth maps $\Phi_1 : M_1 \to N_1$ and $\Phi_2 : M_2 \to N_2$ between smooth manifolds. Show that:

(a) If both Φ_1 and Φ_2 are immersions, then so does $\Phi_1 \times \Phi_2 : M_1 \times M_2 \to N_1 \times N_2$.

(b) If Φ_1 is not an immersion, then $\Phi_1 \times \Phi_2$ cannot be an immersion.

A nice property of an immersion $\Phi : M^m \to N^n$ is that for every $\Phi(p) \in N$, one can find a *special* local parametrization G of N such that $G^{-1} \circ \Phi \circ F$ is an inclusion map from $\mathbb{R}^m$ to $\mathbb{R}^n$, which is a map which takes $(x_1, \ldots, x_m)$ to $(x_1, \ldots, x_m, 0, \ldots, 0)$. Let's state the result in a precise way:

Theorem 2.42 (Immersion Theorem). *Let $\Phi : M^m \to N^{m+k}$ be an immersion at $p \in M$ between two smooth manifolds M^m and N^{n+k} with $k \geq 1$. Given any local parametrization $F : \mathcal{U}_M \to \mathcal{O}_M$ of M near $p \in M$, and any local parametrization $G : \mathcal{U}_N \to \mathcal{O}_N$ of N near $\Phi(p) \in N$, there exists a smooth reparametrization map $\psi : \widetilde{\mathcal{U}}_N \to \mathcal{U}_N$ such that:*

$$(G \circ \psi)^{-1} \circ \Phi \circ F(u_1, \ldots, u_m) = (u_1, \ldots, u_m, \underbrace{0, \ldots, 0}_{k}).$$

See Figure 2.8 for an illustration.

Proof. The proof uses the Inverse Function Theorem. By translation, we may assume that $F(0) = p$ and $G(0) = \Phi(p)$. Given that $(\Phi_*)_p$ is injective, there are n linearly independent rows in the matrix $[(\Phi_*)_p]$. WLOG we may assume that the first m rows of $[(\Phi_*)_p]$ are linearly independent. As such, the matrix can be decomposed into the form:

$$[(\Phi_p)_*] = \begin{bmatrix} A \\ * \end{bmatrix}$$

where A is an invertible $m \times m$ matrix, and $*$ denotes any $k \times m$ matrix.

Now define $\psi : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ as:

(*)

$$\psi(u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+k}) = G^{-1} \circ \Phi \circ F(u_1, \ldots, u_m) + (0, \ldots, 0, u_{m+1}, \ldots, u_{m+k}).$$

We claim that this is the map ψ that we want. First note that $\psi(0) = G^{-1} \circ \Phi(p) = 0$ by our earlier assumption. Next we show that ψ has a smooth inverse near 0. The Jacobian matrix of this map at 0 is given by:

$$[(D\psi)_0] = \begin{bmatrix} A & 0 \\ * & I_k \end{bmatrix}.$$

As rows of A are linearly independent, it is easy to see then all rows of $[(D\psi)_0]$ are linearly independent, and hence $[(D\psi)_0]$ is invertible. By Inverse Function Theorem, ψ is locally invertible near 0, i.e. there exists an open set $\widetilde{\mathcal{U}}_N \subset \mathbb{R}^{m+k}$ containing 0 such that the restricted map:

$$\psi|_{\widetilde{\mathcal{U}}_N} : \widetilde{\mathcal{U}}_N \to \psi(\widetilde{\mathcal{U}}_N) \subset \mathcal{U}_N.$$

has a smooth inverse.

Finally, we verify that this is the map ψ that we want. We compute:

$$(G \circ \psi)^{-1} \circ \Phi \circ F(u_1, \ldots, u_m) = \psi^{-1}\left((G^{-1} \circ \Phi \circ F)(u_1, \ldots, u_m)\right).$$

By (*), we have $\psi(u_1, \ldots, u_m, 0, \ldots, 0) = G^{-1} \circ \Phi \circ F(u_1, \ldots, u_m)$, and hence:

$$\psi^{-1}\left((G^{-1} \circ \Phi \circ F)(u_1, \ldots, u_m)\right) = (u_1, \ldots, u_m, 0, \ldots, 0).$$

It completes our proof. □

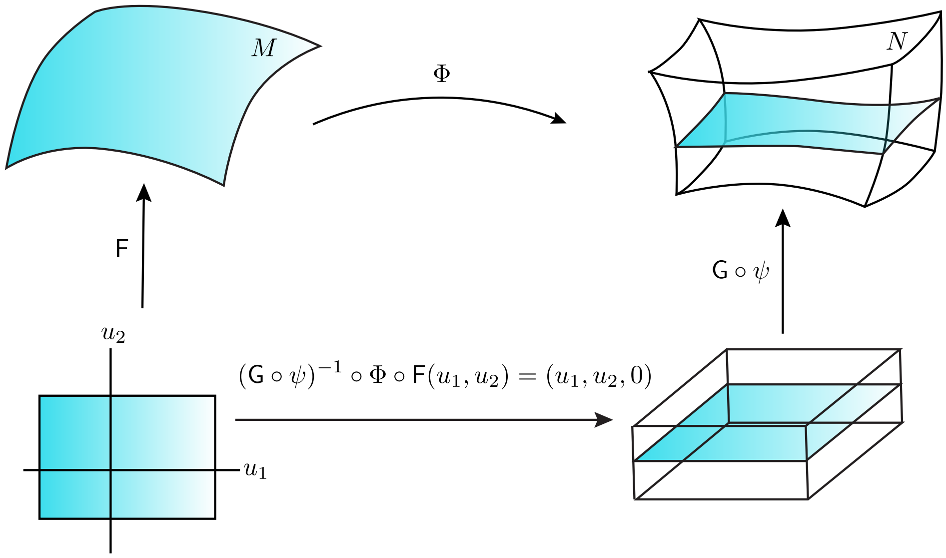


Figure 2.8. Geometric illustration of the Immersion Theorem.

Example 2.43. Consider the map $\Phi : \mathbb{R} \to \mathbb{R}^2$ defined by:

$$\Phi(\theta) = (\cos\theta, \sin\theta).$$

It is easy to see that $[\Phi_*] = (-\sin\theta, \cos\theta) \neq (0,0)$ for any θ. Hence Φ is an immersion. We can locally parametrize $\mathbb{R}^2$ near image of Φ by:

$$\widetilde{G}(\theta, r) := ((1-r)\cos\theta, (1-r)\sin\theta),$$

then $\widetilde{G}^{-1} \circ \Phi(\theta) = \widetilde{G}^{-1}(\cos\theta, \sin\theta) = (\theta, 0)$. Note that the Immersion Theorem (Theorem 2.42) asserts that such $\widetilde{G}$ exists, it fails to give an explicit form of such a $\widetilde{G}$. □

Exercise 2.30. Consider the sphere $\mathbb{S}^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$. Find local parametrizations F for $\mathbb{S}^2$, and G for $\mathbb{R}^3$ such that the composition

$$G^{-1} \circ \iota \circ F$$

takes (u_1, u_2) to $(u_1, u_2, 0)$. Here $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map.

2.5.3. Submersions. Another important type of smooth maps are submersions. Loosely speaking, a submersion is a map that is "locally surjective". Here is the rigorous definition:

Definition 2.44 (Submersions)**.** Let $\Phi : M \to N$ be a smooth map between smooth manifolds M and N. We say Φ is a *submersion at* $p \in M$ if the tangent map $(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$ is surjective. If Φ is a submersion at every point on M, then we simply say Φ is a *submersion*.

Remark 2.45. Clearly, in order for $\Phi : M \to N$ to be a submersion at any point $p \in M$, it is necessary that $\dim M \geq \dim N$. □

Example 2.46. Given two smooth manifolds M and N, the projection maps $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are both submersions. To verify this, recall that

$T_{(p,q)}(M \times N) = T_pM \oplus T_qN$, and from Exercise 2.22 that $(\pi_M)_* = \pi_{TM}$ where π_{TM} is the projection map of the tangent space:

$$\pi_{TM}(v, w) = v \quad \text{for any } v \in T_pM \text{ and } w \in T_qN.$$

The matrix $[\pi_{TM}]$ is then of the form: $\begin{bmatrix} I & 0 \end{bmatrix}$ where I is the identity matrix of dimension $\dim M$ and 0 is the $\dim M \times \dim N$ zero matrix. There are pivots in every row so π_{TM} is surjective. Similarly we can also show $(\pi_N)_* = \pi_{TN}$ is also surjective. □

Example 2.47. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, and at the point $p \in \mathbb{R}^n$ such that $\nabla f(p) \neq 0$, one can show f is a submersion at p. To show this, let $\{\hat{e}_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$, then

$$f_*(\hat{e}_i) = f_*\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$$

and so the matrix of $[f_*]$ is given by $\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$. At the point p, we have $\nabla f(p) \neq 0$ which is equivalent to show $[f_*]$ at p is a non-zero $1 \times n$ matrix, which always have 1 pivot in its RREF. Therefore, $(f_*)_p$ is surjective and f is a submersion at p. □

Exercise 2.31. Show that if M and N are two smooth manifolds of equal dimension, then the following are equivalent:

(i) $\Phi : M \to N$ is a local diffeomorphism.

(ii) $\Phi : M \to N$ is an immersion.

(iii) $\Phi : M \to N$ is a submersion.

Exercise 2.32. Find a smooth map $\Phi : \mathbb{R} \to \mathbb{R}$ which is a submersion but is not surjective.

Exercise 2.33. Show that the map $\Phi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{RP}^n$ defined as:

$$\Phi(x_0, \ldots, x_n) = [x_0 : \cdots : x_n]$$

is a submersion.

One nice property of a submersion $\Phi : M^m \to N^n$ that locally around every $p \in M$, one can find *special* local parametrizations F of M near p, and G of N near $\Phi(p)$ such that $G^{-1} \circ \Phi \circ F$ is a projection map. We will see later that this result will show any level-set of Φ, if non-empty, must be a smooth manifold. Let's state this result in a precise way:

Theorem 2.48 (Submersion Theorem)**.** *Let $\Phi : M^{n+k} \to N^n$ be a submersion at $p \in M$ between two smooth manifolds M^{n+k} and N^n with $k \geq 1$. Given any local parametrization $F : \mathcal{U}_M \to \mathcal{O}_M$ of M near $p \in M$, and any local parametrization $G : \mathcal{U}_N \to \mathcal{O}_N$ of N near $\Phi(p) \in N$, there exists a smooth reparametrization map $\psi : \widetilde{\mathcal{U}}_M \to \mathcal{U}_M$ such that:*

$$G^{-1} \circ \Phi \circ (F \circ \psi)(u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k}) = (u_1, \ldots, u_n).$$

See Figure 2.9 for illustration.

Proof. The proof uses again the Inverse Function Theorem. First by translation we may assume that $F(0) = p$ and $G(0) = \Phi(p)$. Given that $(\Phi_*)_p$ is surjective, there are n linearly independent columns in the matrix $[(\Phi_*)_p]$. WLOG assume that they are the first n columns, then $[(\Phi_*)_p]$ is of the form:

$$[(\Phi_*)_p] = \begin{bmatrix} D(G^{-1} \circ \Phi \circ F)_0 \end{bmatrix} = \begin{bmatrix} A & * \end{bmatrix}$$

where A is an $n \times n$ invertible matrix, and $*$ is any $n \times k$ matrix.

Now define $\phi : \mathcal{U}_M \to \mathbb{R}^{n+k}$ as:

$$(*) \qquad \phi(u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k}) = (\underbrace{G^{-1} \circ \Phi \circ F(u_1, \ldots, u_{n+k})}_{\in \mathbb{R}^n}, u_{n+1}, \ldots, u_{n+k}).$$

This map is has an invertible Jacobian matrix at $F^{-1}(p)$ since:

$$[(D\phi)_0] = \begin{bmatrix} A & * \\ 0 & I \end{bmatrix}.$$

By Inverse Function Theorem, there exists a local inverse $\phi^{-1} : \widetilde{\mathcal{U}}_M \to \phi^{-1}(\widetilde{\mathcal{U}}_M) \subset \mathcal{U}_M$. Finally, we verify that:

$$\begin{aligned} & G^{-1} \circ \Phi \circ (F \circ \phi^{-1})(u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k}) \\ &= (G^{-1} \circ \Phi \circ F) \left(\phi^{-1}(u_1, \ldots, u_{n+k})\right) \end{aligned}$$

Let $\phi^{-1}(u_1, \ldots, u_{n+k}) = (v_1, \ldots, v_{n+k})$, then $\phi(v_1, \ldots, v_{n+k}) = (u_1, \ldots, u_{n+k})$. From (*), we get:

$$\begin{aligned} (G^{-1} \circ \Phi \circ F(v_1, \ldots, v_{n+k}), v_{n+1}, \ldots, v_{n+k}) &= \phi(v_1, \ldots, v_{n+k}) \\ &= (u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k}) \end{aligned}$$

which implies $G^{-1} \circ \Phi \circ F(v_1, \ldots, v_{n+k}) = (u_1, \ldots, u_n)$. Combine with previous result, we get:

$$\begin{aligned} & (G^{-1} \circ \Phi \circ F) \left(\phi^{-1}(u_1, \ldots, u_{n+k})\right) \\ &= (G^{-1} \circ \Phi \circ F)(v_1, \ldots, v_{n+k}) \\ &= (u_1, \ldots, u_n). \end{aligned}$$

Hence, $G^{-1} \circ \Phi \circ (F \circ \phi^{-1})$ is the projection from $\mathbb{R}^{n+k}$ onto $\mathbb{R}^n$. It completes the proof by taking $\psi = \phi^{-1}$. $\square$

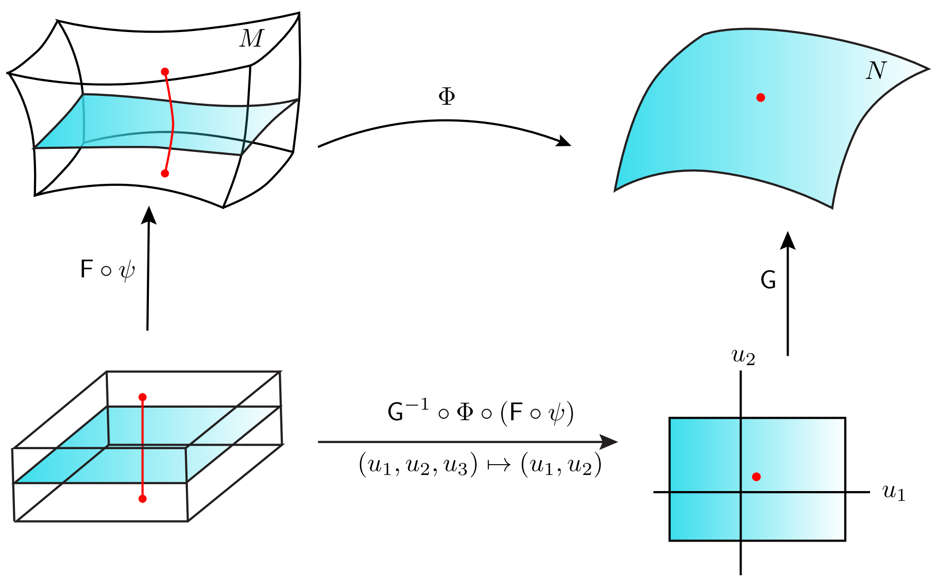


Figure 2.9. Geometric illustration of the Submersion Theorem

2.6. Submanifolds

In this section we talk about submanifolds. A subspace W of a vector space V is a subset of V and is itself a vector space. A subgroup H of a group G is a subset of G and is itself a group. It *seems* that a smooth submanifold N of a smooth manifold M *might* be defined as a subset of M and is itself a smooth manifold. However, it is just one side of the full story – we need more than that because we hope that the local coordinates of a submanifold is in some sense *compatible* with the local coordinates of the manifold M.

Definition 2.49 (Submanifolds)**.** Let M be a smooth n-manifold. A subset $N \subset M$ is said to be a *smooth k-submanifold of M* if N is a smooth k-manifold and the inclusion map $\iota : N \to M$ is an smooth immersion.

Example 2.50. Let $\Phi : M^m \to N^n$ be a smooth map. Define Γ_Φ to be the graph of Φ. Precisely:

$$\Gamma_\Phi = \{(p, \Phi(p)) \in M \times N : p \in M\}.$$

We are going to show that the graph Γ_Φ is a submanifold of $M \times N$, with $\dim \Gamma_\Phi = \dim M$. To show this, consider an arbitrary point $(p, \Phi(p)) \in \Gamma_\Phi$ where $p \in M$. The product manifold $M \times N$ can be locally parametrized by $F \times G$ where F is a smooth local parametrization of M near p and G is a smooth local parametrization of N around $\Phi(p)$.

Γ_Φ is locally parametrized around $(p, \Phi(p))$ by:

$$\widetilde{F}(u) := (F(u), \Phi \circ F(u)).$$

Here for simplicity, we denote $u := (u_1, \ldots, u_m)$ where $m = \dim M$. It can be verified that if F_1 and F_2 are compatible (i.e. with smooth transition maps) parametrizations of M around p, then the induced parametrizations $\widetilde{F}_1$ and $\widetilde{F}_2$ are also compatible (see exercise below).

Recall that for any $u = (u_1, \ldots, u_m)$ and $\mathsf{v} = (v_1, \ldots, v_n)$, the product map $F \times G$ defined as:

$$(F \times G)(u, \mathsf{v}) = (F(u), G(\mathsf{v}))$$

is a local parametrization of $M \times N$. Now we show that the inclusion $\iota : \Gamma_\Phi \to M \times N$ is an immersion:

$$\begin{aligned}(F \times G)^{-1} \circ \iota \circ \widetilde{F}(u) &= (F \times G)^{-1} \circ \iota\,(F(u), \Phi(F(u))) \\ &= (F \times G)^{-1}\,(F(u), \Phi(F(u))) \\ &= \left(u, G^{-1} \circ \Phi \circ F(u)\right).\end{aligned}$$

Its Jacobian matrix has the form:

$$[\iota_*] = \begin{bmatrix} I \\ [\Phi_*] \end{bmatrix}$$

which is injective since its RREF does not have any free column. This completes the proof that ι is an immersion and so Γ_Φ is a submanifold of $M \times N$.

□

Exercise 2.34. Complete the *exercise* stated in Example 2.50 that if F_1 and F_2 are compatible parametrizations of M around p, then the induced parametrizations $\widetilde{F}_1$ and $\widetilde{F}_2$ of Γ_Φ are also compatible.

Exercise 2.35. Let M^m be a smooth manifold. Consider the *diagonal* subset $\Delta_M \subset M \times M$ defined as:

$$\Delta_M := \{(x, x) \in M \times M : x \in M\}.$$

Show that Δ_M is a submanifold of $M \times M$.

Exercise 2.36. Show that if N is a submanifold of M, and P is a submanifold of N, then P is also a submanifold of M.

Exercise 2.37. Show that any non-empty open subset N of a smooth manifold M is a submanifold of M with $\dim N = \dim M$.

We require a submanifold to have the inclusion map being smooth because we want to rule out some pathological cases. Consider the graph of an absolute function, i.e. $\Gamma = \{(x, |x|) : x \in \mathbb{R}\}$, and $\mathbb{R}^2$. The graph Γ can be parametrized by a single parametrization $F : \mathbb{R} \to \Gamma$ defined by:

$$F(t) = (t, |t|).$$

Then, since Γ equipped with this *single* parametrization, it is considered as a *smooth* manifold (although quite difficult to accept) since there is essentially no transition map. However, we (fortunately) can show that Γ is not a submanifold of $\mathbb{R}^2$ (with usual differential structure, parametrized by the identity map). It is because the inclusion map is not smooth:

$$\mathrm{id}_{\mathbb{R}^2}^{-1} \circ \iota \circ F(t) = (t, |t|)$$

Exercise 2.38. Show that if $\mathbb{R}^2$ is (pathologically) parametrized by

$$G : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x, y + |x|)$$

and $\Gamma = \{(x, |x|) : x \in \mathbb{R}\}$ is parametrized by $F(t) = (t, |t|)$, then with differential structures generated by these parametrizations, Γ becomes a submanifold of $\mathbb{R}^2$.

That says: the "pathologically" smooth manifold Γ is a submanifold of this "pathological" $\mathbb{R}^2$.

We require the inclusion map is an immersion because we want a submanifold N of M to be equipped with local coordinates "compatible" with that of M in the following sense:

Proposition 2.51. *If N^n is a submanifold of M^m, then near every point $p \in N$, there exists a smooth local parametrization $G(u_1, \ldots, u_m) : \mathcal{U} \to \mathcal{O}$ of M near p such that $G(0) = p$ and*

$$N \cap \mathcal{O} = \{G(u_1, \ldots, u_n, 0, \ldots, 0) : (u_1, \ldots, u_n, 0, \ldots, 0) \in \mathcal{U}\}.$$

Proof. By Theorem 2.42 (Immersion Theorem), given that $\iota : N \to M$ is an immersion, then around every point $p \in N$ one can find a local parametrization $F : \mathcal{U}_N \to \mathcal{O}_N$ of N near p, and another local parametrization $G(u_1, \ldots, u_m) : \mathcal{U}_M \to \mathcal{O}_M$ of M near $\iota(p) = p$ such that:

$$G^{-1} \circ \iota \circ F(u_1, \ldots, u_n) = (u_1, \ldots, u_n, \underbrace{0, \ldots, 0}_{m-n})$$

and so $F(u_1, \ldots, u_n) = G(u_1, \ldots, u_n, 0, \ldots, 0)$. Note that in order for $G^{-1} \circ \iota \circ F$ to be well-defined, we assume (by shrinking the domains if necessary) that $\mathcal{O}_N = N \cap \mathcal{O}_M$.

Therefore,

$$\begin{aligned}&\{G(u_1,\ldots,u_n,0,\ldots,0):(u_1,\ldots,u_n,0,\ldots,0)\in\mathcal{U}\}\\&=\{F(u_1,\ldots,u_n):(u_1,\ldots,u_n)\in\mathcal{U}_N\}\\&=\mathcal{O}_N=N\cap\mathcal{O}_M\end{aligned}$$

It completes our proof. □

We introduced submersions because the level-set of a submersion, if non-empty, can in fact shown to be a submanifold! We will state and prove this result. Using this fact, one can show a lot of sets are in fact manifolds.

Proposition 2.52. *Let $\Phi : M^m \to N^n$ be a smooth map between two smooth manifolds M and N. Suppose $q \in N$ such that $\Phi^{-1}(q)$ is non-empty, and that Φ is a submersion at any $p \in \Phi^{-1}(q)$, then the level-set $\Phi^{-1}(q)$ is a submanifold of M with $\dim \Phi^{-1}(q) = m - n$.*

Proof. Using Theorem 2.48 (Submersion Theorem), given any point $p \in \Phi^{-1}(q) \subset M$, there exist a local parametrization $F : \mathcal{U}_M \to \mathcal{O}_M$ of M near p, and a local parametrization G of N near $\Phi(p) = q$, such that:

$$G^{-1}\circ\Phi\circ F(u_1,\ldots,u_n,u_{n+1},\ldots,u_m)=(u_1,\ldots,u_n)$$

and that $F(0) = p$, $G(0) = q$.

We first show that $\Phi^{-1}(q)$ is a smooth manifold. Note that we have:

$$\Phi\left(F(0,\ldots,0,u_{n+1},\ldots,u_m)\right)=G(0,\ldots,0)=q.$$

Therefore, $F(0,\ldots,0,u_{n+1},\ldots,u_m) \in \Phi^{-1}(q)$. Hence, $\Phi^{-1}(q)$ can be locally parametrized by $\widetilde{F}(u_{n+1},\ldots,u_m) := F(0,\ldots,0,u_{n+1},\ldots,u_m)$. One can also verify that compatible F's gives compatible $\widetilde{F}$'s. This shows $\Phi^{-1}(q)$ is a smooth manifold of dimension $m - n$.

To show it is a submanifold of M, we need to compute the tangent map ι_*. First consider the composition:

$$F^{-1}\circ\iota\circ\widetilde{F}(u_{n+1},\ldots,u_m)=F^{-1}(F(0,\ldots,0,u_{n+1},\ldots,u_m))=(0,\ldots,0,u_{n+1},\ldots,u_m).$$

The matrix $[\iota_*]$ with respect to local parametrizations $\widetilde{F}$ of $\Phi^{-1}(q)$, and F of M is given by the Jacobian:

$$[\iota_*]=[D(F^{-1}\circ\iota\circ\widetilde{F})]=\begin{bmatrix}0\\I\end{bmatrix}$$

which shows ι_* is injective. Therefore, $\Phi^{-1}(q)$ is a submanifold of M. □

Using Proposition 2.52, one can produce a lot of examples of manifolds which are level-sets of smooth functions.

Example 2.53. In $\mathbb{R}^4$, the set $\Sigma := \{x^3 + y^3 + z^3 + w^3 = 1\}$ is a smooth 3-manifold. It can be shown by consider $\Phi : \mathbb{R}^4 \to \mathbb{R}$ defined by:

$$\Phi(x,y,z,w)=x^3+y^3+z^3+w^3.$$

Then, $\Sigma = \Phi^{-1}(1)$. To show it is a manifold, we show Φ is a submersion at every $p \in \Sigma$. By direct computation, we get:

$$[\Phi_*]=[3x^2\quad 3y^2\quad 3z^2\quad 3w^2]$$

Since $[\Phi_*] = 0$ only when $(x, y, z, w) = (0, 0, 0, 0)$ which is not contained in Σ, we have shown $(\Phi_*)_p$ is injective for any $p \in \Sigma$. By Proposition 2.52, we have proved $\Sigma = \Phi^{-1}(1)$ is a smooth manifold of dimension $4 - 1 = 3$. □

Example 2.54. The set $M_{n\times n}(\mathbb{R})$ of all $n \times n$ real matrices can be regarded as $\mathbb{R}^{2n}$ equipped with the usual differential structure. Consider these subsets of $M_{n\times n}(\mathbb{R})$:

(a) $\text{GL}(n, \mathbb{R})$ = the set of all invertible $n \times n$ matrices;

(b) $\text{Sym}(n, \mathbb{R})$ = the set of all symmetric $n \times n$ matrices;

(c) $\text{O}(n, \mathbb{R})$ = the set of all orthogonal matrices;

We are going to show that they are all submanifolds of $M_{n\times n}(\mathbb{R})$. Consider the determinant function $f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined as:

$$f(A) := \det(A).$$

Since f is a continuous function, the set $\text{GL}(n, \mathbb{R}) = f^{-1}(\mathbb{R}\backslash\{0\})$ is an open subset of $M_{n\times n}(\mathbb{R})$. Any (non-empty) open subset N of a smooth manifold M is a submanifold of M with $\dim N = \dim M$ (see Exercise 2.37).

For $\text{Sym}(n, \mathbb{R})$, we first label the coordinates of $\mathbb{R}^{\frac{n(n+1)}{2}}$ by (x_{ij}) where and $1 \leq i \leq j \leq n$. Then one can parametrize $\text{Sym}(n, \mathbb{R})$ by $F : \mathbb{R}^{\frac{n(n+1)}{2}} \to \text{Sym}(n, \mathbb{R})$ taking $(x_{ij})_{i\leq j} \in \mathbb{R}^{\frac{n(n+1)}{2}}$ to the matrix A with (i, j)-th entry x_{ij} when $i \leq j$, and x_{ji} when $i > j$. For instance, when $n = 2$, $\mathbb{R}^{\frac{n(n+1)}{2}}$ becomes $\mathbb{R}^3$ with coordinates labelled by (x_{11}, x_{12}, x_{22}). The parametrization F will take the point $(x_{11}, x_{12}, x_{22}) = (a, b, c) \in \mathbb{R}^3$ to the matrix:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \text{Sym}(n, \mathbb{R}).$$

Back to the general n, this F is a global parametrization and it makes $\text{Sym}(n, \mathbb{R})$ a smooth $\frac{n(n+1)}{2}$-manifold. To show that it is a submanifold of $M_{n\times n}(\mathbb{R})$, we computed the tangent map ι_* of the inclusion map $\iota : \text{Sym}(n, \mathbb{R}) \to M_{n\times n}(\mathbb{R})$:

$$\begin{aligned} \iota_* \left(\frac{\partial}{\partial x_{ij}} \right) = \frac{\partial \iota}{\partial x_{ij}} &= \frac{\partial}{\partial x_{ij}}(\iota \circ F) \\ &= \frac{1}{2}(E_{ij} + E_{ji}) \end{aligned}$$

where E_{ij} is the $n \times n$ matrix with 1 in the (i, j)-th entry, and 0 elsewhere. The tangent space $T\text{Sym}(n, \mathbb{R})$ at each point is spanned by the basis $\left\{ \frac{\partial}{\partial x_{ij}} \right\}_{1\leq i\leq j\leq n}$. Its image $\left\{ \frac{1}{2}(E_{ij} + E_{ji}) \right\}_{1\leq i\leq j\leq n}$ under the map ι_* is linearly independent (why?). This shows ι_* is injective, and hence $\text{Sym}(n, \mathbb{R})$ is a submanifold of $M_{n\times n}(\mathbb{R})$. The image of the inclusion map is the set of all symmetric matrices in $M_{n\times n}(\mathbb{R})$, hence we conclude that $T_{A_0}\text{Sym}(n, \mathbb{R}) \cong T\text{Sym}(n, \mathbb{R})$ for any $A_0 \in \text{Sym}(n, \mathbb{R})$.

The set of all orthogonal matrices $\text{O}(n)$ can be regarded as the level-set $\Phi^{-1}(I)$ of the following map:

$$\begin{aligned} \Phi : M_{n\times n}(\mathbb{R}) &\to \text{Sym}(n, \mathbb{R}) \\ A &\mapsto A^T A \end{aligned}$$

We are going to show that Φ is a submersion at every $A_0 \in \Phi^{-1}(I)$, we compute its tangent map:

$$(\Phi_*) \left(\frac{\partial}{\partial x_{ij}} \right) = \frac{\partial}{\partial x_{ij}} A^T A = E_{ij}^T A + A^T E_{ij} = (A^T E_{ij})^T + A^T E_{ij}.$$

From now on we denote $[A]_{ij}$ to be the (i, j)-th entry of any matrix A (Be cautious that E_{ij} without the square brackets is a matrix, not the (i, j)-th entry of E). In fact, any matrix A can be written as:

$$A = \sum_{i,j=1}^{n} [A]_{ij} E_{ij}.$$

At $A_0 \in \Phi^{-1}(I)$, we have $A_0^T A_0 = I$ and so for any symmetric matrix B, we have:

$$(\Phi_*)_{A_0}\left(\frac{\partial}{\partial x_{ij}}\right) = (A_0^T E_{ij})^T + A_0^T E_{ij}$$

$$\begin{aligned}
(\Phi_*)_{A_0}\left(\frac{1}{2}\sum_{i,j=1}^n [A_0 B]_{ij}\frac{\partial}{\partial x_{ij}}\right) &= \frac{1}{2}\sum_{i,j=1}^n [A_0 B]_{ij}\left((A_0^T E_{ij})^T + A_0^T E_{ij}\right) \\
&= \frac{1}{2}\left(A_0^T \sum_{i,j=1}^n [A_0 B]_{ij} E_{ij}\right)^T + \frac{1}{2}\left(A_0^T \sum_{i,j=1}^n [A_0 B]_{ij} E_{ij}\right) \\
&= \frac{1}{2}(A_0^T A_0 B)^T + \frac{1}{2}(A_0^T A_0 B) \\
&= \frac{1}{2}B^T + \frac{1}{2}B = B.
\end{aligned}$$

Therefore, $(\Phi_*)_{A_0}$ is surjective. This shows Φ_* is a submersion at every point $A_0 \in \Phi^{-1}(I)$. This shows $\mathrm{Sym}(n, \mathbb{R}) = \Phi^{-1}(I)$ is a submanifold of $M_{n\times n}(\mathbb{R})$ of dimension $\dim M_{n\times n}(\mathbb{R}) - \dim \mathrm{Sym}(n, \mathbb{R})$, which is $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

$\square$

Exercise 2.39. Show that the subset Σ of $\mathbb{R}^3$ defined by the two equations below is a 1-dimensional manifold:

$$\begin{aligned}
x^3 + y^3 + z^3 &= 1 \\
x + y + z &= 0
\end{aligned}$$

Exercise 2.40. Define

- $\mathrm{SL}(n, \mathbb{R})$ = the set of all $n \times n$ matrices with determinant 1
- $\mathfrak{sl}(n, \mathbb{R})$ = the set of all $n \times n$ skew-symmetric matrices (i.e. the set of matrices $A \in M_{n\times n}(\mathbb{R})$ such that $A^T = -A$).

Show that they are both submanifolds of $M_{n\times n}(\mathbb{R})$, and find their dimensions.

Exercise 2.41. Consider the map $\Phi : \mathbb{S}^3\backslash\{(0,0)\} \to \mathbb{CP}^1$ defined by:

$$\Phi(x_1, x_2, x_3, x_4) := [x_1 + ix_2 : x_3 + ix_4].$$

Show that $\Phi^{-1}([1:0])$ is a smooth manifold of (real) dimension 1, and show that $\Phi^{-1}([1:0])$ is diffeomorphic to a circle.

Chapter 3

Tensors and Differential Forms

> "In the beginning, God said, the four-dimensional divergence of an antisymmetric, second-rank tensor equals zero, and there was light."
>
> *Michio Kaku*

In Multivariable Calculus, we learned about gradient, curl and divergence of a vector field, and three important theorems associated to them, namely Green's, Stokes' and Divergence Theorems. In this and the next chapters, we will generalize these theorems to higher dimensional manifolds, and unify them into one single theorem (called the *Generalized Stokes' Theorem*). In order to carry out this generalization and unification, we need to introduce tensors and differential forms. The reasons of doing so are *many-folded*. We will explain it in detail. Meanwhile, one obvious reason is that the curl of a vector field is only defined in $\mathbb{R}^3$ since it uses the cross product. In this chapter, we will develop the language of *differential forms* which will be used in place of gradient, curl, divergence and all that in Multivariable Calculus.

3.1. Cotangent Spaces

3.1.1. Review of Linear Algebra: dual spaces. Let V be an n-dimensional real vector space, and $\mathcal{B} = \{e_1, \ldots, e_n\}$ be a basis for V. The set of all linear maps $T : V \to \mathbb{R}$ from V to the scalar field $\mathbb{R}$ (they are commonly called *linear functionals*) forms a vector space with dimension n. This space is called the *dual space* of V, denoted by V^*.

Associated to the basis $\mathcal{B} = \{e_i\}_{i=1}^n$ for V, there is a basis $\mathcal{B}^* = \{e^i\}_{i=1}^n$ for V^*:

$$e^i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The basis $\mathcal{B}^*$ for V^* (do Exericse 3.1 to verify it is indeed a basis) is called the *dual basis* of V^* with respect to $\mathcal{B}$.

Exercise 3.1. Given that V is a finite-dimensional real vector space, show that:

(a) V^* is a vector space

(b) $\dim V^* = \dim V$

(c) If $\mathcal{B} = \{e_i\}_{i=1}^n$ is a basis for V, then $\mathcal{B}^* := \{e^i\}_{i=1}^n$ is a basis for V^*.

Given $T \in V^*$ and that $T(e_i) = a_i$, verify that:

$$T = \sum_{i=1}^n a_i e^i.$$

3.1.2. Cotangent Spaces of Smooth Manifolds. Let M^n be a smooth manifold. Around $p \in M$, suppose there is a local parametrization $F(u_1, \ldots, u_n)$. Recall that the tangent space T_pM at p is defined as the span of partial differential operators $\left\{\dfrac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$. The *cotangent space* denoted by T_p^*M is defined as follows:

Definition 3.1 (Cotangent Spaces)**.** Let M^n be a smooth manifold. At every $p \in M$, the *cotangent space of M at p* is the dual space of the tangent space T_pM, i.e.:

$$T_p^*M = (T_pM)^*.$$

The elements in T_p^*M are called *cotangent vectors* of M at p.

Remark 3.2. Some authors use T_pM^* to denote the cotangent space. □

Associated to the basis $\mathcal{B}_p = \left\{\dfrac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$ of T_pM, there is a dual basis $\mathcal{B}_p^* = \{(du^1)_p, \ldots, (du^n)_p\}$ for T_p^*M, which is defined as follows:

$$(du^i)_p\left(\frac{\partial}{\partial u_j}(p)\right) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

As $(du^i)_p$ is a linear map from T_pM to $\mathbb{R}$, from the above definition we have:

$$(du^i)_p\left(\sum_{j=1}^n a_j \frac{\partial}{\partial u_j}(p)\right) = \sum_{j=1}^n a_j \delta_{ij} = a_i.$$

Occasionally (just for aesthetic purpose), $(du^i)_p$ can be denoted as $du^i\big|_p$. Moreover, whenever p is clear from the context (or not significant), we may simply write du^i and $\frac{\partial}{\partial u_i}$.

Note that both $\mathcal{B}_p$ and $\mathcal{B}_p^*$ depend on the choice of local coordinates. Suppose $(v_1, \ldots, v_n)$ is another local coordinates around p, then by chain rule we have:

$$\begin{aligned} \frac{\partial}{\partial v_j} &= \sum_{k=1}^n \frac{\partial u_k}{\partial v_j}\frac{\partial}{\partial u_k} \\ \frac{\partial}{\partial u_j} &= \sum_{k=1}^n \frac{\partial v_k}{\partial u_j}\frac{\partial}{\partial v_k}. \end{aligned}$$

We are going to express dv^i in terms of du^j's:

$$\begin{aligned} dv^i\left(\frac{\partial}{\partial u_j}\right) &= dv^i\left(\sum_{k=1}^n \frac{\partial v_k}{\partial u_j}\frac{\partial}{\partial v_k}\right) \\ &= \sum_{k=1}^n \frac{\partial v_k}{\partial u_j} dv^i\left(\frac{\partial}{\partial v_k}\right) \\ &= \sum_{k=1}^n \frac{\partial v_k}{\partial u_j}\delta_{ik} \\ &= \frac{\partial v_i}{\partial u_j}. \end{aligned}$$

This proves the transition formula for the cotangent basis:

$$dv^i = \sum_{k=1}^n \frac{\partial v_i}{\partial u_k} du^k. \tag{3.1}$$

Example 3.3. Consider $M = \mathbb{R}^2$ which can be parametrized by

$$\begin{aligned} F_1(x,y) &= (x,y) \\ F_2(r,\theta) &= (r\cos\theta, r\sin\theta). \end{aligned}$$

From (3.1), the conversion between $\{dr, d\theta\}$ and $\{dx, dy\}$ is given by:

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ &= (\cos\theta)\ dr - (r\sin\theta)\ d\theta \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \\ &= (\sin\theta)\ dr + (r\cos\theta)\ d\theta \end{aligned}$$

□

Exercise 3.2. Consider $M = \mathbb{R}^3$ which can be parametrized by:

$$\begin{aligned} F_1(x,y,z) &= (x,y,z) \\ F_2(r,\theta,z) &= (r\cos\theta, r\sin\theta, z) \\ F_3(\rho,\phi,\theta) &= (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \end{aligned}$$

Express $\{dx, dy, dz\}$ in terms of $\{dr, d\theta, dz\}$ and $\{d\rho, d\phi, d\theta\}$.

Exercise 3.3. Suppose $F(u_1, \ldots, u_n)$ and $G(v_1, \ldots, v_n)$ are two local parametrizations of a smooth manifold M. Let $\omega : M \to TM$ be a smooth differential 1-form such that on the overlap of local coordinates we have:

$$\omega = \sum_j a_j du^j = \sum_i b_i dv^i.$$

Find a conversion formula between a_j's and b_i's.

3.2. Tangent and Cotangent Bundles

3.2.1. Definitions. Let M be a smooth manifold. Loosely speaking, the *tangent bundle* (resp. *cotangent bundle*) are defined as the disjoint union of all tangent (resp. cotangent) spaces over the whole M. Precisely:

Definition 3.4 (Tangent and Cotangent Bundles)**.** Let M be a smooth manifold. The *tangent bundle*, denoted by TM, is defined to be:

$$TM = \bigcup_{p \in M} (\{p\} \times T_pM).$$

Elements in TM can be written as (p, V) where $V \in T_pM$.

Similarly, the *cotangent bundle*, denoted by T^*M, is defined to be:

$$T^*M = \bigcup_{p \in M} \left(\{p\} \times T_p^*M\right).$$

Elements in T^*M can be written as (p, ω) where $\omega \in T_p^*M$.

Suppose $F(u_1, \ldots, u_n) : \mathcal{U} \to M$ is a local parametrization of M, then a general element of TM can be written as:

$$\left(p, \sum_{i=1}^n V^i \frac{\partial}{\partial u_i}(p)\right)$$

and a general element of T^*M can be written as:

$$\left(p, \sum_{i=1}^n a_i \, du^i\big|_p\right).$$

We are going to explain why both TM and T^*M are smooth manifolds. The local parametrization $F(u_1, \ldots, u_n)$ of M induces a local parametrization $\widetilde{F}$ of TM defined by:

$$\begin{aligned} \widetilde{F} &: \mathcal{U} \times \mathbb{R}^n \to TM \\ (u_1, \ldots, u_n; V^1, \ldots, V^n) &\mapsto \left(F(u_1, \ldots, u_n), \sum_{i=1}^n V^i \left.\frac{\partial}{\partial u_i}\right|_{F(u_1,\ldots,u_n)}\right). \end{aligned} \tag{3.2}$$

Likewise, it also induces a local parametrization $\widetilde{\mathsf{F}}^*$ of T^*M defined by:

$$\begin{aligned} \widetilde{\mathsf{F}}^* &: \mathcal{U} \times \mathbb{R}^n \to T^*M \\ (u_1, \ldots, u_n; a_1, \ldots, a_n) &\mapsto \left(F(u_1, \ldots, u_n), \sum_{i=1}^n a_i \, du^i\big|_{F(u_1,\ldots,u_n)}\right). \end{aligned} \tag{3.3}$$

It suggests that TM and T^*M are both smooth manifolds of dimension $2 \dim M$. To do so, we need to verify compatible F's induce compatible $\widetilde{F}$ and $\widetilde{\mathsf{F}}^*$. Let's state this as a proposition and we leave the proof as an exercise for readers:

Proposition 3.5. *Let M^n be a smooth manifold. Suppose F and G are two overlapping smooth local parametrizations of M, then their induced local parametrizations $\widetilde{F}$ and $\widetilde{G}$ defined as in* (3.2) *on the tangent bundle TM are compatible, and also that $\widetilde{\mathsf{F}}^*$ and $\widetilde{\mathsf{G}}^*$ defined as in* (3.3) *on the cotangent bundle T^*M are also compatible.*

Corollary 3.6. *The tangent bundle TM and the cotangent bundle T^*M of a smooth manifold M are both smooth manifolds of dimension $2 \dim M$.*

Exercise 3.4. Prove Proposition 3.5.

Exercise 3.5. Show that the bundle map $\pi : TM \to M$ taking $(p, V) \in TM$ to $p \in M$ is a submersion. Show also that the set:

$$\Sigma_0 := \{(p, 0) \in TM : p \in M\}$$

is a submanifold of TM.

3.2.2. Vector Fields. Intuitively, a vector field V on a manifold M is an assignment of a vector to each point on M. Therefore, it can be regarded as a map $V : M \to TM$ such that $V(p) \in \{p\} \times T_pM$. Since we have shown that the tangent bundle TM is also a smooth manifold, one can also talk about C^k and *smooth* vector fields.

Definition 3.7 (Vector Fields of Class C^k)**.** Let M be a smooth manifold. A map $V : M \to TM$ is said to be a *vector field* if for each $p \in M$, we have $V(p) = (p, V_p) \in \{p\} \times T_pM$.

If V is of class C^k as a map between M and TM, then we say V is a C^k *vector field*. If V is of class C^∞, then we say V is a *smooth vector field*.

Remark 3.8. In the above definition, we used $V(p)$ to be denote the element (p, V_p) in TM, and V_p to denote the vector in T_pM. We will distinguish between them for a short while. After getting used to the notations, we will abuse the notations and use V_p and $V(p)$ interchangeably. □

Remark 3.9. Note that a vector field can also be defined *locally* on an open set $\mathcal{O} \subset M$. In such case we say V *is a* C^k *on* $\mathcal{O}$ if the map $V : \mathcal{O} \to TM$ is C^k. □

Under a local parametrization $F(u_1, \ldots, u_n) : \mathcal{U} \to M$ of M, a vector field $V : M \to TM$ can be expressed in terms of local coordinates as:

$$V(p) = \left(p,\ \sum_{i=1}^{n} V^i(p)\, \frac{\partial}{\partial u_i}(p)\right).$$

The functions $V^i : F(\mathcal{U}) \subset M \to \mathbb{R}$ are all locally defined and are commonly called the *components* of V with respect to local coordinates $(u_1, \ldots, u_n)$.

Let $\widetilde{F}(u_1, \ldots, u_n; V^1, \ldots, V^n)$ be the induced local parametrization of TM defined as in (3.2). Then, one can verify that:

$$\begin{aligned}\widetilde{F}^{-1} \circ V \circ F(u_1, \ldots, u_n) &= \widetilde{F}^{-1}\left(F(u_1, \ldots, u_n),\ \sum_{i=1}^{n} V^i(F(u_1, \ldots, u_n))\, \frac{\partial}{\partial u_i}\bigg|_{F(u_1,\ldots,u_n)}\right) \\ &= \big(u_1, \ldots, u_n; V^1(F(u_1, \ldots, u_n)), \ldots, V^n(F(u_1, \ldots, u_n))\big).\end{aligned}$$

Therefore, $\widetilde{F}^{-1} \circ V \circ F(u_1, \ldots, u_n)$ is smooth if and only if the components V^i's are all smooth. Similarly for class C^k. In short, a vector field V is smooth if and only if the components V^i in every its local expression:

$$V(p) = \left(p,\ \sum_{i=1}^{n} V^i(p)\, \frac{\partial}{\partial u_i}(p)\right)$$

are all smooth.

3.2.3. Differential 1-Forms. Differential 1-forms are the *dual* counterpart of vector fields. It is essentially an assignment of a cotangent vector to each point on M. Precisely:

Definition 3.10 (Differential 1-Forms of Class C^k). Let M be a smooth manifold. A map $\omega : M \to T^*M$ is said to be a *differential 1-form* if for each $p \in M$, we have $\omega(p) = (p, \omega_p) \in \{p\} \times T_p^*M$.

If ω is of class C^k as a map between M and T^*M, then we say ω is a C^k *differential 1-form*. If ω is of class C^∞, then we say ω is a *smooth differential 1-form*.

Remark 3.11. At this moment we use $\omega(p)$ to denote an element in $\{p\} \times T_pM$, and ω_p to denote an element in T_p^*M. We will abuse the notations later on and use them interchangeably, since such a distinction is unnecessary for many practical purposes. □

Remark 3.12. If a differential 1-form ω is locally defined on an open set $\mathcal{O} \subset M$, we may say ω is C^k on $\mathcal{O}$ to mean the map $\omega : \mathcal{O} \to T^*M$ is of class C^k. □

Under a local parametrization $F(u_1, \ldots, u_n) : \mathcal{U} \to M$ of M, a differential 1-form $\omega : M \to T^*M$ has a local coordinate expression given by:

$$\omega(p) = \left(p,\ \sum_{i=1}^{n} \omega_i(p)\ du^i\big|_p\right)$$

where $\omega_i : F(\mathcal{U}) \subset M \to \mathbb{R}$ are locally defined functions and are commonly called the *components* of ω with respect to local coordinates $(u_1, \ldots, u_n)$. Similarly to vector fields, one can show that ω is a C^∞ differential 1-form if and only if all ω_i's are smooth under any local coordinates in the atlas of M (see Exercise 3.6).

Exercise 3.6. Show that a differential 1-form ω is C^k on M if and only if all components ω_i's are C^k under any local coordinates in the atlas of M.

Example 3.13. The differential 1-form:

$$\omega = -\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

is smooth on $\mathbb{R}^2 \backslash \{0\}$, but is not smooth on $\mathbb{R}^2$. □

3.2.4. Push-Forward and Pull-Back. Consider a smooth map $\Phi : M \to N$ between two smooth manifolds M and N. The tangent map at p denoted by $(\Phi_*)_p$ is the induced map between tangent spaces T_pM and $T_{\Phi(p)}N$. Apart from calling it the tangent map, we often call Φ_* to be the *push-forward by* Φ, since Φ and Φ_* are both from the space M to the space N.

The push-forward map Φ_* takes tangent vectors on M to tangent vectors on N. There is another induced map Φ^*, called the *pull-back by* Φ, which is *loosely* defined as follows:

$$(\Phi^*\omega)(V) = \omega(\Phi_* V)$$

where ω is a cotangent vector and V is a tangent vector. In order for the above to make sense, V has to be a tangent vector on M (say at p). Then, Φ_*V is a tangent vector in $T_{\Phi(p)}N$. Therefore, $\Phi^*\omega$ needs to act on V and hence is a cotangent vector in T_p^*M; whereas ω acts on Φ_*V and so it should be a cotangent vector in $T_{\Phi(p)}^*N$. It is precisely defined as follows:

Definition 3.14 (Pull-Back of Cotangent Vectors)**.** Let $\Phi : M \to N$ be a smooth map between two smooth manifolds M and N. Given any cotangent vector $\omega_{\Phi(p)} \in T^*_{\Phi(p)}N$, the *pull-back of* ω *by* Φ *at* p denoted by $(\Phi^*\omega)_p$ is an element in T^*_pM and is defined to be the following linear functional on T_pM:

$$\begin{aligned}(\Phi^*\omega)_p &: T_pM \to \mathbb{R}\\ (\Phi^*\omega)_p (V_p) &:= \omega_{\Phi(p)} \left((\Phi_*)_p(V_p)\right)\end{aligned}$$

Therefore, one can think of Φ^* is a map which takes a *cotangent vector* $\omega_{\Phi(p)} \in T^*_{\Phi(p)}N$ to a cotangent vector $(\Phi^*\omega)_p$ on T^*_pM. As it is in the opposite direction to $\Phi : M \to N$, we call Φ^* the *pull-back* whereas Φ_* is called the *push-forward*.

Remark 3.15. In many situations, the points p and $\Phi(p)$ are clear from the context. Therefore, we often omit the subscripts p and $\Phi(p)$ when dealing with pull-backs and push-forwards. □

Example 3.16. Consider the map $\Phi : \mathbb{R} \to \mathbb{R}^2\backslash\{0\}$ defined by:

$$\Phi(\theta) = (\cos\theta, \sin\theta).$$

Let ω be the following 1-form on $\mathbb{R}^2\backslash\{0\}$:

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

First note that

$$\Phi_*\left(\frac{\partial}{\partial\theta}\right) = \frac{\partial\Phi}{\partial\theta} = \frac{\partial\overbrace{(\cos\theta)}^{x}}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial\overbrace{(\sin\theta)}^{y}}{\partial\theta}\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Therefore, one can compute:

$$\begin{aligned}(\Phi^*\omega)\left(\frac{\partial}{\partial\theta}\right) &= \omega\left(\Phi_*\left(\frac{\partial}{\partial\theta}\right)\right) = \omega\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\\ &= -y\left(\frac{-y}{x^2+y^2}\right) + x\left(\frac{x}{x^2+y^2}\right)\\ &= 1.\end{aligned}$$

Therefore, $\Phi^*\omega = d\theta$. □

Example 3.17. Let $M := \mathbb{R}^2\backslash\{(0,0)\}$ (equipped with polar (r,θ)-coordinates) and $N = \mathbb{R}^2$ (with (x,y)-coordinates), and define:

$$\begin{aligned}\Phi &: M \to N\\ \Phi(r,\theta) &:= (r\cos\theta, r\sin\theta)\end{aligned}$$

One can verify that:

$$\begin{aligned}\Phi_*\left(\frac{\partial}{\partial r}\right) &= \frac{\partial\Phi}{\partial r} = (\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}\\ \Phi_*\left(\frac{\partial}{\partial\theta}\right) &= \frac{\partial\Phi}{\partial\theta} = (-r\sin\theta)\frac{\partial}{\partial x} + (r\cos\theta)\frac{\partial}{\partial y}\\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\end{aligned}$$

Hence, we have:

$$\begin{aligned}(\Phi^* dx)\left(\frac{\partial}{\partial r}\right) &= dx\left(\Phi_*\left(\frac{\partial}{\partial r}\right)\right)\\ &= dx\left((\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}\right)\\ &= \cos\theta\\ (\Phi^* dx)\left(\frac{\partial}{\partial \theta}\right) &= dx\left(\Phi_*\left(\frac{\partial}{\partial \theta}\right)\right)\\ &= dx\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\\ &= -y = -r\sin\theta\end{aligned}$$

We conclude:

$$\Phi^* dx = \cos\theta\, dr - r\sin\theta\, d\theta.$$

□

Given a smooth map $\Phi : M^m \to N^n$, and local coordinates $(u_1, \ldots, u_m)$ of M around p and local coordinates $(v_1, \ldots, v_n)$ of N around $\Phi(p)$. One can compute a local expression for Φ^*:

$$\Phi^* dv^i = \sum_{j=1}^{n} \frac{\partial v_i}{\partial u_j} du^j \tag{3.4}$$

where $(v_1, \ldots, v_n)$ is regarded as a function of $(u_1, \ldots, u_m)$ via the map $\Phi : M \to N$.

Exercise 3.7. Prove (3.4).

Exercise 3.8. Express $\Phi^* dy$ in terms of dr and $d\theta$ in Example 3.17. Try computing it directly and then verify that (3.4) gives the same result.

Exercise 3.9. Denote (x_1, x_2) the coordinates for $\mathbb{R}^2$ and (y_1, y_2, y_3) the coordinates for $\mathbb{R}^3$. Define the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ by:

$$\Phi(x_1, x_2) = (x_1 x_2, x_2 x_3, x_3 x_1).$$

Compute $\Phi^*(dy^1)$, $\Phi^*(dy^2)$ and $\Phi^*(dy^3)$.

Exercise 3.10. Consider the map $\Phi : \mathbb{R}^3\backslash\{0\} \to \mathbb{RP}^2$ defined by:

$$\Phi(x, y, z) = [x : y : z].$$

Consider the local parametrization $F(u_1, u_2) = [1 : u_1 : u_2]$ of $\mathbb{RP}^2$. Compute $\Phi^*(du^1)$ and $\Phi^*(du^2)$.

3.2.5. Lie Derivatives. Derivatives of a function f or a vector field Y in Euclidean spaces along a curve $\gamma(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ are defined as follows:

$$\begin{aligned}D_{\gamma'(t)} f &:= \frac{d}{dt}(f \circ \gamma)(t) = \lim_{\delta\to 0} \frac{f(\gamma(t+\delta)) - f(\gamma(t))}{\delta}\\ D_{\gamma'(t)} Y &:= \frac{d}{dt}(Y \circ \gamma)(t) = \lim_{\delta\to 0} \frac{Y(\gamma(t+\delta)) - Y(\gamma(t))}{\delta}\end{aligned}$$

Now given any vector field X and any point $p \in \mathbb{R}^n$, if one can find a curve $\gamma(t) : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(t) = X(\gamma(t))$ for $t \in (-\varepsilon, \varepsilon)$ and $\gamma(0) = 0$, then it is well-defined to denote:

$$(D_X Y)_p := D_{\gamma'(t)} Y, \quad (D_X f)_p := D_{\gamma'(t)} f \qquad \text{at } t = 0$$

By the existence and uniqueness theorems of ODE, such a curve $\gamma(t)$ exists uniquely provided that the vector field X is C^1.

Furthermore, it can also be checked that if $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ are two curves with $\gamma_1(0) = \gamma_2(0) = p$ and with the same velocity vectors $\gamma_1'(0) = \gamma_2'(0)$ at p, then it is necessarily that $D_{\gamma_1'} f = D_{\gamma_2'} f$ and $D_{\gamma_1'} Y = D_{\gamma_2'} Y$ at p. Therefore, just the existence theorem of ODE is sufficient to argue that $D_X Y$ and $D_X f$ are well-defined.

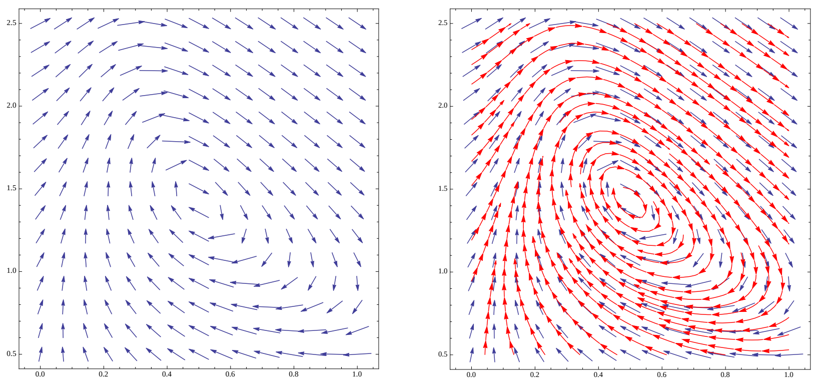
Exercise 3.11. Prove the above claim that $D_{\gamma_1'} f = D_{\gamma_2'} f$ and $D_{\gamma_1'} Y = D_{\gamma_2'} Y$ at p.

Remark 3.18. Consult any standard textbook about theory of ODEs for a proof of existence and uniqueness of the curve $\gamma(t)$ given any vector field X. Most standard textbook uses contraction mapping to prove existence, and Gronwall's inequality to prove uniqueness. □

Now let M be a smooth manifold, and X be a smooth vector field on M. Then, one can also extend the existence and uniqueness theorem of ODE to manifolds to prove that for any point $p \in M$, there exists a smooth curve $\gamma(t) : (-\varepsilon, \varepsilon) \to M$ on M with $\gamma(0) = p$ such that:

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)).$$

Recall that $\frac{d}{dt}\gamma(t)$ is defined as $\gamma_*\left(\frac{\partial}{\partial t}\right)$. This curve γ is called the *integral curve* of X passing through p. This extension can be justified by applying standard ODE theorems on the local coordinate chart covering p. Then one solves for the integral curve within this chart until the curve approaches the boundary of the chart (say at point q). Since the boundary of one chart must be the interior of another local coordinate chart, one can then continue solving for the integral curve starting from q.



(a) vector field X **(b)** integral curves of X

Figure 3.1. a vector field and its integral curves

Now one can still talk about integral curves $\gamma(t)$ given a vector field X on a manifold, so one can define $D_{\gamma'(t)} f$ and $D_X f$ in the same way as in $\mathbb{R}^n$ (as it makes perfect sense to talk about $f(\gamma(t+\delta)) - f(\gamma(t))$. However, it is not straight-forward how to generalize the definitions of $D_{\gamma'(t)} Y$ and $D_X Y$ where Y is a vector field on a manifold. The vectors $Y(\gamma(t+\delta))$ and $Y(\gamma(t))$ are at different based points, so one cannot make sense of $Y(\gamma(t+\delta)) - Y(\gamma(t))$.

One notion of differentiating a vector field by another one is called the *Lie derivatives*. The key idea is to push-forward tangent vectors in a natural way so that they become vectors at the same based point. Then, it makes sense to consider subtraction of vectors and also derivatives.

To begin with, we denote the integral curves using a map. First fix a vector field X on a manifold M. Then, given any point $p \in M$, as discussed before, one can find an integral curve $\gamma(t)$ so that $\gamma(0) = p$ and $\gamma'(t) = X(\gamma(t))$. We denote this curve $\gamma(t)$ by $\Phi_t(p)$, indicating that it depends on p. Now, for any fixed t, we can view $\Phi_t : M \to M$ as a map. One nice way to interpret this map is to regard $\Phi_t(p)$ as the point on M reached by flowing p along the vector field X for t unit time. As such, this map Φ_t is often called the *flow map*.

Exercise 3.12. Consider the unit sphere S^2 parametrized by spherical coordinates. (θ, φ), and the vector field $X = \dfrac{\partial}{\partial\theta}$. Describe the flow map Φ_t for this vector field, i.e. state how Φ_t maps the point with coordinates (θ, φ).

There are many meaningful purposes of this interpretation of integral curves. Standard theory of ODE shows Φ_t is smooth as long as X is a smooth vector field. Moreover, given $s, t \in \mathbb{R}$ and $p \in M$, we can regard $\Phi_s(\Phi_t(p))$ as the point obtained by flowing p along X first for t unit time, then for s unit time. Naturally, one would expect that the point obtained is exactly $\Phi_{s+t}(p)$. It is indeed true provided that X is independent of t.

Proposition 3.19. *Given any smooth vector field X on a smooth manifold M, and denote its flow map by $\Phi_t : M \to M$. Then, given any $t, s \in \mathbb{R}$ and $p \in M$, we have:*

$$\Phi_t(\Phi_s(p)) = \Phi_{t+s}(p), \qquad \text{or equivalently} \quad \Phi_t \circ \Phi_s = \Phi_{t+s}. \tag{3.5}$$

Consequently, for each fixed $t \in \mathbb{R}$, the flow map Φ_t is a diffeomorphism with inverse Φ_{-t}.

Proof. The proof is a direct consequence of the uniqueness theorem of ODE. Consider s as fixed and t as the variable, then $\Phi_t(\Phi_s(p))$ and $\Phi_{t+s}(p)$ can be regarded as curves on M. When $t = 0$, both curves pass through the point $\Phi_s(p)$. It remains to show that both curves satisfy the same ODE, then uniqueness theorem of ODE guarantee that the two curves must be the same. We leave the detail as an exercise for readers. □

Exercise 3.13. Complete the detail of the above proof that the curves $\{\Phi_t(\Phi_s(p))\}_{t\in\mathbb{R}}$ and $\{\Phi_{t+s}(p)\}_{t\in\mathbb{R}}$ both satisfy the same ODE.

Now we are ready to introduce Lie derivatives of vector fields. Given two vector fields X and Y, we want to develop a notion of differentiating Y along X, i.e. the rate of change of Y when moving along integral curves of X. Denote the flow map of X by Φ_t. Fix a point $p \in M$, we want to compare $Y_{\Phi_t(p)}$ with Y_p. However, they are at different base points, so we push-forward $Y_{\Phi_t(p)}$ so that it becomes a vector based at p. To do so, the natural way is to push it forward by the map Φ_{-t} as it maps tangent vectors at $\Phi_t(p)$ to tangent vectors at $\Phi_{-t}(\Phi_t(p)) = p$.

Definition 3.20 (Lie Derivatives of Vector Fields)**.** Let X and Y be smooth vector fields on a manifold M. We define the *Lie derivative* of X along Y by:

$$(\mathcal{L}_X Y)_p := \left.\frac{d}{dt}\right|_{t=0} (\Phi_{-t})_* \left(Y_{\Phi_t(p)}\right)$$

where Φ_t denotes the flow map of X.

It sounds like a very technical definition that is very difficult to compute! Fortunately, we will prove that $\mathcal{L}_X Y$ is simply the commutator $[X, Y]$, to be defined below. First recall that a vector field on a manifold is a differential operator acting on scalar functions f. After differentiating f by a vector field Y, we get another scalar function $Y(f)$. Then, we can differentiate $Y(f)$ by another vector field X and obtaining $X(Y(f))$ (which for simplicity we denote it by XYf. The commutator, or the Lie brackets, measure the difference between XYf and YXf:

Definition 3.21 (Lie Brackets). Given two vector fields X and Y on a manifold M, we define the *Lie brackets* $[X, Y]$ to be the vector field such that for any smooth function $f : M \to \mathbb{R}$, we have:

$$[X, Y]f := XYf - YXf.$$

Remark 3.22. Suppose under local coordinates $(u_1, \dots, u_n)$, the vector fields X and Y can be written as:

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial u_i} \qquad\qquad Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial u_i},$$

then $[X, Y]$ has following the local expression:

$$[X, Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial u_i} - Y^i \frac{\partial X^j}{\partial u_i} \right) \frac{\partial}{\partial u_j} \tag{3.6}$$

□

Exercise 3.14. Verify (3.6), i.e. show that for any smooth function $f : M \to \mathbb{R}$, we have:

$$XYf - YXf = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial u_i} - Y^i \frac{\partial X^j}{\partial u_i} \right) \frac{\partial f}{\partial u_j}$$

Exercise 3.15. Let $(u_1, \dots, u_n)$ be a local coordinate of a manifold M, and define

$$X = \frac{\partial}{\partial u_i} \qquad\qquad Y = \frac{\partial}{\partial u_j}$$

Then, what is $[X, Y]$?

Exercise 3.16. Let X, Y, Z be vector fields on a manifold M, and $\varphi : M \to \mathbb{R}$ be a smooth scalar function. Show that:

(i) $[X + Y, Z] = [X, Z] + [Y, Z]$
(ii) $[X, \varphi Y] = (X\varphi)Y + \varphi[X, Y]$

It appears that $[X, Y]$ is more like an algebraic operation whereas Lie derivative $\mathcal{L}_X Y$ is a differential operation. Amazingly, they are indeed equal!

Proposition 3.23. *Let X and Y be smooth vector fields on a manifold M. Then, we have:*

$$\mathcal{L}_X Y = [X, Y].$$

Proof. Denote Φ_t to be the flow map of the vector field X. Fix a point $p \in M$ and let $F(u_1, \dots, u_n) : \mathcal{U} \to M$ be a local parametrization covering p. In order to compute $\mathcal{L}_X Y$ at p, we may assume that t is sufficiently small so that $\Phi_t(p)$ is also covered by F. Denote that coordinate representation of Φ_t by:

$$F^{-1} \circ \Phi_t \circ F(u_1, \dots, u_n) = (v_t^1(u_1, \dots, u_n), \dots, v_t^n(u_1, \dots, u_n)).$$

In local coordinates, the flow map Φ_t is then related to X under the relation:

$$\underbrace{\sum_{i=1}^{n} \frac{\partial v_t^i}{\partial t} \frac{\partial}{\partial u^i}\bigg|_{\Phi_t(p)}}_{\frac{\partial \Phi_t}{\partial t}\big|_p} = \underbrace{\sum_{i=1}^{n} X^i(\Phi_t(p)) \frac{\partial}{\partial u^i}\bigg|_{\Phi_t(p)}}_{X_{\Phi_t(p)}}.$$

Equating the coefficients, we have

$$\frac{\partial v_t^i}{\partial t} = X^i(\Phi_t(p)) \tag{3.7}$$

for $i = 1, \ldots, n$. Recall that the Lie derivative $(\mathcal{L}_X Y)_p$ is the time derivative at $t = 0$ of $(\Phi_{-t})_*\big(Y(\Phi_t(p))\big)$, which is given by:

$$\begin{aligned}
(\Phi_{-t})_*\big(Y_{\Phi_t(p)}\big) &= (\Phi_t)_*^{-1}\left(\sum_{i=1}^{n} Y^i(\Phi_t(p)) \frac{\partial}{\partial u_i}(\Phi_t(p))\right) \\
&= \sum_{i=1}^{n} Y^i(\Phi_t(p)) \cdot (\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right).
\end{aligned}$$

It then follows that:

$$\begin{aligned}
(\mathcal{L}_X Y)_p &= \frac{\partial}{\partial t}\bigg|_{t=0} (\Phi_{-t})_*\big(Y_{\Phi_t(p)}\big) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial t}\bigg|_{t=0} Y^i(\Phi_t(p)) \cdot (\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right)\bigg|_{t=0} \\
&\quad + \sum_{i=1}^{n} Y^i(\Phi_t(p))\bigg|_{t=0} \cdot \frac{\partial}{\partial t}\bigg|_{t=0} (\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial t}\bigg|_{t=0} Y^i(\Phi_t(p)) \cdot \frac{\partial}{\partial u_i}(p) + \sum_{i=1}^{n} Y^i(p) \cdot \frac{\partial}{\partial t}\bigg|_{t=0} (\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right).
\end{aligned} \tag{3.8}$$

To compute $\frac{\partial}{\partial t} Y^i(\Phi_t(p))$, we use the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial t} Y^i(\Phi_t(p)) &= \sum_{j=1}^{n} \frac{\partial Y^i}{\partial u_j} \frac{\partial v_t^j}{\partial t} \\
&= \sum_{j=1}^{n} \frac{\partial Y^i}{\partial u_j} X^j(\Phi_t(p)) \qquad \text{(from (3.7))}.
\end{aligned} \tag{3.9}$$

The term $\frac{\partial}{\partial t}\big|_{t=0} (\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right)$ is a bit more tricky. We first define Z_i^k by the coefficients of:

$$\sum_{k=1}^{n} Z_i^k \frac{\partial}{\partial u_k}(p) = (\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right) \implies \sum_{k=1}^{n} Z_i^k \cdot (\Phi_t)_*\left(\frac{\partial}{\partial u_k}\bigg|_p\right) = \frac{\partial}{\partial u_i}\bigg|_{\Phi_t(p)}.$$

Recall that the coordinate representation of Φ_t is given by v_t^k's, so we get:

$$\sum_{j,k=1}^{n} Z_i^k \frac{\partial v_t^j}{\partial u_k} \frac{\partial}{\partial u_j}\bigg|_{\Phi_t(p)} = \frac{\partial}{\partial u_i}\bigg|_{\Phi_t(p)}.$$

By the linear independence of coordinate vectors, the coefficients Z_i^k satisfy:

$$\sum_{k=1}^{n} Z_i^k \frac{\partial v_t^j}{\partial u_k} = \delta_{ij}.$$

Differentiate both sides with respect to t, we get:

$$\sum_{k=1}^{n}\left(\frac{\partial Z_i^k}{\partial t}\frac{\partial v_t^j}{\partial u_k}+Z_i^k\frac{\partial X^j}{\partial u_k}\right)\bigg|_{t=0}=0$$

where we have used (3.7). At $t=0$, we have $v_t^j=u_j$, hence

$$\frac{\partial v_t^j}{\partial u_k}\bigg|_{t=0}=\delta_{jk} \implies \underbrace{Z_i^j(p)=\delta_{ij}}_{\text{from definition}} \text{ and } \sum_{k=1}^{n}\left(\frac{\partial Z_i^k}{\partial t}\delta_{jk}+\delta_{ik}\frac{\partial X^j}{\partial u_k}\right)\bigg|_{t=0}=0.$$

It implies that:

$$\frac{\partial Z_i^j}{\partial t}\bigg|_{t=0}=-\frac{\partial X^j}{\partial u_i}.$$

Then, we can compute that:

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{t=0}(\Phi_t)_*^{-1}\left(\frac{\partial}{\partial u_i}(\Phi_t(p))\right) &= \frac{\partial}{\partial t}\bigg|_{t=0}\sum_{k=1}^{n}Z_i^k\frac{\partial}{\partial u_k}(p)\\ &= -\sum_{k=1}^{n}\frac{\partial X^k}{\partial u_i}\frac{\partial}{\partial u_k}(p).\end{aligned} \tag{3.10}$$

Finally, substitute (3.9) and (3.10) back into (3.8), we get:

$$(\mathcal{L}_X Y)_p=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial Y^i}{\partial u_j}X^j(p)\frac{\partial}{\partial u_i}(p)-\sum_{i=1}^{n}\sum_{k=1}^{n}Y^i(p)\frac{\partial X^k}{\partial u_i}\frac{\partial}{\partial u_k}(p),$$

which is exactly $[X,Y]$ at p according to (3.6). □

Now we know the geometric meaning of two commuting vector fields X and Y, i.e. $[X,Y]=0$. According to Proposition 3.23, it is equivalent to saying $\mathcal{L}_X Y=0$ at any $p\in M$. Then by the definition of Lie derivatives, we can conclude that:

$$(\Phi_{-t})_*(Y_{\Phi_t(p)})=(\Phi_{-0})_*(Y_{\Phi_0(p)})=Y_p \quad \text{for any } t.$$

In other words, we have $Y_{\Phi_t(p)}=(\Phi_t)_*(Y_p)$ for any t, meaning that pushing Y at p forward by the flow map Φ_t of X will yield the vector field Y at the point $\Phi_t(p)$. This result can further extends to show the flow maps of X and Y commute:

Exercise 3.17. Let X and Y be two vector fields on M such that $[X,Y]=0$. Denote Φ_t and Ψ_t be the flow maps of X and Y respectively, show that for any $s,t\in\mathbb{R}$, we have:

$$\Phi_s\circ\Psi_t=\Psi_t\circ\Phi_s.$$

Sketch a diagram to illustrate its geometric meaning.

Lie derivatives on 1-forms can be defined similarly as on vector fields, except that we uses *pull-backs* instead of *push-forwards* this time.

Definition 3.24 (Lie Derivatives of Differential 1-Forms)**.** Let X and a smooth vector field and α be a smooth 1-form on a manifold M. Denote the flow map of X by Φ_t, then we define the *Lie derivative* of α at $p\in M$ along X by:

$$(\mathcal{L}_X\alpha)_p:=\frac{d}{dt}\bigg|_{t=0}(\Phi_t)^*\alpha_{\Phi_t(p)}.$$

One can compute similarly as in Proposition 3.23 that the Lie derivative of a 1-form $\alpha = \sum_{i=1}^{n} \alpha_i \, du^i$ can be locally expressed as:

$$\mathcal{L}_X \alpha = \sum_{i,j=1}^{n} \left(X^i \frac{\partial \alpha_j}{\partial u_i} + \alpha_i \frac{\partial X^i}{\partial u_j} \right) du^j \tag{3.11}$$

Exercise 3.18. Verify (3.11).

Exercise 3.19. Let X and Y be two vector fields on M, and α be a 1-form on M. Show that:

$$X\left(\alpha(Y)\right) = \left(\mathcal{L}_X \alpha\right)(Y) + \alpha\left(\mathcal{L}_X Y\right).$$

3.3. Tensor Products

In Differential Geometry, tensor products are often used to produce bilinear, or in general multilinear, maps between tangent and cotangent spaces. The first and second fundamental forms of a regular surface, the Riemann curvature, etc. can all be expressed using tensor notations.

3.3.1. Tensor Products in Vector Spaces. Given two vector spaces V and W, their dual spaces V^* and W^* are vector spaces of all linear functionals $T : V \to \mathbb{R}$ and $S : W \to \mathbb{R}$ respectively. Pick two linear functionals $T \in V^*$ and $S \in W^*$, their tensor product $T \otimes S$ is a map from $V \times W$ to $\mathbb{R}$ defined by:

$$\begin{aligned} T \otimes S &: V \times W \to \mathbb{R} \\ (T \otimes S)(X, Y) &:= T(X)\, S(Y) \end{aligned}$$

It is easy to verify that $T \otimes S$ is bilinear, meaning that it is linear at each slot:

$$\begin{aligned} &(T \otimes S)\left(a_1 X_1 + a_2 X_2,\, b_1 Y_1 + b_2 Y_2\right) \\ &= a_1 b_1 (T \otimes S)(X_1, Y_1) + a_2 b_1 (T \otimes S)(X_2, Y_1) \\ &\quad + a_1 b_2 (T \otimes S)(X_1, Y_2) + a_2 b_2 (T \otimes S)(X_1, Y_2) \end{aligned}$$

Given three vector spaces U, V, W, and linear functionals $T_U \in U^*$, $T_V \in V^*$ and $T_W \in W^*$, one can define a triple tensor product $T_U \otimes (T_V \otimes T_W)$ by:

$$\begin{aligned} T_U \otimes (T_V \otimes T_W) &: U \times (V \times W) \to \mathbb{R} \\ (T_U \otimes (T_V \otimes T_W))(X, Y, Z) &:= T_U(X)\, (T_V \otimes T_W)(Y, Z) \\ &= T_U(X)\, T_V(Y)\, T_W(Z) \end{aligned}$$

One check easily that $(T_U \otimes T_V) \otimes T_W = T_U \otimes (T_V \otimes T_W)$. Since there is no ambiguity, we may simply write $T_U \otimes T_V \otimes T_W$. Inductively, given finitely many vector spaces $V_1, \ldots, V_k$, and linear functions $T_i \in V_i^*$, we can define the tensor product $T_1 \otimes \cdots \otimes T_k$ as a k-linear map by:

$$\begin{aligned} T_1 \otimes \cdots \otimes T_k &: V_1 \times \cdots \times V_k \to \mathbb{R} \\ (T_1 \otimes \cdots \otimes T_k)(X_1, \ldots, X_k) &:= T_1(X_1) \cdots T_k(X_k) \end{aligned}$$

Given two tensor products $T_1 \otimes S_1 : V \times W \to \mathbb{R}$ and $T_2 \otimes S_2 : V \times W \to \mathbb{R}$, one can form a linear combination of them:

$$\begin{aligned} \alpha_1(T_1 \otimes S_1) + \alpha_2(T_2 \otimes S_2) &: V \times W \to \mathbb{R} \\ (\alpha_1(T_1 \otimes S_1) + \alpha_2(T_2 \otimes S_2))(X, Y) &:= \alpha_1(T_1 \otimes S_1)(X, Y) + \alpha_2(T_2 \otimes S_2)(X, Y) \end{aligned}$$

The tensor products $T \otimes S$ with $T \in V^*$ and $S \in W^*$ generate a vector space. We denote this vector space by:

$$V^* \otimes W^* := \text{span}\{T \otimes S : T \in V^* \text{ and } S \in W^*\}.$$

Exercise 3.20. Verify that $\alpha\,(T \otimes S) = (\alpha T) \otimes S = T \otimes (\alpha S)$. Therefore, we can simply write $\alpha T \otimes S$.

Exercise 3.21. Show that the tensor product is bilinear in a sense that:

$$T \otimes (\alpha_1 S_1 + \alpha_2 S_2) = \alpha_1 T \otimes S_1 + \alpha_2 T \otimes S_2$$

and similar for the T slot.

Let's take the dual basis as an example to showcase the use of tensor products. Consider a vector space V with a basis $\{e_i\}_{i=1}^n$. Let $\{e^i\}_{i=1}^n$ be its dual basis for V^*. Then, one can check that:

$$\begin{aligned}(e^i \otimes e^j)(e_k, e_l) &= e^i(e_k)\, e^k(e_l) \\ &= \delta_{ik}\, \delta_{jl} \\ &= \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Generally, the sum $\displaystyle\sum_{i,j=1}^n A_{ij} e^i \otimes e^j$ will act on vectors in V by:

$$\begin{aligned}&\left(\sum_{i,j=1}^n A_{ij} e^i \otimes e^j\right)\left(\sum_{k=1}^n \alpha_k e_k, \sum_{l=1}^n \beta_l e_l\right) \\ &= \sum_{i,j,k,l=1}^n A_{ij}\alpha_k\beta_l (e^i \otimes e^j)(e_k, e_l) = \sum_{i,j,k,l=1}^n A_{ij}\alpha_k\beta_l\delta_{ik}\delta_{jl} = \sum_{k,l=1}^n A_{kl}\alpha_k\beta_l\end{aligned}$$

In other words, the sum of tensor products $\displaystyle\sum_{i,j=1}^n A_{ij} e^i \otimes e^j$ is the inner product on V represented by the matrix $[A_{kl}]$ with respect to the basis $\{e_i\}_{i=1}^n$ of V. For example, when $A_{kl} = \delta_{kl}$, then $\displaystyle\sum_{i,j=1}^n A_{ij} e^i \otimes e^j = \sum_{i=1}^n e^i \otimes e^i$. It is the usual dot product on V.

Exercise 3.22. Show that $\{e^i \otimes e^j\}_{i,j=1}^n$ is a basis for $V^* \otimes V^*$. What is the dimension of $V^* \otimes V^*$?

Exercise 3.23. Suppose $\dim V = 2$. Let $\omega \in V^* \otimes V^*$ satisfy:

$$\begin{aligned}\omega(e_1, e_1) &= 0 & \omega(e_1, e_2) &= 3 \\ \omega(e_2, e_1) &= -3 & \omega(e_2, e_2) &= 0\end{aligned}$$

Express ω in terms of e^i's.

To describe linear or multilinear map between two vector spaces V and W (where W is not necessarily the one-dimensional space $\mathbb{R}$), one can also use tensor products. Given a linear functional $f \in V^*$ and a vector $w \in W$, we can form a tensor $f \otimes w$, which is regarded as a linear map $f \otimes w : V \to W$ defined by:

$$(f \otimes w)(v) := f(v)w.$$

Let $\{e_i\}$ be a basis for V, and $\{F_j\}$ be a basis for W. Any linear map $T : V \to W$ can be expressed in terms of these bases. Suppose:

$$T(e_i) = \sum_j A_i^j F_j.$$

Then, we claim that T can be expressed using the following tensor notations:

$$T = \sum_{i,j} A_i^j e^i \otimes F_j$$

Let's verify this. Note that a linear map is determined by its action on the basis $\{e_i\}$ for V. It suffices to show:

$$\left(\sum_{i,j} A_i^j e^i \otimes F_j\right)(e_k) = T(e_k).$$

Using the fact that:

$$(e^i \otimes F_j)(e_k) = e^i(e_k)F_j = \delta_{ik}F_j,$$

one can compute:

$$\begin{aligned}\left(\sum_{i,j} A_i^j e^i \otimes F_j\right)(e_k) &= \sum_{i,j} A_i^j (e^i \otimes F_j)(e_k)\\ &= \sum_{i,j} A_i^j \delta_{ik} F_j = \sum_j A_k^j F_j = T(e_k)\end{aligned}$$

as desired.

Generally, if $T_1, \ldots, T_k \in V^*$ and $X \in V$, then

$$T_1 \otimes \cdots \otimes T_k \otimes X$$

is regarded to be a k-linear map from $V \times \ldots \times V$ to V, defined by:

$$T_1 \otimes \cdots \otimes T_k \otimes X : \underbrace{V \times \ldots \times V}_{k} \to V$$

$$(T_1 \otimes \cdots \otimes T_k \otimes X)(Y_1, \ldots, Y_k) := T_1(Y_1) \cdots T_k(Y_k)\, X$$

Example 3.25. One can write the cross-product in $\mathbb{R}^3$ using tensor notations. Think of the cross product as a bilinear map $\omega : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ that takes two input vectors u and v, and outputs the vector $u \times$ v. Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbb{R}^3$ (i.e. $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$). Then one can write:

$$\begin{aligned}\omega = e^1 \otimes e^2 \otimes e_3 - e^2 \otimes e^1 \otimes e_3\\ + e^2 \otimes e^3 \otimes e_1 - e^3 \otimes e^2 \otimes e_1\\ + e^3 \otimes e^1 \otimes e_2 - e^1 \otimes e^3 \otimes e_2\end{aligned}$$

One can check that, for instance, $\omega(e_1, e_2) = e_3$, which is exactly $e_1 \times e_2 = e_3$. □

3.3.2. Tensor Products on Smooth Manifolds. In the previous subsection we take tensor products on a general abstract vector space V. In this course, we will mostly deal with the case when V is the tangent or cotangent space of a smooth manifold M.

Recall that if $F(u_1, \ldots, u_n)$ is a local parametrization of M, then there is a local coordinate basis $\left\{\frac{\partial}{\partial u_i}(p)\right\}_{j=1}^n$ for the tangent space T_pM at every $p \in M$ covered by F. The cotangent space T_p^*M has a dual basis $\left\{du^j\big|_p\right\}_{j=1}^n$ defined by $du_j\left(\frac{\partial}{\partial u_i}\right) = \delta_{ij}$ at every $p \in M$.

Then, one can take tensor products of du^i's and $\frac{\partial}{\partial u_i}$'s to express multilinear maps between tangent and cotangent spaces. For instance, the tensor product $g = \sum_{i,j=1}^n g_{ij}\, du^i \otimes du^j$, where g_{ij}'s are scalar functions, means that it is a bilinear map at each point $p \in M$ such that:

$$g(X, Y) = \sum_{i,j=1}^n g_{ij}(du^i \otimes du^j)(X, Y) = \sum_{i,j=1}^n g_{ij}\, du^i(X)\, du^j(Y)$$

for any vector field $X, Y \in TM$. In particular, we have:

$$g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = g_{ij}.$$

We can also express multilinear maps from $T_pM \times T_pM \times T_pM$ to T_pM. For instance, we let:

$$\mathrm{Rm} = \sum_{i,j,k,l=1}^{n} R_{ijk}^{l} du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u_l}.$$

Then, Rm is a mutlilinear map at each $p \in M$ such that:

$$\mathrm{Rm}(X, Y, Z) = \sum_{i,j,k,l=1}^{n} R_{ijk}^{l} du^i(X)\, du^j(Y)\, du^k(Z)\, \frac{\partial}{\partial u_l}.$$

It is a trilinear map such that:

$$\mathrm{Rm}\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = \sum_{l=1}^{n} R_{ijk}^{l} \frac{\partial}{\partial u_l}.$$

We call g a $(2, 0)$-tensor (meaning that it maps two vectors to a scalar), and Rm a $(3, 1)$-tensor (meaning that it maps three vectors to one vector). In general, we can also define $(k, 0)$-tensor ω on M which has the general form:

$$\omega_p = \sum_{i_1,\ldots,i_k=1}^{n} \omega_{i_1 i_2 \cdots i_k}(p)\, du^{i_1}\big|_p \otimes \cdots \otimes du^{i_k}\big|_p$$

Here $\omega_{i_1 i_2 \cdots i_k}$'s are scalar functions. This tensor maps the tangent vectors $\left(\frac{\partial}{\partial u_{i_1}}, \ldots, \frac{\partial}{\partial u_{i_k}}\right)$ to the scalar $\omega_{i_1 i_2 \ldots i_k}$ at the corresponding point.

Like the Rm-tensor, we can also generally define $(k, 1)$-tensor Ω on M which has the general form:

$$\Omega_p = \sum_{i_1,\ldots,i_k,j=1}^{n} \Omega_{i_1 i_2 \cdots i_k}^{j}(p)\, du^{i_1}\big|_p \otimes \cdots \otimes du^{i_k}\big|_p \otimes \frac{\partial}{\partial u_j}(p)$$

where $\Omega_{i_1 i_2 \ldots i_k}^{j}$'s are scalar functions. This tensor maps the tangent vectors $\left(\frac{\partial}{\partial u_{i_1}}, \ldots, \frac{\partial}{\partial u_{i_k}}\right)$ to the tangent vector $\sum_j \Omega_{i_1 i_2 \ldots i_k}^{j} \frac{\partial}{\partial u_j}$ at the corresponding point.

Note that these g_{ij}, R_{ijk}^{l}, $\omega_{i_1 i_2 \cdots i_k}$ and $\Omega_{i_1 i_2 \ldots i_k}^{j}$ are scalar functions locally defined on the open set covered by the local parametrization F, so we can talk about whether they are smooth or not:

Definition 3.26 (Smooth Tensors on Manifolds)**.** A *smooth* $(k,0)$-*tensor* ω on M is a k-linear map $\omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \to \mathbb{R}$ at each $p \in M$ such that under any local parametrization $F(u_1, \ldots, u_n) : \mathcal{U} \to M$, it can be written in the form:

$$\omega_p = \sum_{i_1,\ldots,i_k=1}^{n} \omega_{i_1 i_2 \cdots i_k}(p)\, du^{i_1}\big|_p \otimes \cdots \otimes du^{i_k}\big|_p$$

where $\omega_{i_1 i_2 \ldots i_k}$'s are smooth scalar functions locally defined on $F(\mathcal{U})$.

A *smooth* $(k,1)$-*tensor* Ω on M is a k-linear map $\Omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \to T_pM$ at each $p \in M$ such that under any local parametrization $F(u_1, \ldots, u_n) : \mathcal{U} \to M$, it can be written in the form:

$$\Omega_p = \sum_{i_1,\ldots,i_k,j=1}^{n} \Omega^j_{i_1 i_2 \cdots i_k}(p)\, du^{i_1}\big|_p \otimes \cdots \otimes du^{i_k}\big|_p \otimes \frac{\partial}{\partial u_j}(p)$$

where $\Omega^j_{i_1 i_2 \ldots i_k}$'s are smooth scalar functions locally defined on $F(\mathcal{U})$.

Remark 3.27. Since T_pM is finite dimensional, from Linear Algebra we know $(T_pM)^{**}$ is isomorphic to T_pM. Therefore, a tangent vector $\frac{\partial}{\partial u_i}(p)$ can be regarded as a linear functional on cotangent vectors in T_p^*M, meaning that:

$$\left.\frac{\partial}{\partial u_i}\right|_p \left(du^j\big|_p \right) = \delta_{ij}.$$

Under this interpretation, one can also view a $(k,1)$-tensor Ω as a $(k+1)$-linear map $\Omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \times T_p^*M \to \mathbb{R}$, which maps $\left(du^{i_1}, \ldots, du^{i_k}, \frac{\partial}{\partial u_j}\right)$ to $\Omega^j_{i_1 i_2 \ldots i_k}$. However, we will not view a $(k,1)$-tensor this way in this course.

Generally, we can also talk about (k,s)-tensors, which is a $(k+s)$-linear map $\Omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \times \underbrace{T_p^*M \times \ldots \times T_p^*M}_{s} \to \mathbb{R}$ taking $\left(du^{i_1}, \ldots, du^{i_k}, \frac{\partial}{\partial u_{j_1}}, \ldots, \frac{\partial}{\partial u_{j_s}}\right)$ to a scalar. However, we seldom deal with these tensors in this course. □

Exercise 3.24. Let M be a smooth manifold with local coordinates (u_1, u_2). Consider the tensor products:

$$T_1 = du^1 \otimes du^2 \quad \text{and} \quad T_2 = du^1 \otimes \frac{\partial}{\partial u_2}.$$

Which of the following is well-defined?

(a) $T_1\left(\frac{\partial}{\partial u_1}\right)$

(b) $T_2\left(\frac{\partial}{\partial u_1}\right)$

(c) $T_1\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$

(d) $T_2\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$

Exercise 3.25. let M be a smooth manifold with local coordinates (u_1, u_2). The linear map $T : T_pM \to T_pM$ satisfies:

$$\begin{aligned}
T\left(\frac{\partial}{\partial u_1}\right) &= \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} \\
T\left(\frac{\partial}{\partial u_2}\right) &= \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}.
\end{aligned}$$

Express T using tensor products.

One advantage of using tensor notations, instead of using matrices, to denote a multilinear map between tangent or cotangent spaces is that one can figure out the conversion rule between local coordinate systems easily (when compared to using matrices)

Example 3.28. Consider the extended complex plane $M := \mathbb{C} \cup \{\infty\}$ defined in Example 2.12. We cover M by two local parametrizations:

$$\begin{aligned}
F_1 &: \mathbb{R}^2 \to \mathbb{C} \subset M & F_2 &: \mathbb{R}^2 \to (\mathbb{C}\backslash\{0\}) \cup \{\infty\} \subset M \\
(x, y) &\mapsto x + yi & (u, v) &\mapsto \frac{1}{u + vi}
\end{aligned}$$

The transition maps on the overlap are given by:

$$\begin{aligned}
(u, v) = F_2^{-1} \circ F_1(x, y) &= \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right) \\
(x, y) = F_1^{-1} \circ F_2(u, v) &= \left(\frac{u}{u^2 + v^2}, -\frac{v}{u^2 + v^2}\right)
\end{aligned}$$

Consider the $(2, 0)$-tensor ω defined using local coordinates (x, y) by:

$$\omega = e^{-(x^2+y^2)}\, dx \otimes dy.$$

Using the chain rule, we can express dx and dy in terms of du and dv:

$$\begin{aligned}
dx = d\left(\frac{u}{u^2 + v^2}\right) &= \frac{(u^2 + v^2)\, du - u(2u\, du + 2v\, dv)}{(u^2 + v^2)^2} \\
&= \frac{v^2 - u^2}{(u^2 + v^2)^2} du - \frac{2uv}{(u^2 + v^2)^2} dv \\
dy = -d\left(\frac{v}{u^2 + v^2}\right) &= -\frac{(u^2 + v^2)dv - v(2u\, du + 2v\, dv)}{(u^2 + v^2)^2} \\
&= -\frac{2uv}{(u^2 + v^2)^2} du + \frac{v^2 - u^2}{(u^2 + v^2)^2} dv
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
dx \otimes dy = &\frac{2uv(u^2 - v^2)}{(u^2 + v^2)^4} du \otimes du + \frac{(u^2 - v^2)^2}{(u^2 + v^2)^4} du \otimes dv \\
&+ \frac{4u^2v^2}{(u^2 + v^2)^4} dv \otimes du + \frac{2uv(u^2 - v^2)}{(u^2 + v^2)^4} dv \otimes dv
\end{aligned}$$

Recall that $\omega = e^{-(x^2+y^2)}\, dx \otimes dy$, and in terms of (u, v), we have:

$$e^{-(x^2+y^2)} = e^{-\frac{1}{u^2+v^2}}.$$

Hence, in terms of (u, v), ω is expressed as:

$$\omega = e^{-\frac{1}{u^2+v^2}} \left\{ \frac{2uv(u^2-v^2)}{(u^2+v^2)^4} du \otimes du + \frac{(u^2-v^2)^2}{(u^2+v^2)^4} du \otimes dv \right.$$
$$\left. + \frac{4u^2v^2}{(u^2+v^2)^4} dv \otimes du + \frac{2uv(u^2-v^2)}{(u^2+v^2)^4} dv \otimes dv \right\}$$

□

Exercise 3.26. Consider the extended complex plane $\mathbb{C} \cup \{\infty\}$ as in Example 3.28, and the $(1, 1)$-tensor of the form:

$$\Omega = e^{-(x^2+y^2)}\, dx \otimes \frac{\partial}{\partial y}.$$

Express Ω in terms of (u, v).

Generally, if $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are two overlapping local coordinates on a smooth manifold M, then given a $(2, 0)$-tensor:

$$g = \sum_{i,j} g_{ij} du^i \otimes du^j$$

written using the u_i's coordinates, one can convert it to v_α's coordinates by the chain rule:

$$g = \sum_{i,j} g_{ij} du^i \otimes du^j = \sum_{i,j} g_{ij} \left(\sum_\alpha \frac{\partial u_i}{\partial v_\alpha} dv^\alpha \right) \otimes \left(\sum_\beta \frac{\partial u_j}{\partial v_\beta} dv^\beta \right)$$
$$= \sum_{\alpha,\beta} \left(\sum_{i,j} g_{ij} \frac{\partial u_i}{\partial v_\alpha} \frac{\partial u_j}{\partial v_\beta} \right) dv^\alpha \otimes dv^\beta$$

Exercise 3.27. Given that u_i's and v_α's are overlapping local coordinates of a smooth manifold M. Using these coordinates, one can express the following $(3, 1)$-tensor in two ways:

$$\mathrm{Rm} = \sum_{i,j,k,l} R^l_{ijk} du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u_l} = \sum_{\alpha,\beta,\gamma,\eta} \widetilde{R}^\eta_{\alpha\beta\gamma} dv^\alpha \otimes dv^\beta \otimes dv^\gamma \otimes \frac{\partial}{\partial v_\eta}$$

Express R^l_{ijk} in terms of $R^\eta_{\alpha\beta\gamma}$'s.

Exercise 3.28. Given that u_i's and v_α's are overlapping local coordinates of a smooth manifold M. Suppose g and h are two $(2, 0)$-tensors expressed in terms of local coordinates as:

$$g = \sum_{i,j} g_{ij}\, du^i \otimes du^j = \sum_{\alpha,\beta} \widetilde{g}_{\alpha\beta} dv^\alpha \otimes dv^\beta$$
$$h = \sum_{i,j} h_{ij}\, du^i \otimes du^j = \sum_{\alpha,\beta} \widetilde{h}_{\alpha\beta} dv^\alpha \otimes dv^\beta.$$

Let G be the matrix with g_{ij} as its (i, j)-th entry, and let g^{ij} be the (i, j)-th entry of G^{-1}. Similarly, define $\widetilde{g}^{\alpha\beta}$ to be the inverse of $\widetilde{g}_{\alpha\beta}$. Show that:

$$\sum_{i,j} g^{ik} h_{kj}\, du^i \otimes du^j = \sum_{\alpha,\beta} \widetilde{g}^{\alpha\gamma} \widetilde{h}_{\gamma\beta}\, dv^\alpha \otimes dv^\beta.$$

3.4. Wedge Products

Recall that in Multivariable Calculus, the cross product plays a crucial role in many aspects. It is a bilinear map which takes two vectors to one vectors, and so it is a $(2,1)$-tensor on $\mathbb{R}^3$.

The determinant is another important quantity in Multivariable Calculus and Linear Algebra. Using tensor languages, an $n \times n$ determinant can be regarded as a n-linear map taking n vectors in $\mathbb{R}^n$ to a scalar. For instance, for the 2×2 case, one can view:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a bilinear map taking column vectors $(a,c)^T$ and $(b,d)^T$ in $\mathbb{R}^2$ to a number $ad - bc$. Therefore, it is a $(2,0)$-tensor on $\mathbb{R}^2$; and generally for $n \times n$, the determinant is an $(n,0)$-tensor on $\mathbb{R}^n$.

Both the cross product in $\mathbb{R}^3$ and determinant ($n \times n$ in general) are *alternating*, in a sense that interchanging any pair of inputs will give a negative sign for the output. For the cross product, we have $\mathsf{a} \times \mathsf{b} = -\mathsf{b} \times \mathsf{a}$; and for the determinant, switching any pair of columns will give a negative sign:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}.$$

In the previous section we have seen how to express k-linear maps over tangent vectors using tensor notations. To deal with the above alternating tensors, it is more elegant and concise to use *alternating tensors*, or *wedge products* that we are going to learn in this section.

3.4.1. Wedge Product on Vector Spaces. Let's start from the easiest case. Suppose V is a finite dimensional vector space and V^* is the dual space of V. Given any two elements $T, S \in V^*$, the tensor product $T \otimes S$ is a map given by:

$$(T \otimes S)(X,Y) = T(X)\,S(Y)$$

for any $X, Y \in V$. The *wedge product* $T \wedge S$, where $T, S \in V^*$, is a bilinear map defined by:

$$T \wedge S := T \otimes S - S \otimes T$$

meaning that for any $X, Y \in V$, we have:

$$\begin{aligned}(T \wedge S)(X,Y) &= (T \otimes S)(X,Y) - (S \otimes T)(X,Y) \\ &= T(X)\,S(Y) - S(X)\,T(Y)\end{aligned}$$

It is easy to note that $T \wedge S = -S \wedge T$.

Take the cross product in $\mathbb{R}^3$ as an example. Write the cross product as a bilinear map $\omega(\mathsf{a}, \mathsf{b}) := \mathsf{a} \times \mathsf{b}$. It is a $(2,1)$-tensor on $\mathbb{R}^3$ which can be represented as:

$$\begin{aligned}\omega = e^1 \otimes e^2 \otimes e_3 &- e^2 \otimes e^1 \otimes e_3 \\ + e^2 \otimes e^3 \otimes e_1 &- e^3 \otimes e^2 \otimes e_1 \\ + e^3 \otimes e^1 \otimes e_2 &- e^1 \otimes e^3 \otimes e_2\end{aligned}$$

Now using the wedge product notations, we can express ω as:

$$\omega = (e^1 \wedge e^2) \otimes e_3 + (e^2 \wedge e^3) \otimes e_1 + (e^3 \wedge e^1) \otimes e_2$$

which is a half shorter than using tensor products alone.

Now given three elements $T_1, T_2, T_3 \in V^*$, one can also form a triple wedge product $T_1 \wedge T_2 \wedge T_3$ which is a $(3,0)$-tensor so that switching any pair of T_i and T_j (with $i \neq j$) will give a negative sign. For instance:

$$T_1 \wedge T_2 \wedge T_3 = -T_2 \wedge T_1 \wedge T_3 \quad \text{and} \quad T_1 \wedge T_2 \wedge T_3 = -T_3 \wedge T_2 \wedge T_1.$$

It can be defined in a precise way as:

$$\begin{aligned} T_1 \wedge T_2 \wedge T_3 := T_1 \otimes T_2 \otimes T_3 &- T_1 \otimes T_3 \otimes T_2 \\ + T_2 \otimes T_3 \otimes T_1 &- T_2 \otimes T_1 \otimes T_3 \\ + T_3 \otimes T_1 \otimes T_2 &- T_3 \otimes T_2 \otimes T_1 \end{aligned}$$

Exercise 3.29. Verify that the above definition of triple wedge product will result in $T_1 \wedge T_2 \wedge T_3 = -T_3 \wedge T_2 \wedge T_1$.

Exercise 3.30. Propose the definition of $T_1 \wedge T_2 \wedge T_3 \wedge T_4$. Do this exercise before reading ahead.

We can also define $T_1 \wedge T_2 \wedge T_3$ in a more systematic (yet equivalent) way using symmetric groups. Let S_3 be the permutation group of $\{1,2,3\}$. An element $\sigma \in S_3$ is a bijective map $\sigma : \{1,2,3\} \to \{1,2,3\}$. For instance, a map satisfying $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$ is an example of an element in S_3. We can express this σ by:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{or simply:} \quad (123)$$

A map $\tau \in S_3$ given by $\tau(1) = 2$, $\tau(2) = 1$ and $\tau(3) = 3$ can be expressed as:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \text{or simply:} \quad (12)$$

This element, which switches two of the elements in $\{1,2,3\}$ and fixes the other one, is called a *transposition*.

Multiplication of two elements $\sigma_1, \sigma_2 \in S_3$ is defined by composition. Precisely, $\sigma_1\sigma_2$ is the composition $\sigma_1 \circ \sigma_2$. Note that this means the elements $\{1,2,3\}$ are input into σ_2 first, and then into σ_1. In general, $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. One can check easily that, for instance, we have:

$$\begin{aligned} (12)(23) &= (123) \\ (23)(12) &= (132) \end{aligned}$$

Elements in the permutation group S_n of n elements (usually denoted by $\{1, 2, \ldots, n\}$) can be represented and mutliplied in a similar way.

Exercise 3.31. Convince yourself that in S_5, we have:

$$(12345)(31) = (32)(145) = (32)(15)(14)$$

The above exercise shows that we can decompose $(12345)(31)$ into a product of three transpositions (32), (15) and (14). In fact, any element in S_n can be decomposed this way. Here we state a standard theorem in elementary group theory:

Theorem 3.29. *Every element $\sigma \in S_n$ can be expressed as a product of transpositions: $\sigma = \tau_1\tau_2 \ldots \tau_r$. Such a decomposition is not unique. However, if $\sigma = \widetilde{\tau}_1\widetilde{\tau}_2 \ldots \widetilde{\tau}_k$ is another decomposition of σ into transpositions, then we have $(-1)^k = (-1)^r$.*

Proof. Consult any standard textbook on Abstract Algebra. □

In view of Theorem 3.29, given an element $\sigma \in S_n$ which can be decomposed into the product of r transpositions, we define:

$$\text{sgn}(\sigma) := (-1)^r.$$

For instance, $\text{sgn}(12345) = (-1)^3 = -1$, and $\text{sgn}(123) = (-1)^2 = 1$. Certainly, if τ is a transposition, we have $\text{sgn}(\sigma\tau) = -\text{sgn}(\sigma)$.

Now we are ready to state an equivalent way to define triple wedge product using the above notations:

$$T_1 \wedge T_2 \wedge T_3 := \sum_{\sigma \in S_3} \text{sgn}(\sigma) T_{\sigma(1)} \otimes T_{\sigma(2)} \otimes T_{\sigma(3)}.$$

We can verify that it gives the same expression as before:

$$\begin{aligned}
&\sum_{\sigma \in S_3} \text{sgn}(\sigma) T_{\sigma(1)} \otimes T_{\sigma(2)} \otimes T_{\sigma(3)} \\
&= T_1 \otimes T_2 \otimes T_3 && \sigma = \text{id} \\
&\quad - T_2 \otimes T_1 \otimes T_3 && \sigma = (12) \\
&\quad - T_3 \otimes T_2 \otimes T_1 && \sigma = (13) \\
&\quad - T_1 \otimes T_3 \otimes T_2 && \sigma = (23) \\
&\quad + T_2 \otimes T_3 \otimes T_1 && \sigma = (123) = (13)(12) \\
&\quad + T_3 \otimes T_1 \otimes T_2 && \sigma = (132) = (12)(13)
\end{aligned}$$

In general, we define:

Definition 3.30 (Wedge Product)**.** Let V be a finite dimensional vector space, and V^* be the dual space of V. Then, given any $T_1, \ldots, T_k \in V^*$, we define their *k-th wedge product* by:

$$T_1 \wedge \cdots \wedge T_k := \sum_{\sigma \in S_k} \text{sgn}(\sigma)\, T_{\sigma(1)} \otimes \ldots \otimes T_{\sigma(k)}$$

where S_k is the permutation group of $\{1, \ldots, k\}$. The vector space spanned by $T_1 \wedge \cdots \wedge T_k$'s (where $T_1, \ldots, T_k \in V^*$) is denoted by $\wedge^k V^*$.

Remark 3.31. It is a convention to define $\wedge^0 V^* := \mathbb{R}$. □

If we switch any pair of the T_i's, then the wedge product differs by a minus sign. To show this, let $\tau \in S_k$ be a transposition, then for any $\sigma \in S_k$, we have $\text{sgn}(\sigma \circ \tau) = -\text{sgn}(\sigma)$. Therefore, we get:

$$\begin{aligned}
T_{\tau(1)} \wedge \cdots \wedge T_{\tau(k)} &= \sum_{\sigma \in S_k} \text{sgn}(\sigma) T_{\sigma(\tau(1))} \otimes \ldots \otimes T_{\sigma(\tau(k))} \\
&= -\sum_{\sigma \in S_k} \text{sgn}(\sigma \circ \tau) T_{\sigma \circ \tau(1)} \otimes \ldots \otimes T_{\sigma \circ \tau(k)} \\
&= -\sum_{\sigma \in S_k} \text{sgn}(\sigma') T_{\sigma'(1)} \otimes \ldots \otimes T_{\sigma' \tau(k)} \qquad \text{(where } \sigma' := \sigma \circ \tau\text{)} \\
&= -T_1 \wedge \cdots \wedge T_k.
\end{aligned}$$

The last step follows from the fact that $\sigma \mapsto \sigma \circ \tau$ is a bijection between S_k and itself.

Exercise 3.32. Write down $T_1 \wedge T_2 \wedge T_3 \wedge T_4$ explicitly in terms of tensor products (with no wedge and summation sign).

Exercise 3.33. Show that $\dim \wedge^k V^* = C^n_k$, when $n = \dim V$ and $0 \leq k \leq n$, by writing a basis for $\wedge^k V^*$. Show also that $\wedge^k V^* = \{0\}$ if $k > \dim V$.

Exercise 3.34. Let $\{e_i\}_{i=1}^n$ be a basis for a vector space V, and $\{e^i\}_{i=1}^n$ be the corresponding dual basis for V^*. Show that:

$$\left(e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}\right)(e_{j_1}, \ldots, e_{j_k}) = \delta_{i_1 j_1} \cdots \delta_{i_k j_k}.$$

Remark 3.32. The vector space $\wedge^k V^*$ is *spanned* by $T_1 \wedge \cdots \wedge T_k$'s where $T_1, \ldots, T_k \in V^*$. Note that not all elements in V^* can be expressed in the form of $T_1 \wedge \cdots \wedge T_k$. For instance when $V = \mathbb{R}^4$ with standard basis $\{e_i\}_{i=1}^4$, the element $\sigma = e^1 \wedge e^2 + e^3 \wedge e^4 \in \wedge^2 V^*$ cannot be written in the form of $T_1 \wedge T_2$ where $T_1, T_2 \in V^*$. It is because $(T_1 \wedge T_2) \wedge (T_1 \wedge T_2) = 0$ for any $T_1, T_2 \in V^*$, while $\sigma \wedge \sigma = 2e^1 \wedge e^2 \wedge e^3 \wedge e^4 \neq 0$. □

In the above remark, we take the wedge product between elements in $\wedge^2 V^*$. It is defined in a natural way that for any $T_1, \ldots T_k, S_1, \ldots, S_r \in V^*$, we have:

$$\underbrace{(T_1 \wedge \cdots \wedge T_k)}_{\in \wedge^k V^*} \wedge \underbrace{(S_1 \wedge \cdots \wedge S_r)}_{\in \wedge^r V^*} = \underbrace{T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r}_{\in \wedge^{k+r} V^*}$$

and extended linearly to other elements in $\wedge^k V^*$ and $\wedge^r V^*$. For instance, we have:

$$\underbrace{(T_1 \wedge T_2 + S_1 \wedge S_2)}_{\in \wedge^2 V^*} \wedge \underbrace{\sigma}_{\in \wedge^k V^*} = \underbrace{T_1 \wedge T_2 \wedge \sigma + S_1 \wedge S_2 \wedge \sigma}_{\in \wedge^{k+2} V^*}.$$

While it is true that $T_1 \wedge T_2 = -T_2 \wedge T_1$ for any $T_1, T_2 \in V^*$, it is generally *not* true that $\sigma \wedge \eta = -\eta \wedge \sigma$ where $\sigma \in \wedge^k V^*$ and $\eta \in \wedge^r V^*$. For instance, let $T_1, \ldots, T_5 \in V^*$ and consider $\sigma = T_1 \wedge T_2$ and $\eta = T_3 \wedge T_4 \wedge T_5$. Then we can see that:

$$\begin{aligned}
\sigma \wedge \eta &= T_1 \wedge T_2 \wedge T_3 \wedge T_4 \wedge T_5 \\
&= -T_1 \wedge T_3 \wedge T_4 \wedge T_5 \wedge T_2 && \text{(switching } T_2 \text{ subsequently with } T_3, T_4, T_5) \\
&= T_3 \wedge T_4 \wedge T_5 \wedge T_1 \wedge T_2 && \text{(switching } T_1 \text{ subsequently with } T_3, T_4, T_5) \\
&= \eta \wedge \sigma.
\end{aligned}$$

Proposition 3.33. *Let V be a finite dimensional vector space, and V^* be the dual space of V. Given any $\sigma \in \wedge^k V^*$ and $\eta \in \wedge^r V^*$, we have:*

$$\sigma \wedge \eta = (-1)^{kr} \eta \wedge \sigma. \tag{3.12}$$

Clearly from (3.12)*, any $\omega \in \wedge^{\text{even}} V^*$ commutes with any $\sigma \in \wedge^k V^*$.*

Proof. By linearity, it suffices to prove that case $\sigma = T_1 \wedge \cdots \wedge T_k$ and $\eta = S_1 \wedge \cdots \wedge S_r$ where $T_i, S_j \in V^*$, in which we have:

$$\sigma \wedge \eta = T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r$$

In order to switch all the T_i's with the S_j's, we can first switch T_k subsequently with each of $S_1, \ldots, S_r$ and each switching contributes to a factor of (-1). Precisely, we have:

$$T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r = (-1)^r T_1 \wedge \cdots \wedge T_{k-1} \wedge S_1 \wedge \cdots \wedge S_r \wedge T_k.$$

By repeating this sequence of switching on each of T_{k-1}, T_{k-2}, etc., we get a factor of $(-1)^r$ for each set of switching, and so we finally get the following as desired:

$$T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r = \left[(-1)^r\right]^k S_1 \wedge \cdots \wedge S_r \wedge T_1 \wedge \cdots \wedge T_k$$

□

From Exercise 3.33, we know that $\dim \wedge^n V^* = 1$ if $n = \dim V$. In fact, every element $\sigma \in \dim \wedge^n V^*$ is a constant multiple of $e^1 \wedge \cdots \wedge e^n$, and it is interesting (and important) to note that this constant multiple is related to a determinant! Precisely, for each $i = 1, \ldots, n$, we consider the elements:

$$\omega_i = \sum_{j=1}^n a_{ij} e^j \in V^*$$

where a_{ij} are real constants. Then, the wedge product of all ω_i's are given by:

$$\begin{aligned}
\omega_1 \wedge \cdots \wedge \omega_n &= \left(\sum_{j_1=1}^n a_{1j_1} e^*_{j_1}\right) \wedge \left(\sum_{j_2=1}^n a_{2j_2} e^*_{j_2}\right) \wedge \cdots \wedge \left(\sum_{j_n=1}^n a_{nj_n} e^*_{j_n}\right) \\
&= \sum_{j_1,\ldots,j_n \text{ distinct}} a_{1j_1} a_{2j_2} \ldots a_{nj_n} e^*_{j_1} \wedge \cdots \wedge e^*_{j_n} \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)} e^*_{\sigma(1)} \wedge \cdots \wedge e^*_{\sigma(n)}
\end{aligned}$$

Next we want to find a relation between $e^*_{\sigma(1)} \wedge \cdots \wedge e^*_{\sigma(n)}$ and $e^1 \wedge \cdots \wedge e^n$. $\sigma \in S_n$, by decomposing it into transpositions $\sigma = \tau_1 \circ \cdots \circ \tau_k$, then we have:

$$\begin{aligned}
e^*_{\sigma(1)} \wedge \cdots \wedge e^*_{\sigma(n)} &= e^*_{\tau_1 \circ \cdots \circ \tau_k(1)} \wedge \cdots \wedge e^*_{\tau_1 \circ \cdots \circ \tau_k(n)} \\
&= (-1) e^*_{\tau_2 \circ \cdots \circ \tau_k(1)} \wedge \cdots \wedge e^*_{\tau_2 \circ \cdots \circ \tau_k(n)} \\
&= (-1)^2 e^*_{\tau_3 \circ \cdots \circ \tau_k(1)} \wedge \cdots \wedge e^*_{\tau_3 \circ \cdots \circ \tau_k(n)} \\
&= \ldots \\
&= (-1)^{k-1} e^*_{\tau_k(1)} \wedge \cdots \wedge e^*_{\tau_k(n)} \\
&= (-1)^k e^1 \wedge \cdots \wedge e^n \\
&= \operatorname{sgn}(\sigma) e^1 \wedge \cdots \wedge e^n.
\end{aligned}$$

Therefore, we have:

$$\omega_1 \wedge \cdots \wedge \omega_n = \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}\right) e^1 \wedge \cdots \wedge e^n.$$

Note that the sum:

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

is exactly the determinant of the matrix A whose (i, j)-th entry is a_{ij}. To summarize, let's state it as a proposition:

Proposition 3.34. *Let V^* be the dual space of a vector space V of dimension n, and let $\{e_i\}_{i=1}^n$ be a basis for V, and $\{e^i\}_{i=1}^n$ be the corresponding dual basis for V^*. Given any n elements $\omega_i = \sum_{j=1}^n a_{ij} e^j \in V^*$, we have:*

$$\omega_1 \wedge \cdots \wedge \omega_n = (\det A)\ e^1 \wedge \cdots \wedge e^n,$$

where A is the $n \times n$ matrix whose (i, j)-th entry is a_{ij}.

Exercise 3.35. Given an n-dimensional vector space V. Show that $\omega_1, \ldots, \omega_n \in V^*$ are linearly independent if and only if $\omega_1 \wedge \cdots \wedge \omega_n \neq 0$.

Exercise 3.36. Generalize Proposition 3.34. Precisely, now given

$$\omega_i = \sum_{j=1}^{n} a_{ij} e^j \in V^*$$

where $1 \leq i \leq k < \dim V$, express $\omega_1 \wedge \cdots \wedge \omega_k$ in terms of e^i's.

Exercise 3.37. Regard $\det : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as a multilinear map:

$$\det(\mathsf{v}_1, \ldots, \mathsf{v}_n) := \begin{vmatrix} | & & | \\ \mathsf{v}_1 & \cdots & \mathsf{v}_n \\ | & & | \end{vmatrix}.$$

Denote $\{e_i\}$ the standard basis for $\mathbb{R}^n$. Show that:

$$\det = e^1 \wedge \cdots \wedge e^n.$$

3.4.2. Differential Forms on Smooth Manifolds. In the simplest term, differential forms on a smooth manifold are wedge products of cotangent vectors in T^*M. At each point $p \in M$, let $(u_1, \ldots, u_n)$ be the local coordinates near p, then the cotangent space T_p^*M is spanned by $\left\{ du^1\big|_p, \ldots, du^n\big|_p \right\}$, and a smooth differential 1-form α is a map from M to T^*M such that it can be locally expressed as:

$$\alpha(p) = \left(p, \sum_{i=1}^{n} \alpha_i(p)\, du^i\big|_p \right)$$

where α_i are smooth functions locally defined near p. Since the based point p can usually be understood from the context, we usually denote α by simply:

$$\alpha = \sum_{i=1}^{n} \alpha_i\, du^i.$$

Since T_p^*M is a finite dimensional vector space, we can consider the wedge products of its elements. A *differential k-form* ω on a smooth manifold M is a map which assigns each point $p \in M$ to an element in $\wedge^k T_p^*M$. Precisely:

Definition 3.35 (Smooth Differential k-Forms)**.** Let M be a smooth manifold. A *smooth differential k-form* ω on M is a map $\omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k \text{ times}} \to \mathbb{R}$ at each $p \in M$ such that under any local parametrization $F(u_1, \ldots, u_n) : \mathcal{U} \to M$, it can be written in the form:

$$\omega = \sum_{i_1, \ldots, i_k = 1}^{n} \omega_{i_1 i_2 \cdots i_k}\, du^{i_1} \wedge \cdots \wedge du^{i_k}$$

where $\omega_{i_1 i_2 \ldots i_k}$'s are smooth scalar functions locally defined in $F(\mathcal{U})$, and they are commonly called the *local components* of ω. The vector space of all smooth differential k-forms on M is denoted by $\wedge^k T^*M$.

Remark 3.36. It is a convention to denote $\wedge^0 T^*M := C^\infty(M, \mathbb{R})$, the vector space of all smooth scalar functions defined on M. □

We will mostly deal with differential k-forms that are smooth. Therefore, we will very often call a *smooth differential k-form* simply by a *differential k-form*, or even simpler, a *k-form*. As we will see in the next section, the language of differential forms will unify and generalize the curl, grad and div in Multivariable Calculus and Physics courses.

From algebraic viewpoint, the manipulations of differential k-forms on a manifold are similar to those for wedge products of a finite-dimensional vector space. The major difference is a manifold is usually covered by more than one local parametrizations, hence there are conversion rules for differential k-forms from one local coordinate system to another.

Example 3.37. Consider $\mathbb{R}^2$ with (x, y) and (r, θ) as its two local coordinates. Given a 2-form $\omega = dx \wedge dy$, for instance, we can express it in terms of the polar coordinates (r, θ):

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial \theta}\, d\theta \\ &= (\cos\theta)\, dr - (r\sin\theta)\, d\theta \\ dy &= \frac{\partial y}{\partial r}\, dr + \frac{\partial y}{\partial \theta}\, d\theta \\ &= (\sin\theta)\, dr + (r\cos\theta)\, d\theta \end{aligned}$$

Therefore, using $dr \wedge dr = 0$ and $d\theta \wedge d\theta = 0$, we get:

$$\begin{aligned} dx \wedge dy &= (r\cos^2\theta) dr \wedge d\theta - (r\sin^2) d\theta \wedge dr \\ &= (r\cos^2\theta + r\sin^2\theta)\, dr \wedge d\theta \\ &= r\, dr \wedge d\theta. \end{aligned}$$

□

Exercise 3.38. Define a 2-form on $\mathbb{R}^3$ by:

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

Express ω in terms of spherical coordinates (ρ, θ, φ), defined by:

$$(x, y, z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

Exercise 3.39. Let ω be the 2-form on $\mathbb{R}^{2n}$ given by:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2n-1} \wedge dx^{2n}.$$

Compute $\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$.

Exercise 3.40. Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two local coordinates of a smooth manifold M. Show that:

$$du^1 \wedge \cdots \wedge du^n = \det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)}\, dv^1 \wedge \cdots \wedge dv^n.$$

Exercise 3.41. Show that on $\mathbb{R}^3$, a $(2, 0)$-tensor T is in $\wedge^2(\mathbb{R}^2)^*$ if and only if $T(\mathsf{v}, \mathsf{v}) = 0$ for any $\mathsf{v} \in \mathbb{R}^3$.

3.5. Exterior Derivatives

Exterior differentiation is an important operations on differential forms. It not only generalizes and unifies the curl, grad, div operators in Multivariable Calculus and Physics, but also leads to the development of de Rham cohomology to be discussed in Chapter 5.

3.5.1. Definition of Exterior Derivatives. Exterior differentiation, commonly denoted by the symbol d, takes a k-form to a $(k+1)$-form. To begin, let's define it on scalar functions first. Suppose $(u_1, \ldots, u_n)$ are local coordinates of M^n, then given any smooth scalar function $f \in C^\infty(M, \mathbb{R})$, we define:

$$df := \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\, du^i \tag{3.13}$$

Although (3.13) involves local coordinates, it can be easily shown that df is independent of local coordinates. Suppose $(v_1, \ldots, v_n)$ is another local coordinates of M which overlap with $(u_1, \ldots, u_n)$. By the chain rule, we have:

$$\frac{\partial f}{\partial u_i} = \sum_{k=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_i}$$

$$dv^k = \sum_{i=1}^{n} \frac{\partial v_k}{\partial u_i}\, du^i$$

which combine to give:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\, du^i = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_i}\, du^i = \sum_{k=1}^{n} \frac{\partial f}{\partial v_k}\, dv^k.$$

Therefore, if f is smooth on M then df is a smooth 1-form on M. The components of df are $\frac{\partial f}{\partial u_i}$'s, and so df is analogous to ∇f in Multivariable Calculus. Note that as long as f is C^∞ just in an open set $\mathcal{U} \subset M$, we can also define df locally on $\mathcal{U}$ since (3.13) is a local expression.

Exterior derivatives can also be defined on differential forms of higher degrees. Let $\alpha \in \wedge^1 T^*M$, which can be locally written as:

$$\alpha = \sum_{i=1}^{n} \alpha_i\, du^i$$

where α_i's are smooth functions locally defined in a local coordinate chart. Then, we define:

$$d\alpha := \sum_{i=1}^{n} d\alpha_i \wedge du^i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \alpha_i}{\partial u_j}\, du^j \wedge du^i. \tag{3.14}$$

Using the fact that $du^j \wedge du^i = -du^i \wedge du^j$ and $du^i \wedge du^i = 0$, we can also express $d\alpha$ as:

$$d\alpha = \sum_{1 \le j < i \le n} \left(\frac{\partial \alpha_i}{\partial u_j} - \frac{\partial \alpha_j}{\partial u_i} \right) du^j \wedge du^i.$$

Example 3.38. Take $M = \mathbb{R}^3$ as an example, and let (x, y, z) be the (usual) coordinates of $\mathbb{R}^3$, then given any 1-form $\alpha = P\,dx + Q\,dy + R\,dz$ (which is analogous to the vector field $P\hat{i} + Q\hat{j} + R\hat{k}$), we have:

$$\begin{aligned}
d\alpha &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy + \frac{\partial P}{\partial z}\,dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy + \frac{\partial Q}{\partial z}\,dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial y}\,dy + \frac{\partial R}{\partial z}\,dz\right) \wedge dz \\
&= \frac{\partial P}{\partial y}\,dy \wedge dx + \frac{\partial P}{\partial z}\,dz \wedge dx + \frac{\partial Q}{\partial x}\,dx \wedge dy + \frac{\partial Q}{\partial z}\,dz \wedge dy \\
&\quad + \frac{\partial R}{\partial x}\,dx \wedge dz + \frac{\partial R}{\partial y}\,dy \wedge dz \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz
\end{aligned}$$

which is analogous to $\nabla \times \left(P\hat{i} + Q\hat{j} + R\hat{k}\right)$ by declaring the correspondence $\{\hat{i}, \hat{j}, \hat{k}\}$ with $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$. □

One can check that the definition of $d\alpha$ stated in (3.14) is independent of local coordinates. On general k-forms, the exterior derivatives are defined in a similar way as:

Definition 3.39 (Exterior Derivatives). Let M^n be a smooth manifold and $(u_1, \ldots, u_n)$ be local coordinates on M. Given any (smooth) k-form

$$\omega = \sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\,du^{j_1} \wedge \cdots \wedge du^{j_k},$$

we define:

$$\begin{aligned}
d\omega &:= \sum_{j_1,\cdots,j_k=1}^{n} d\omega_{j_1\cdots j_k} \wedge du^{j_1} \wedge \cdots \wedge du^{j_k} \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i} du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}
\end{aligned} \tag{3.15}$$

In particular, if ω is an n-form (where $n = \dim V$), we have $d\omega = 0$.

Exercise 3.42. Show that $d\omega$ defined as in (3.15) does not depend on the choice of local coordinates.

Example 3.40. Consider $\mathbb{R}^2$ equipped with polar coordinates (r, θ). Consider the 1-form:

$$\omega = (r \sin\theta)\,dr.$$

Then, we have

$$\begin{aligned}
d\omega &= \frac{\partial (r\sin\theta)}{\partial r}\,dr \wedge dr + \frac{\partial (r\sin\theta)}{\partial \theta}\,d\theta \wedge dr \\
&= 0 + (r\cos\theta)\,d\theta \wedge dr \\
&= -(r\cos\theta)\,dr \wedge d\theta.
\end{aligned}$$

□

Exercise 3.43. Let $\omega = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$ be a smooth 2-form on $\mathbb{R}^3$. Compute $d\omega$. What operator in Multivariable Calculus is the d analogous to in this case?

Exercise 3.44. Let ω, η, θ be the following differential forms on $\mathbb{R}^3$:

$$\begin{aligned}\omega &= x\,dx - y, dy\\ \eta &= z\,dx \wedge dy + x\,dy \wedge dz\\ \theta &= z\,dy\end{aligned}$$

Compute: $\omega \wedge \eta$, $\omega \wedge \eta \wedge \theta$, $d\omega$, $d\eta$ and $d\theta$.

3.5.2. Properties of Exterior Derivatives. The exterior differentiation d can hence be regarded as a chain of maps:

$$\wedge^0 T^*M \xrightarrow{d} \wedge^1 T^*M \xrightarrow{d} \wedge^2 T^*M \xrightarrow{d} \cdots \xrightarrow{d} \wedge^{n-1} T^*M \xrightarrow{d} \wedge^n T^*M.$$

Here we abuse the use of the symbol d a little bit – we use the same symbol d for all the maps $\wedge^k T^*M \xrightarrow{d} \wedge^{k+1} T^*M$ in the chain. The following properties about exterior differentiation are not difficult to prove:

Proposition 3.41. *For any k-forms ω and η, and any smooth scalar function f, we have the following:*

(1) $d(\omega + \eta) = d\omega + d\eta$

(2) $d(f\omega) = df \wedge \omega + f\,d\omega$

Proof. (1) is easy to prove (left as an exercise for readers). To prove (2), we consider local coordinates $(u_1, \ldots, u_n)$ and let $\omega = \sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\,du^{j_1} \wedge \cdots \wedge du^{j_k}$. Then, we have:

$$\begin{aligned}d(f\omega) &= \sum_{j_1,\ldots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial}{\partial u_i}(f\omega_{j_1\cdots j_k})\,du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\\ &= \sum_{j_1,\ldots,j_k=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i}\,\omega_{j_1\cdots j_k} + f\,\frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i}\right) du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\\ &= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\,du^i\right) \wedge \left(\sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\,du^{j_1} \wedge \cdots \wedge du^{j_k}\right)\\ &\quad + f \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i} du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\end{aligned}$$

as desired. □

Identity (2) in Proposition 3.41 can be regarded as a kind of product rule. Given a k-form α and a r-form β, the general product rule for exterior derivative is stated as:

Proposition 3.42. *Let $\alpha \in \wedge^k T^*M$ and $\beta \in \wedge^r T^*M$ be smooth differential forms on M, then we have:*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k\,\alpha \wedge d\beta.$$

Exercise 3.45. Prove Proposition 3.42. Based on your proof, explain briefly why the product rule does not involve any factor of $(-1)^r$.

Exercise 3.46. Given three differential forms α, β and γ such that $d\alpha = 0$, $d\beta = 0$ and $d\gamma = 0$. Show that:
$$d(\alpha \wedge \beta \wedge \gamma) = 0.$$

An crucial property of exterior derivatives is that the *composition* is zero. For instance, given a smooth scalar function $f(x,y,z)$ defined on $\mathbb{R}^3$, we have:
$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$
Taking exterior derivative one more time, we get:
$$\begin{aligned} d(df) &= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\,dz\right) \wedge dx \\ &\quad + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\,dz\right) \wedge dy \\ &\quad + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial z}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial z}\,dz\right) \wedge dz \\ &= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right) dx \wedge dy + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right) dz \wedge dx \\ &\quad + \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right) dy \wedge dz \end{aligned}$$
Since partial derivatives commute, we get $d(df) = 0$, or in short $d^2 f = 0$, for any scalar function f. The fact that $d^2 = 0$ is generally true on smooth differential forms, not only for scalar functions. Precisely, we have:

Proposition 3.43. *Let ω be a smooth k-form defined on a smooth manifold M, then we have:*
$$d^2\omega := d(d\omega) = 0.$$

Proof. Let $\omega = \displaystyle\sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\, du^{j_1} \wedge \cdots \wedge du^{j_k}$, then:
$$d\omega = \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i}\, du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}.$$
$$\begin{aligned} d^2\omega &= d\left(\sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i}\, du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\right) \\ &= \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n}\sum_{l=1}^{n} \frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_l \partial u_i}\, du^l \wedge du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k} \end{aligned}$$
For each fixed k-tuple $(j_1,\ldots,j_k)$, the term $\displaystyle\sum_{i,l=1}^{n} \frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_l \partial u_i}\, du^l \wedge du^i$ can be rewritten as:
$$\sum_{1\le i<l\le n} \left(\frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_l \partial u_i} - \frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_i \partial u_l}\right) du^l \wedge du^i$$

which is zero since partial derivatives commute. It concludes that $d^2\omega = 0$. $\square$

Proposition 3.43 is a important fact that leads to the development of de Rham cohomology in Chapter 5.

In Multivariable Calculus, we learned that given a vector field $F = P\hat{i} + Q\hat{j} + R\hat{k}$ and a scalar function f, we have:

$$\nabla \times \nabla f = 0$$
$$\nabla \cdot (\nabla \times F) = 0$$

These two formulae can be unified using the language of differential forms. The one-form df corresponds to the vector field ∇f:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$
$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Define a one-form $\omega = P\,dx + Q\,dy + R\,dz$ on $\mathbb{R}^3$, which corresponds to the vector field F, then we have discussed that $d\omega$ corresponds to taking curl of F:

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz$$
$$\nabla \times F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \hat{k} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \hat{j} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \hat{i}$$

If one takes $\omega = df$, and $F = \nabla f$, then we have $d\omega = d(df) = 0$, which corresponds to the fact that $\nabla \times G = \nabla \times \nabla f = 0$ in Multivariable Calculus.

Taking exterior derivative on a two-form $\beta = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$ corresponds to taking the divergence on the vector field $G = A\hat{i} + B\hat{j} + C\hat{k}$ according to Exercise 3.43:

$$d\beta = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$$
$$\nabla \cdot G = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)$$

By taking $\beta = d\omega$, and $G = \nabla \times F$, then we have $d\beta = d(d\omega) = 0$ corresponding to $\nabla \cdot G = \nabla \cdot (\nabla \times F) = 0$ in Multivariable Calculus.

Here is a summary of the correspondences:

Differential Form on $\mathbb{R}^3$	Multivariable Calculus
$f(x, y, z)$	$f(x, y, z)$
$\omega = P\,dx + Q\,dy + R\,dz$	$F = P\hat{i} + Q\hat{j} + R\hat{k}$
$\beta = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$	$G = A\hat{i} + B\hat{j} + C\hat{k}$
df	∇f
$d\omega$	$\nabla \times F$
$d\beta$	$\nabla \cdot G$
$d^2 f = 0$	$\nabla \times \nabla f = 0$
$d^2\omega = 0$	$\nabla \cdot (\nabla \times F) = 0$

3.5.3. Exact and Closed Forms. In Multivariable Calculus, we discussed various concepts of vector fields including potential functions, conservative vector fields, solenoidal vector fields, curl-less and divergence-less vector fields, etc. All these concepts can be unified using the language of differential forms.

As a reminder, a conservative vector field F is one that can be expressed as $F = \nabla f$ where f is a scalar function. It is equivalent to saying that the 1-form ω can be expressed as $\omega = df$. Moreover, a solenoidal vector field G is one that can be expressed as $G = \nabla \times F$ for some vector field F. It is equivalent to saying that the 2-form β can be expressed as $\beta = d\omega$ for some 1-form ω.

Likewise, a curl-less vector field F (i.e. $\nabla \times F = 0$) corresponds to a 1-form ω satisfying $d\omega = 0$; and a divergence-less vector field G (i.e. $\nabla \cdot G = 0$) corresponds to a 2-form β satisfying $d\beta = 0$.

In view of the above correspondence, we introduce two terminologies for differential forms, namely *exact-ness* and *closed-ness*:

Definition 3.44 (Exact and Closed Forms)**.** Let ω be a smooth k-form defined on a smooth manifold M, then we say:

- ω is *exact* if there exists a $(k-1)$-form η defined on M such that $\omega = d\eta$;
- ω is *closed* if $d\omega = 0$.

Remark 3.45. By the fact that $d^2 = 0$ (Proposition 3.43), it is clear that every exact form is a closed form (but *not* vice versa). □

The list below showcases the corresponding concepts of exact/closed forms in Multivariable Calculus.

Differential Form on $\mathbb{R}^3$	Multivariable Calculus
exact 1-form	conservative vector field
closed 1-form	curl-less vector field
exact 2-form	solenoidal vector field
closed 2-form	divergence-less vector field

Example 3.46. On $\mathbb{R}^3$, the 1-form:

$$\alpha = yz\,dx + zx\,dy + xy\,dz$$

is exact since $\alpha = df$ where $f(x, y, z) = xyz$. By Proposition 3.43, we immediately get $d\alpha = d(df) = 0$, so α is a closed form. One can also verify this directly:

$$\begin{aligned} d\alpha &= (z\,dy + y\,dz) \wedge dx + (z\,dx + x\,dz) \wedge dy + (y\,dx + x\,dy) \wedge dz \\ &= (z - z)\,dx \wedge dy + (y - y)\,dz \wedge dx + (x - x)\,dy \wedge dz = 0. \end{aligned}$$

□

Example 3.47. The 1-form:

$$\alpha := -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy$$

defined on $\mathbb{R}^2\backslash\{(0, 0)\}$ is closed:

$$\begin{aligned} d\alpha &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) dx \wedge dy \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2}\,dy \wedge dx + \frac{y^2 - x^2}{(x^2 + y^2)^2}\,dx \wedge dy \\ &= 0 \end{aligned}$$

as $dx \wedge dy = -dy \wedge dx$. However, we will later see that α is not exact.

Note that even though we have $\alpha = df$ where $f(x, y) = \tan^{-1} \dfrac{y}{x}$, such an f is NOT smooth on $\mathbb{R}^2\backslash\{(0, 0)\}$. In order to claim α is exact, we require such an f to be smooth on the domain of α. □

Exercise 3.47. Consider the forms ω, η and θ on $\mathbb{R}^3$ defined in Exercise 3.44. Determine whether each of them is closed and/or exact on $\mathbb{R}^3$.

Exercise 3.48. The purpose of this exercise is to show that any closed 1-form ω on $\mathbb{R}^3$ must be exact. Let

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

be a closed 1-form on $\mathbb{R}^3$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by:

$$f(x,y,z) = \int_{t=0}^{t=1} \left(xP(tx,ty,tz) + yQ(tx,ty,tz) + zR(tx,ty,tz)\right)\,dt$$

Show that $\omega = df$. Point out exactly where you have used the fact that $d\omega = 0$.

3.5.4. Pull-Back of Tensors. Let's first begin by reviewing the *push-forward* and *pull-back* of tangent and cotangent vectors. Given a smooth map $\Phi : M \to N$ between two smooth manifolds M^m and N^n, its tangent map Φ_* takes a tangent vector in T_pM to a tangent vector in $T_{\Phi(p)}N$. If we let $F(u_1, \ldots, u_m)$ be local coordinates of M, $G(v_1, \ldots, v_n)$ be local coordinates of N and express the map Φ locally as:

$$(v_1, \ldots, v_n) = G^{-1} \circ \Phi \circ F(u_1, \ldots, u_m),$$

then Φ_* acts on the basis vectors $\left\{\dfrac{\partial}{\partial u_i}\right\}$ by:

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial \Phi}{\partial u_i} = \sum_j \frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}.$$

The tangent map Φ_* is also commonly called the *push-forward* map. It is important to note that the v_j's in the partial derivatves $\frac{\partial v_j}{\partial u_i}$ can sometimes cause confusion if we talk about the push-forwards of two different smooth maps $\Phi : M \to N$ and $\Psi : M \to N$. Even with the same input $(u_1, \ldots, u_m)$, the output $\Phi(u_1, \ldots, u_m)$ and $\Psi(u_1, \ldots, u_m)$ are generally different and have different v_j-coordinates. To avoid this confusion, it is best to write:

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \sum_j \frac{\partial(v_j \circ \Phi)}{\partial u_i}\frac{\partial}{\partial v_j}$$

$$\Psi_*\left(\frac{\partial}{\partial u_i}\right) = \sum_j \frac{\partial(v_j \circ \Psi)}{\partial u_i}\frac{\partial}{\partial v_j}$$

Here each v_j in the partial derivatives $\frac{\partial v_j}{\partial u_i}$ are considered to be a locally defined function taking a point $p \in N$ to its v_j-coordinate.

For cotangent vectors (i.e. 1-forms), we talk about *pull-back* instead. According to Definition 3.14, Φ^* takes a cotangent vector in $T^*_{\Phi(p)}N$ to a cotangent vector in T^*_pM, defined as follows:

$$\Phi^*(dv^i)(X) = dv^i\,(\Phi_* X) \quad \text{for any } X \in T_pM.$$

In terms of local coordinates, it is given by:

$$\Phi^*(dv^i) = \sum_j \frac{\partial(v_i \circ \Phi)}{\partial u_j}\,du^j.$$

The pull-back action by a smooth $\Phi : M \to N$ between manifolds can be extended to $(k,0)$-tensors (and hence to differential forms):

Definition 3.48 (Pull-Back on $(k,0)$-Tensors)**.** Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. Given T a smooth $(k,0)$-tensor on N, then we define:

$$(\Phi^*T)_p\,(X_1,\ldots,X_k) = T_{\Phi(p)}\,(\Phi_*(X_1),\ldots,\Phi_*(X_k)) \quad \text{for any } X_1,\ldots,X_k \in T_pM$$

Remark 3.49. An equivalent way to state the definition is as follows: let $T_1, \ldots T_k \in TN$ be 1-forms on N, then we define:

$$\Phi^*(T_1 \otimes \cdots \otimes T_k) = (\Phi^*T_1) \otimes \cdots \otimes (\Phi^*T_k).$$

□

Remark 3.50. It is easy to verify that Φ^* is linear, in a sense that:

$$\Phi^*(aT + bS) = a\Phi^*T + b\Phi^*S$$

for any $(k,0)$-tensors T and S, and scalars a and b. □

Example 3.51. Let's start with an example on $\mathbb{R}^2$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by:

$$\Phi(x_1,x_2) = \left(e^{x_1+x_2}, \sin(x_1^2x_2), x_1\right).$$

To avoid confusion, we use (x_1,x_2) to label the coordinates of the domain $\mathbb{R}^2$, and use (y_1,y_2,y_3) to denote the coordinates of the codomain $\mathbb{R}^3$. Then, we have:

$$\begin{aligned}
\Phi^*(dy^1)\left(\frac{\partial}{\partial x_1}\right) &= dy^1\left(\Phi_*\left(\frac{\partial}{\partial x_1}\right)\right) = dy^1\left(\frac{\partial \Phi}{\partial x_1}\right) \\
&= dy^1\left(\frac{\partial(y_1\circ\Phi)}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial(y_2\circ\Phi)}{\partial x_1}\frac{\partial}{\partial y_2} + \frac{\partial(y_3\circ\Phi)}{\partial x_1}\frac{\partial}{\partial y_3}\right) \\
&= \frac{\partial(y_1\circ\Phi)}{\partial x_1} = \frac{\partial}{\partial x_1}e^{x_1+x_2} = e^{x_1+x_2}.
\end{aligned}$$

Similarly, we have:

$$\Phi^*(dy^1)\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial(y_1\circ\Phi)}{\partial x_2} = \frac{\partial}{\partial x_2}e^{x_1+x_2} = e^{x_1+x_2}.$$

Therefore, $\Phi^*(dy^1) = e^{x_1+x_2}dx^1 + e^{x_1+x_2}dx^2 = e^{x_1+x_2}(dx^1 + dx^2)$. We leave it as an exercise for readers to verify that:

$$\begin{aligned}
\Phi^*(dy^2) &= 2x_1x_2\cos(x_1^2x_2)\,dx^1 + x_1^2\cos(x_1^2x_2)\,dx^2 \\
\Phi^*(dy^3) &= dx^1
\end{aligned}$$

Let $f(y_1,y_2,y_3)$ be a scalar function on $\mathbb{R}^3$, and consider the $(2,0)$-tensor on $\mathbb{R}^3$:

$$T = f(y_1,y_2,y_3)\,dy^1\otimes dy^2$$

The pull-back of T by Φ is given by:

$$\begin{aligned}
\Phi^*T &= f(y_1,y_2,y_3)\,\Phi^*(dy^1)\otimes\Phi^*(dy^2) \\
&= f(\Phi(x_1,x_2))\left(e^{x_1+x_2}(dx^1+dx^2)\right)\otimes\left(2x_1x_2\cos(x_1^2x_2)\,dx^1 + x_1^2\cos(x_1^2x_2)\,dx^2\right)
\end{aligned}$$

The purpose of writing $f(y_1,y_2,y_3)$ as $f(\Phi(x_1,x_2))$ is to leave the final expression in terms of functions and tensors in (x_1,x_2)-coordinates. □

Example 3.52. Let Σ be a regular surface in $\mathbb{R}^3$. The standard dot product in $\mathbb{R}^3$ is given by the following $(2,0)$-tensor:

$$\omega = dx\otimes dx + dy\otimes dy + dz\otimes dz.$$

Consider the inclusion map $\iota : \Sigma \to \mathbb{R}^3$. Although the input and output are the same under the map ι, the cotangents dx and $\iota^*(dx)$ are different! The former is a cotangent vector on $\mathbb{R}^3$, while $\iota^*(dx)$ is a cotangent vector on the surface Σ. If $(x,y,z) = F(u,v)$ is a local parametrization of Σ, then $\iota^*(dx)$ should be in terms of du and dv, but not dx, dy and dz. Precisely, we have:

$$\begin{aligned}
\iota_*\left(\frac{\partial F}{\partial u}\right) &= \frac{\partial \iota}{\partial u} := \frac{\partial(\iota \circ F)}{\partial u} = \frac{\partial F}{\partial u} \\
\iota^*(dx)\left(\frac{\partial F}{\partial u}\right) &= dx\left(\iota_*\left(\frac{\partial F}{\partial u}\right)\right) = dx\left(\frac{\partial F}{\partial u}\right) \\
&= dx\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right) \\
&= \frac{\partial x}{\partial u}.
\end{aligned}$$

Similarly, we also have $\iota^*(dx)\left(\frac{\partial F}{\partial v}\right) = \frac{\partial x}{\partial v}$, and hence:

$$\iota^*(dx) = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv.$$

As a result, we have:

$$\begin{aligned}
\iota^*\omega &= \iota^*(dx) \otimes \iota^*(dx) + \iota^*(dy) \otimes \iota^*(dy)\iota^*(dz) \otimes \iota^*(dz) \\
&= \left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right) \otimes \left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right) \\
&\quad + \left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right) \otimes \left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right) \\
&\quad + \left(\frac{\partial z}{\partial u}\,du + \frac{\partial z}{\partial v}\,dv\right) \otimes \left(\frac{\partial z}{\partial u}\,du + \frac{\partial z}{\partial v}\,dv\right).
\end{aligned}$$

After expansion and simplification, one will get:

$$\iota^*\omega = \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial u}\,du \otimes du + \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v}\,du \otimes dv + \frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial u}\,dv \otimes du + \frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial v}\,dv \otimes dv,$$

which is the first fundamental form in Differential Geometry. □

Exercise 3.49. Let the unit sphere $\mathbb{S}^2$ be locally parametrized by spherical coordinates (θ, φ). Consider the $(2,0)$-tensor on $\mathbb{R}^3$:

$$\omega = x\,dy \otimes dz$$

Express the pull-back $\iota^*\omega$ in terms of (θ, φ).

One can derive a general formula (which you do not need to remember in practice) for the local expression of pull-backs. Consider local coordinates $\{u_i\}$ for M and $\{v_i\}$ for N, and write $(v_1, \ldots, v_n) = \Phi(u_1, \ldots, u_m)$ and

$$T = \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1\cdots i_k}(v_1, \ldots, v_n)\,dv^{i_1} \otimes \cdots \otimes dv^{i_k}.$$

The pull-back $\Phi^* T$ then has the following local expression:

(3.16)

$$\begin{aligned}
\Phi^* T &= \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1\cdots i_k}(v_1,\ldots,v_n)\, \Phi^*(dv^{i_1}) \otimes \cdots \otimes \Phi^*(dv^{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1\cdots i_k}(\Phi(u_1,\ldots,u_m)) \left(\sum_{j_1=1}^{m} \frac{\partial v_{i_1}}{\partial u_{j_1}}\, du^{j_1} \right) \otimes \cdots \otimes \left(\sum_{j_k=1}^{m} \frac{\partial v_{i_k}}{\partial u_{j_k}}\, du^{j_k} \right) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{m} T_{i_1\cdots i_k}(\Phi(u_1,\ldots,u_m)) \frac{\partial v_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_k}}\, du^{j_1} \otimes \cdots \otimes du^{j_k}.
\end{aligned}$$

In view of $T_{i_1\cdots i_k}(v_1,\ldots,v_n) = T_{i_1\cdots i_k}(\Phi(u_1,\ldots,u_m))$ and the above local expression, we define

$$\Phi^* f := f \circ \Phi$$

for any scalar function of f. Using this notation, we then have $\Phi^*(fT) = (\Phi^* f)\, \Phi^* T$ for any scalar function f and $(k,0)$-tensor T.

Exercise 3.50. Let $\Phi : M \to N$ be a smooth map between smooth manifolds M and N, f be a smooth scalar function defined on N. Show that

$$\Phi^*(df) = d(\Phi^* f).$$

In particular, if $(v_1,\ldots,v_n)$ are local coordinates of N, we have $\Phi^*(dv^j) = d(\Phi^* v^j)$.

Example 3.53. Using the result from Exercise 3.50, one can compute the pull-back by inclusion map $\iota : \Sigma \to \mathbb{R}^3$ for regular surfaces Σ in $\mathbb{R}^3$. Suppose $F(u,v)$ is a local parametrization of Σ, then:

$$\iota^*(dx) = d(\iota^* x) = d(x \circ \iota).$$

Although $x \circ \iota$ and x (as a coordinate function) have the same output, their domains are different! Namely, $x \circ \iota : \Sigma \to \mathbb{R}$ while $x : \mathbb{R}^3 \to \mathbb{R}$. Therefore, when computing $d(x \circ \iota)$, one should express it in terms of local coordinates (u,v) of Σ:

$$d(x \circ \iota) = \frac{\partial (x \circ \iota)}{\partial u}\, du + \frac{\partial (x \circ \iota)}{\partial v}\, dv = \frac{\partial x}{\partial u}\, du + \frac{\partial x}{\partial v}\, dv.$$

□

Recall that the tangent maps (i.e. push-forwards) acting on tangent vectors satisfy the chain rule: if $\Phi : M \to N$ and $\Psi : N \to P$ are smooth maps between smooth manifolds, then we have $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$. It is easy to extend the chain rule to $(k,0)$-tensors:

Theorem 3.54 (Chain Rule for $(k,0)$-tensors)**.** *Let $\Phi : M \to N$ and $\Psi : N \to P$ be smooth maps between smooth manifolds M, N and P, then the pull-back maps Φ^* and Ψ^* acting on $(k,0)$-tensors for any $k \geq 1$ satisfy the following chain rule:*

(3.17) $$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*.$$

Exercise 3.51. Prove Theorem 3.54.

Exercise 3.52. Denote id_M and id_{TM} to be the identity maps of a smooth manifold M and its tangent bundle respectively. Show that $(\mathrm{id}_M)^* = \mathrm{id}_{TM}$. Hence, show that if M and N are diffeomorphic, then for $k \geq 1$ the vector spaces of $(k, 0)$-tensors $\otimes^k T^*M$ and $\otimes^k T^*N$ are isomorphic.

3.5.5. Pull-Back of Differential Forms. By linearity of the pull-back map, and the fact that differential forms are linear combinations of tensors, the pull-back map acts on differential forms by the following way:

$$\Phi^*(T_1 \wedge \cdots \wedge T_k) = \Phi^* T_1 \wedge \cdots \wedge \Phi^* T_k$$

for any 1-forms $T_1, \ldots, T_k$.

Example 3.55. Consider the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$\underbrace{\Phi(x_1, x_2)}_{(y_1, y_2)} = (x_1^2 - x_2, x_2^3).$$

By straight-forward computations, we have:

$$\Phi^*(dy^1) = 2x_1\, dx^1 - dx^2$$
$$\Phi^*(dy^2) = 3x_2\, dx^2$$

Therefore, we have:

$$\Phi^*(dy^1 \wedge dy^2) = \Phi^*(dy^1) \wedge \Phi^*(dy^2) = 6x_1 x_2\, dx^1 \wedge dx^2.$$

Note that $6x_1x_2$ is the Jacobian determinant $\det[\Phi_*]$. We will see soon that it is not a coincident, and it holds true in general. □

Although the computation of pull-back on differential forms is not much different from that on tensors, there are several distinctive features for pull-back on forms. One feature is that the pull-back on forms is closely related to Jacobian determinants:

Proposition 3.56. *Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. Suppose $(u_1, \ldots, u_m)$ are local coordinates of M, and $(v_1, \ldots, v_n)$ are local coordinates of N, then for any $1 \leq i_1, \ldots, i_k \leq n$, we have:*

$$\Phi^*(dv^{i_1} \wedge \cdots \wedge dv^{i_k}) = \sum_{1 \leq j_1 < \cdots < j_k \leq m} \det \frac{\partial(v_{i_1}, \ldots, v_{i_k})}{\partial(u_{j_1}, \ldots, u_{j_k})}\, du^{j_1} \wedge \cdots \wedge du^{j_k}. \tag{3.18}$$

In particular, if $\dim M = \dim N = n$, then we have:

$$\Phi^*(dv^1 \wedge \cdots \wedge dv^n) = \det[\Phi_*]\, du^1 \wedge \cdots \wedge du^n \tag{3.19}$$

where $[\Phi_*]$ is the Jacobian matrix of Φ with respect to local coordinates $\{u_i\}$ and $\{v_i\}$, i.e.

$$[\Phi_*] = \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)}.$$

Proof. Proceed as in the derivation of (3.16) by simply replacing all tensor products by wedge products, we get:

$$\begin{aligned}
\Phi^*(dv^{i_1} \wedge \cdots \wedge dv^{i_k}) &= \sum_{j_1, \ldots, j_k = 1}^{m} \left(\frac{\partial v_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_k}}\, du^{j_1} \wedge \cdots \wedge du^{j_k} \right) \\
&= \sum_{\substack{j_1, \ldots, j_k = 1 \\ j_1, \ldots, j_k \text{ distinct}}}^{m} \left(\frac{\partial v_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_k}}\, du^{j_1} \wedge \cdots \wedge du^{j_k} \right)
\end{aligned}$$

The second equality follows from the fact that $du^{j_1} \wedge \cdots \wedge du^{j_k} = 0$ if $\{j_1, \ldots, j_k\}$ are not all distinct. Each k-tuples $(j_1, \ldots, j_k)$ with distinct j_i's can be obtained by permuting a strictly increasing sequence of j's. Precisely, we have:

$$\begin{aligned}
&\{(j_1, \ldots, j_k) : 1 \leq j_1, \ldots, j_k \leq n \text{ and } j_1, \ldots, j_k \text{ are all distinct}\} \\
&= \bigcup_{\sigma \in S_k} \{(j_{\sigma(1)}, \ldots, j_{\sigma(k)}) : 1 \leq j_1 < j_2 < \ldots < j_k \leq n\}
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
&\Phi^*(dv^{i_1} \wedge \cdots \wedge dv^{i_k}) \\
&= \sum_{1 \leq j_1 < \ldots < j_k \leq m} \sum_{\sigma \in S_k} \left(\frac{\partial v_{i_1}}{\partial u_{j_{\sigma(1)}}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_{\sigma(k)}}} du^{j_{\sigma(1)}} \wedge \cdots \wedge du^{j_{\sigma(k)}} \right) \\
&= \sum_{1 \leq j_1 < \ldots < j_k \leq m} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \frac{\partial v_{i_1}}{\partial u_{j_{\sigma(1)}}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_{\sigma(k)}}} du^{j_1} \wedge \cdots \wedge du^{j_k}
\end{aligned}$$

By observing that $\displaystyle\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \frac{\partial v_{i_1}}{\partial u_{j_{\sigma(1)}}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_{\sigma(k)}}}$ is the determinant of $\displaystyle\left[\frac{\partial v_{i_p}}{\partial u_{j_q}}\right]_{1 \leq p,q \leq k}$, the desired result (3.18) follows easily.

The second result (3.19) follows directly from (3.18). In case of $\dim M = \dim N = n$ and $k = n$, the only possible strictly increasing sequence $1 \leq j_1 < \ldots < j_n \leq n$ is $(j_1, \ldots, j_n) = (1, 2, \ldots, n)$. □

Proposition 3.57. *Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. For any $\omega \in \wedge^k T^*N$, we have:*

$$\Phi^*(d\omega) = d(\Phi^*\omega). \tag{3.20}$$

To be precise, we say $\Phi^*(d_N\omega) = d_M(\Phi^*\omega)$, where $d_N : \wedge^k T^*N \to \wedge^{k+1} T^*N$ and $d_M : \wedge^k T^*M \to \wedge^{k+1} T^*M$ are the exterior derivatives on N and M respectively.

Proof. Let $\{u_j\}$ and $\{v_i\}$ be local coordinates of M and N respectively. By linearity, it suffices to prove (3.20) for the case $\omega = f\, dv^{i_1} \wedge \cdots \wedge dv^{i_k}$ where f is a locally defined scalar function. The proof follows from computing both LHS and RHS of (3.20):

$$\begin{aligned}
d\omega &= df \wedge dv^{i_1} \wedge \cdots \wedge dv^{i_k} \\
\Phi^*(d\omega) &= \Phi^*(df) \wedge \Phi^*(dv^{i_1}) \wedge \cdots \wedge \Phi^*(dv^{i_k}) \\
&= d(\Phi^* f) \wedge d(\Phi^* v^{j_1}) \wedge \cdots \wedge d(\Phi^* v^{j_k}).
\end{aligned}$$

Here we have used Exercise 3.50. On the other hand, we have:

$$\begin{aligned}
\Phi^*\omega &= (\Phi^* f)\, \Phi^*(dv^{j_1}) \wedge \cdots \wedge \Phi^*(dv^{j_k}) \\
&= (\Phi^* f)\, d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k}) \\
d(\Phi^*\omega) &= d(\Phi^* f) \wedge d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k}) \\
&\quad + \Phi^* f\, d\left(d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})\right)
\end{aligned}$$

Since $d^2 = 0$, each of $d(\Phi^* v^{i_q})$ is a closed 1-form. By Proposition 3.42 (product rule) and induction, we can conclude that:

$$d\left(d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})\right) = 0$$

and so $d(\Phi^*\omega) = d(\Phi^* f) \wedge d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})$ as desired. □

Exercise 3.53. Show that the pull-back of any closed form is closed, and the pull-back of any exact form is exact.

Exercise 3.54. Consider the unit sphere $\mathbb{S}^2$ locally parametrized by

$$F(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Define a map $\Phi : \mathbb{S}^2 \to \mathbb{R}^3$ by $\Phi(x, y, z) = (xz, yz, z^2)$, and consider a 2-form $\omega = z\,dx \wedge dy$. Compute $d\omega$, $\Phi^*(d\omega)$, $\Phi^*\omega$ and $d(\Phi^*\omega)$, and verify they satisfy Proposition 3.57.

3.5.6. Unification of Green's, Stokes' and Divergence Theorems. Given a submanifold M^m in $\mathbb{R}^n$, a differential form on $\mathbb{R}^n$ induces a differential form on M^m. For example, let C be a smooth regular curve in $\mathbb{R}^3$ parametrized by $\gamma(t) = (x(t), y(t), z(t))$. The 1-form:

$$\alpha = \alpha_x\,dx + \alpha_y\,dy + \alpha_z\,dz$$

is *a priori* defined on $\mathbb{R}^3$, but we can regard the coordinates (x, y, z) as functions on the curve C parametrized by $\gamma(t)$, then we have $dx = \frac{dx}{dt}\,dt$ and similarly for dy and dz. As such, dx can now be regarded as a 1-form on C. Therefore, the 1-form α on $\mathbb{R}^3$ induces a 1-form α (abuse in notation) on C:

$$\begin{aligned}\alpha &= \alpha_x(\gamma(t))\frac{dx}{dt}\,dt + \alpha_y(\gamma(t))\frac{dy}{dt}\,dt + \alpha_z(\gamma(t))\frac{dz}{dt}\,dt \\ &= \left(\alpha_x(\gamma(t))\frac{dx}{dt} + \alpha_y(\gamma(t))\frac{dy}{dt} + \alpha_z(\gamma(t))\frac{dz}{dt}\right)dt\end{aligned}$$

In practice, there is often no issue of using α to denote both the 1-form on $\mathbb{R}^3$ and its induced 1-form on C. To be (overly) rigorous over notations, we can use the inclusion map $\iota : C \to \mathbb{R}^3$ to distinguish them. The 1-form α on $\mathbb{R}^3$ is transformed into a 1-form $\iota^*\alpha$ on C by the pull-back of ι. From the previous subsection, we learned that:

$$\iota^*(dx) = d(\iota^*x) = d(x \circ \iota).$$

Note that dx and $d(x \circ \iota)$ are different in a sense that $x \circ \iota : C \to \mathbb{R}$ has the curve C as its domain, while $x : \mathbb{R}^3 \to \mathbb{R}$ has $\mathbb{R}^3$ as its domain. Therefore, we have:

$$d(x \circ \iota) = \frac{d(x \circ \iota)}{dt}\,dt = \frac{dx}{dt}\,dt.$$

In short, we may use $\iota^*(dx) = \frac{dx}{dt}\,dt$ to distinguish it from dx if necessary. Similarly, we may use $\iota^*\alpha$ to denote the induced 1-form of α on C:

$$\iota^*\alpha = \left(\alpha_x(\gamma(t))\frac{dx}{dt} + \alpha_y(\gamma(t))\frac{dy}{dt} + \alpha_z(\gamma(t))\frac{dz}{dt}\right)dt.$$

An induced 1-form on a curve in $\mathbb{R}^3$ is related to line integrals in Multivariable Calculus. Recall that the 1-form $\alpha = \alpha_x\,dx + \alpha_y\,dy + \alpha_z\,dz$ corresponds to the vector field $F = \alpha_x\hat{i} + \alpha_y\hat{j} + \alpha_z\hat{k}$ on $\mathbb{R}^3$. In Multivariable Calculus, we denote $dl = dx\hat{i} + dy\hat{j} + dz\hat{k}$ and

$$F \cdot dl = \left(\alpha_x\hat{i} + \alpha_y\hat{j} + \alpha_z\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \alpha.$$

The line integral $\int_C F \cdot dl$ over the curve $C \subset \mathbb{R}^3$ can be written using differential form notations:

$$\int_C F \cdot dl = \int_C \alpha \quad \text{or more rigorously:} \quad \int_C \iota^*\alpha.$$

Now consider a regular surface $M \subset \mathbb{R}^3$. Suppose $F(u,v) = (x(u,v), y(u,v), z(u,v))$ is a smooth local parametrization of M. Consider a vector $G = \beta_x \hat{i} + \beta_y \hat{j} + \beta_z \hat{k}$ on $\mathbb{R}^3$ and its corresponding 2-form on $\mathbb{R}^3$:

$$\beta = \beta_x \, dy \wedge dz + \beta_y \, dz \wedge dx + \beta_z \, dx \wedge dy.$$

Denote $\iota : M \to \mathbb{R}^3$ the inclusion map. The induced 2-form $\iota^*\beta$ on M is in fact related to the surface flux of G through M. Let's explain why:

$$\begin{aligned} \iota^*(dy \wedge dz) &= (\iota^* dy) \wedge (\iota^* dz) = d(y \circ \iota) \wedge d(z \circ \iota) \\ &= \left(\frac{\partial y}{\partial u} \, du + \frac{\partial y}{\partial v} \, dv \right) \wedge \left(\frac{\partial z}{\partial u} \, du + \frac{\partial z}{\partial v} \, dv \right) \\ &= \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) du \wedge dv \\ &= \det \frac{\partial(y,z)}{\partial(u,v)} \, du \wedge dv. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \iota^*(dz \wedge dx) &= \det \frac{\partial(z,x)}{\partial(u,v)} \, du \wedge dv \\ \iota^*(dx \wedge dy) &= \det \frac{\partial(x,y)}{\partial(u,v)} \, du \wedge dv \end{aligned}$$

All these show:

$$\iota^*\beta = \left(\beta_x \det \frac{\partial(y,z)}{\partial(u,v)} + \beta_y \det \frac{\partial(z,x)}{\partial(u,v)} + \beta_z \det \frac{\partial(x,y)}{\partial(u,v)} \right) du \wedge dv$$

Compared with the flux element $G \cdot \nu \, dS$ in Multivariable Calculus:

$$\begin{aligned} G \cdot \nu \, dS &= \underbrace{\left(\beta_x \hat{i} + \beta_y \hat{j} + \beta_z \hat{k} \right)}_{G} \cdot \underbrace{\frac{\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}}{\left| \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \right|}}_{\nu} \underbrace{\left| \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \right| dudv}_{dS} \\ &= \left(\beta_x \hat{i} + \beta_y \hat{j} + \beta_z \hat{k} \right) \cdot \left(\det \frac{\partial(y,z)}{\partial(u,v)} \hat{i} + \det \frac{\partial(z,x)}{\partial(u,v)} \hat{j} + \det \frac{\partial(x,y)}{\partial(u,v)} \hat{k} \right) \\ &= \left(\beta_x \det \frac{\partial(y,z)}{\partial(u,v)} + \beta_y \det \frac{\partial(z,x)}{\partial(u,v)} + \beta_z \det \frac{\partial(x,y)}{\partial(u,v)} \right) dudv, \end{aligned}$$

the only difference is that $\iota^*\beta$ is in terms of the wedge product $du \wedge dv$ while the flux element $G \cdot \nu \, dS$ is in terms of $dudv$. Ignoring this minor difference (which will be addressed in the next chapter), the surface flux $\iint_M G \cdot \nu \, dS$ can be expressed in terms of differential forms in the following way:

$$\iint_M G \cdot \nu \, dS = \iint_M \beta \quad \text{or more rigorously:} \quad \iint_M \iota^*\beta$$

Recall that the classical Stokes' Theorem is related to line integrals of a curve and surface flux of a vector field. Based on the above discussion, we see that Stokes' Theorem can be restated in terms of differential forms. Consider the 1-form $\alpha = \alpha_x \, dx + \alpha_y \, dy + \alpha_z \, dz$ and its corresponding vector field $F = \alpha_x \hat{i} + \alpha_y \hat{j} + \alpha_z \hat{k}$. We have already discussed that the 2-form $d\alpha$ corresponds to the vector field $\nabla \times F$. Therefore, the surface flux of the vector field $\nabla \times F$ through M can be expressed in terms of differential forms as:

$$\iint_M (\nabla \times F) \cdot \nu \, dS = \iint_M \iota^*(d\alpha) = \iint_M d(\iota^*\alpha).$$

If C is the boundary curve of M, then from our previous discussion we can write:

$$\int_C F \cdot dl = \int_C \iota^* \alpha.$$

The classical Stokes' Theorem asserts that:

$$\int_C F \cdot\, dl = \iint_M (\nabla \times F) \cdot \nu \, dS$$

which can be expressed in terms of differential form as:

$$\int_C \iota^* \alpha = \iint_M d(\iota^* \alpha) \quad \text{or simply:} \quad \int_C \alpha = \iint_M d\alpha.$$

Due to this elegant way (although not very practical for physicists and engineers) of expressing Stokes' Theorem, we often denote the boundary of a surface M as ∂M, then the classical Stokes' Theorem can be expressed as:

$$\int_{\partial M} \alpha = \iint_M d\alpha.$$

Using differential forms, one can also express Divergence Theorem in Multivariable Calculus in a similar way as above. Let D be a solid region in $\mathbb{R}^3$ and ∂D be the boundary surface of D. Divergence Theorem in MATH 2023 asserts that:

$$\iint_{\partial D} G \cdot \nu \, dS = \iiint_D \nabla \cdot G \, dV,$$

where $G = \beta_x \hat{\imath} + \beta_y \hat{\jmath} + \beta_z \hat{k}$. As discussed before, the LHS is $\displaystyle\iint_{\partial D} \beta$ where $\beta = \beta_x \, dy \wedge dz + \beta_y \, dz \wedge dx + \beta_z \, dx \wedge dy$. We have seen that:

$$d\beta = \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} + \frac{\partial \beta_z}{\partial z} \right) dx \wedge dy \wedge dz,$$

which is (almost) the same as:

$$\nabla \cdot G \, dV = \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} + \frac{\partial \beta_z}{\partial z} \right) dxdydz.$$

Hence, the RHS of Divergence Theorem can be expressed as $\displaystyle\iiint_D d\beta$; and therefore we can rewrite Divergence Theorem as:

$$\iint_{\partial D} \beta = \iiint_D d\beta.$$

Again, the same expression! Stokes' and Divergence Theorems can therefore be unified. Green's Theorem can also be unified with Stokes' and Divergence Theorems as well. Try the exercise below:

Exercise 3.55. Let C be a simple closed smooth curve in $\mathbb{R}^2$ and R be the region enclosed by C in $\mathbb{R}^2$. Given a smooth vector field $F = P\hat{\imath} + Q\hat{\jmath}$ on $\mathbb{R}^2$, Green's Theorem asserts that:

$$\int_C F \cdot dl = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Express Green's Theorem using the language of differential forms.

3.5.7. Differential Forms and Maxwell's Equations. The four Maxwell's equations are a set of partial differential equations that form the foundation of electromagnetism. Denote the components of the electric field e, magnetic field B, and current density $\hat{j}$ by

$$\begin{aligned} \mathsf{E} &= E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k} \\ \mathsf{B} &= B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k} \\ \mathsf{J} &= j_x\hat{\imath} + j_y\hat{\jmath} + j_z\hat{k} \end{aligned}$$

All components of e, B and $\hat{j}$ are considered to be time-dependent. Denote ρ to be the charge density. The four Maxwell's equations assert that:

$$\begin{aligned} \nabla \cdot \mathsf{E} &= \rho & \nabla \cdot \mathsf{B} &= 0 \\ \nabla \times \mathsf{E} &= -\frac{\partial \mathsf{B}}{\partial t} & \nabla \times \mathsf{B} &= \mathsf{J} + \frac{\partial e}{\partial t} \end{aligned}$$

These four equations can be rewritten using differential forms in a very elegant way. Consider $\mathbb{R}^4$ with coordinates (t, x, y, z), which is also denoted as (x_0, x_1, x_2, x_3) in this problem. First we introduce the Minkowski Hodge-star operator $*$ on $\mathbb{R}^4$, which is a linear map taking p-forms on $\mathbb{R}^4$ to $(4-p)$-forms on $\mathbb{R}^4$. In particular, for 2-forms $\omega = dx^i \wedge dx^j$ (where $i, j = 0, 1, 2, 3$ and $i \neq j$), we define $*\omega$ to be the unique 2-form on $\mathbb{R}^4$ such that:

$$\omega \wedge *\omega = \begin{cases} dt \wedge dx \wedge dy \wedge dz & \text{if } i, j \neq 0 \\ -dt \wedge dx \wedge dy \wedge dz & \text{otherwise.} \end{cases}$$

For instance, $*(dx \wedge dy) = dt \wedge dz$ since $dx \wedge dy \wedge dt \wedge dz = dt \wedge dx \wedge dy \wedge dz$ and there is no dt term in $dx \wedge dy$. On the other hand, $*(dt \wedge dx) = -dy \wedge dz$ since there is a dt term in $dt \wedge dx$. The operator $*$ then extends linearly to all 2-forms on $\mathbb{R}^4$.

Exercise 3.56. Compute each of the following:

$$\begin{aligned} &*(dt \wedge dx) && *(dt \wedge dy) && *(dt \wedge dz) \\ &*(dx \wedge dy) && *(dy \wedge dz) && *(dz \wedge dx) \end{aligned}$$

To convert the Maxwell's equations using the language of differential forms, we define the following analogue of E, B, J and ρ using differential forms:

$$\begin{aligned} E &= E_x\,dx + E_y\,dy + E_z\,dz \\ B &= B_x\,dy \wedge dz + B_y\,dz \wedge dx + B_z\,dx \wedge dy \\ J &= -(j_x\,dy \wedge dz + j_y\,dz \wedge dx + j_z\,dx \wedge dy) \wedge dt + \rho\,dx \wedge dy \wedge dz \end{aligned}$$

Note that E_i's and B_j's may depend on t although there is no dt above. Define the 2-form:

$$F := B + E \wedge dt.$$

Exercise 3.57. Show that the four Maxwell's equations can be rewritten in an elegant way as:

$$\begin{aligned} dF &= 0 \\ d\,(*F) &= J \end{aligned}$$

where d is the exterior derivative on $\mathbb{R}^4$.

3.5.8. Global Expressions of Exterior Derivatives. We defined exterior differentiation using local coordinates. In fact, using Lie derivatives, one can derive a global expression (i.e. without using local coordinates) of exterior derivatives on differential forms.

We first introduce Lie derivatives on $(p,0)$-tensors, which are similarly defined as those on 1-forms. Let T be a $(p,0)$-tensor and X be a vector field on M. Denote the flow map of X by Φ_t, then the Lie derivative of T along X at $p \in M$ is defined as:

$$(\mathcal{L}_X T)_p := \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* \left(T_{\Phi_t(p)}\right).$$

Exercise 3.58. Guess the definition of Lie derivatives of a general (p,q)-tensor along a vector field X. Check any standard textbook to see if your guess is right.

Remark 3.58. On a regular surface M in $\mathbb{R}^3$ with the first fundamental form denoted by g, if X is a vector field on M such that $\mathcal{L}_X g = 0$, then we call X to be a *Killing vector field*. The geometric meaning of such an X is that g is invariant when M moves along the vector field X, or equivalently, g is symmetric in the direction of X. This concept of Killing vector fields can be generalize to Riemannian manifolds and is important in Differential Geometry and General Relativity, whenever *symmetry* plays an important role. □

Since the pull-back of a tensor product satisfies $\Phi^*(T \otimes S) = \Phi^* T \otimes \Phi^* S$, it is easy to show from definition that the Lie derivative satisfies the product rule:

$$\mathcal{L}_X (T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S). \tag{3.21}$$

Exercise 3.59. Prove (3.21).

Since differential forms are simply linear combinations of tensor products, the definition of their Lie derivatives is the same as that for $(k,0)$-tensors. One nice fact about Lie derivatives on differential forms is so-called the *Cartan's magic formula*, which relates Lie derivatives and exterior derivatives. We first introduce the interior product:

Definition 3.59 (Interior Product)**.** Let α be a k-form (where $k \geq 2$) on a manifold M, and X be a vector field on M. Then, the *interior product* $i_X \alpha$ is a $(k-1)$-form defined as follows. For any vector fields $Y_1, \ldots, Y_{k-1}$ on M, we define:

$$(i_X \alpha)(Y_1, \ldots, Y_{k-1}) := \alpha(X, Y_1, \ldots, Y_{k-1}).$$

Example 3.60. In local coordinates, if a vector field X can be written as $X = \sum_{i=1}^n X^i \frac{\partial}{\partial u_i}$, then $i_X(du^j \wedge du^k)$ is an 1-form and we have:

$$\left(i_X(du^j \wedge du^k)\right)\left(\frac{\partial}{\partial u_l}\right) = (du^j \wedge du^k)\left(X, \frac{\partial}{\partial u_l}\right) = X^j \delta_{kl} - X^k \delta_{jl}.$$

In other words, we have:

$$i_X(du^j \wedge du^k) = \sum_{l=1}^n (X^j \delta_{kl} - X^k \delta_{jl})\, du^l = X^j\, du^k - X^k\, du^j.$$

□

Exercise 3.60. Let $X = \sum_{i=1}^n X^i \frac{\partial}{\partial u_i}$ be a vector field on a manifold M with local coordinates $(u_1, \ldots, u_n)$. Derive the local expression of:

$$i_X(du^{j_1} \wedge du^{j_2} \wedge \cdots \wedge du^{j_k})$$

where $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.

Now we are ready to present a beautiful and elegant formula due to Elie Cartan:

Proposition 3.61 (Cartan's Magic Formula). *Let X be a smooth vector field on a manifold M, then for any differential k-form ω, we have:*

$$\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega) \tag{3.22}$$

Proof. The proof is by induction on k, the degree of ω. We first show that (3.22) holds for 1-forms.

Consider $\omega = \sum_{j=1}^n \omega_j\, du^j$, we have already computed in (3.11) that:

$$\mathcal{L}_X\omega = \sum_{i,j=1}^n \left(X^i \frac{\partial \omega_j}{\partial u_i} + \omega_i \frac{\partial X^i}{\partial u_j} \right) du^j.$$

Next we verify it is equal to the RHS of (3.22).

$$d\omega = \sum_{i,j=1}^n \frac{\partial \omega_j}{\partial u_i}\, du^i \wedge du^j$$

From Example 3.60, we have

$$i_X(d\omega) = \sum_{i,j=1}^n \frac{\partial \omega_j}{\partial u_i} i_X(du^i \wedge du^j) = \sum_{i,j=1}^n \frac{\partial \omega_j}{\partial u_i}(X^i\, du^j - X^j\, du^i).$$

Moreover, we have

$$i_X\omega = \omega(X) = \sum_{j=1}^n X^j \omega_j$$

$$d(i_X\omega) = \sum_{i,j=1}^n \left(\frac{\partial X^j}{\partial u_i}\omega_j + X^j \frac{\partial \omega_j}{\partial u_i} \right) du^i$$

and it follows easily that:

$$i_X(d\omega) + d(i_X\omega) = \sum_{i,j=1}^n X^i \frac{\partial \omega_j}{\partial u_i}\, du^j + \sum_{i,j=1}^n \omega_j \frac{\partial X^j}{\partial u_i}\, du^i$$

which is exactly $\mathcal{L}_X\omega$ after relabelling indices.

Now that (3.22) holds for 1-form. To complete the inductive proof, we just need to show that if (3.22) holds for both differential forms ω and σ, then it also holds for $\omega \wedge \sigma$. It is left as an exercise for readers. $\square$

Exercise 3.61. Complete the above inductive proof. [Note: the proof is somewhat algebraic.]

Exercise 3.62. Show that if ω is closed, then $\mathcal{L}_X\omega$ is exact for any vector field X.

The purpose of introducing Cartan's magic formula is it gives a coordinate-free expression of exterior derivatives. Consider a 1-form ω, and two vector fields X and Y. Then, from (3.22), we have:

$$(\mathcal{L}_X\omega)(Y) = (i_X(d\omega))\,(Y) + (d(i_X\omega))\,(Y),$$

which, from the definition of i_X and (3.19), can be simplified to:

$$X(\omega(Y)) - \omega(\mathcal{L}_X Y) = (d\omega)(X,Y) + d(\omega(X))(Y).$$

As $\omega(X)$ is a scalar function, we also have:

$$d(\omega(X))(Y) = Y\,(\omega(X))\,.$$

[Note that generally, $(df)(Y) = Y(f)$ for any scalar function f.]

Finally, we get:

$$(d\omega)(X,Y) = X\,(\omega(Y)) - Y\,(\omega(X)) - \omega([X,Y]) \tag{3.23}$$

for any vector fields X and Y. This is a global formula for $d\omega$ as it does not involve any local coordinates.

The expression (3.23) can be generalized to k-forms ω. The proof is by induction and the Cartan's magic formula again. For any k-form ω, and vector fields $X_0, X_1, \ldots, X_k$, we have:

$$\begin{aligned}
&(d\omega)(X_0, X_1, \cdots, X_k) \\
&= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \cdots, \hat{X}_i, \cdots, X_k)) \\
&\quad + \sum_{0\le i<j\le k} (-1)^{i+j}\omega\left([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k\right).
\end{aligned}$$

Readers interested in the proof may consult [**Lee13**, P.370, Proposition 14.32].

Chapter 4

Generalized Stokes' Theorem

"It is very difficult for us, placed as we have been from earliest childhood in a condition of training, to say what would have been our feelings had such training never taken place."

Sir George Stokes, 1st Baronet

4.1. Manifolds with Boundary

We have seen in the Chapter 3 that Green's, Stokes' and Divergence Theorem in Multivariable Calculus can be unified together using the language of differential forms. In this chapter, we will generalize Stokes' Theorem to higher dimensional and abstract manifolds.

These classic theorems and their generalizations concern about an integral over a manifold with an integral over its boundary. In this section, we will first rigorously define the notion of a *boundary* for abstract manifolds. Heuristically, an *interior* point of a manifold locally looks like a ball in Euclidean space, whereas a *boundary* point locally looks like an upper-half space.

4.1.1. Smooth Functions on Upper-Half Spaces. From now on, we denote $\mathbb{R}^n_+ := \{(u_1, \ldots, u_n) \in \mathbb{R}^n : u_n \geq 0\}$ which is the upper-half space of $\mathbb{R}^n$. Under the subspace topology, we say a subset $V \subset \mathbb{R}^n_+$ is *open in* $\mathbb{R}^n_+$ if there exists a set $\widetilde{V} \subset \mathbb{R}^n$ open in $\mathbb{R}^n$ such that $V = \widetilde{V} \cap \mathbb{R}^n_+$. It is intuitively clear that if $V \subset \mathbb{R}^n_+$ is disjoint from the subspace $\{u_n = 0\}$ of $\mathbb{R}^n$, then V is open in $\mathbb{R}^n_+$ if and only if V is open in $\mathbb{R}^n$.

Now consider a set $V \subset \mathbb{R}^n_+$ which is open in $\mathbb{R}^n_+$ and that $V \cap \{u_n = 0\} \neq \emptyset$. We need to first develop a notion of differentiability for functions such an V as their domain. Given a vector-valued function $G : V \to \mathbb{R}^m$, then near a point $u \in V \cap \{u_n = 0\}$, we can only approach u from one side only, namely from directions with positive u_n-coordinates. The usual definition of differentiability does not apply at such a point, so we define:

Definition 4.1 (Functions of Class C^k on $\mathbb{R}^n_+$). Let $V \subset \mathbb{R}^n_+$ be open in $\mathbb{R}^n_+$ and that $V \cap \{u_n = 0\} \neq \emptyset$. Consider a vector-valued function $G : V \to \mathbb{R}^m$. We say G is C^k (resp. smooth) at $u \in V \cap \{u_n = 0\}$ if there exists a C^k (resp. smooth) local extension $\widetilde{G} : B_\varepsilon(u) \to \mathbb{R}^m$ such that $\widetilde{G}(y) = G(y)$ for any $y \in B_\varepsilon(u) \cap V$. Here $B_\varepsilon(u) \subset \mathbb{R}^n$ refers to an open ball in $\mathbb{R}^n$.

If G is C^k (resp. smooth) at every $u \in V$ (including those points with $u_n > 0$), then we say G is C^k (resp. smooth) on V.

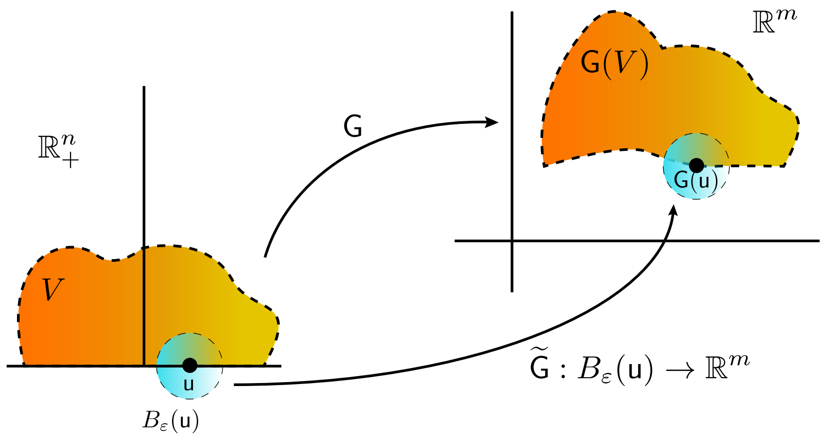


Figure 4.1. G is C^k at u if there exists a local extension $\widetilde{G}$ near u.

Example 4.2. Let $V = \{(x,y) : y \geq 0 \text{ and } x^2 + y^2 < 1\}$, which is an open set in $\mathbb{R}^2_+$ since $V = \underbrace{\{(x,y) : x^2 + y^2 < 1\}}_{\text{open in } \mathbb{R}^2} \cap \mathbb{R}^2_+$. Then $f(x,y) : V \to \mathbb{R}$ defined by $f(x,y) = \sqrt{1 - x^2 - y^2}$ is a smooth function on V since $\sqrt{1 - x^2 - y^2}$ is smoothly on the whole ball $x^2 + y^2 < 1$.

However, the function $g : V \to \mathbb{R}$ defined by $g(x,y) = \sqrt{y}$ is not smooth at every point on the y-axis because $\frac{\partial g}{\partial y} \to \infty$ as $y \to 0^+$. Any extension $\widetilde{g}$ of g will agree with g on the upper-half plane, and hence will also be true that $\frac{\partial \widetilde{g}}{\partial y} \to \infty$ as $y \to 0^+$, which is sufficient to argue that such $\widetilde{g}$ is not smooth. □

Exercise 4.1. Consider $f : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $f(x,y) = |x|$. Is f smooth on $\mathbb{R}^2_+$? If not, at which point(s) in $\mathbb{R}^2_+$ is f not smooth? Do the same for $g : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $g(x,y) = |y|$.

4.1.2. Boundary of Manifolds. After understanding the definition of a smooth function when defined on subsets of the upper-half space, we are ready to introduce the notion of manifolds with boundary:

Definition 4.3 (Manifolds with Boundary)**.** We say M is a smooth *manifold with boundary* if there exist two families of local parametrizations $F_\alpha : \mathcal{U}_\alpha \to M$ where $\mathcal{U}_\alpha$ is open in $\mathbb{R}^n$, and $G_\beta : \mathcal{V}_\beta \to M$ where $\mathcal{V}_\beta$ is open in $\mathbb{R}^n_+$ such that every F_α and G_β is a homeomorphism between its domain and image, and that the transition functions of all types:

$$F_\alpha^{-1} \circ F_{\alpha'} \qquad F_\alpha^{-1} \circ G_\beta \qquad G_\beta^{-1} \circ G_{\beta'} \qquad G_\beta^{-1} \circ F_\alpha$$

are smooth on the overlapping domain for any α, α', β and β'.

Moreover, we denote and define the boundary of M by:

$$\partial M := \bigcup_\beta \{G_\beta(u_1, \ldots, u_{n-1}, 0) : (u_1, \ldots, u_{n-1}, 0) \in \mathcal{V}_\beta\}.$$

Remark 4.4. In this course, we will call these F_α's to be local parametrizations of *interior type*, and these G_β's to be local parametrizations of *boundary type*. □

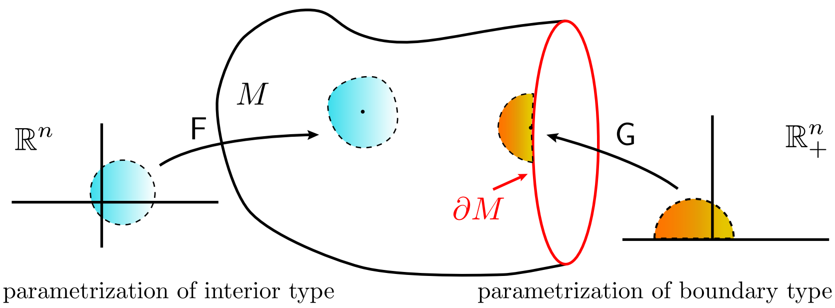


Figure 4.2. A manifold with boundary

Example 4.5. Consider the solid ball $\mathbb{B}^2 := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. It can be locally parametrized using polar coordinates by:

$$G : (0, 2\pi) \times [0, 1) \to \mathbb{B}^2$$
$$G(\theta, r) := (1 - r)(\cos\theta, \sin\theta)$$

Note that the domain of G can be regarded as a subset

$$\mathcal{V} := \{(\theta, r) : \theta \in (0, 2\pi) \text{ and } 0 \leq r < 1\} \subset \mathbb{R}^2_+.$$

Here we used $1 - r$ instead of r so that the boundary of $\mathbb{B}^2$ has zero r-coordinate, and the interior of $\mathbb{B}^2$ has positive r-coordinate.

Note that the image of G does not cover the whole solid ball $\mathbb{B}^2$. Precisely, the image of G is $\mathbb{B}^2 \backslash \{$non-negative x-axis$\}$. In order to complete the proof that $\mathbb{B}^2$ is a manifold with boundary, we cover $\mathbb{B}^2$ by two more local parametrizations:

$$\widetilde{G} : (-\pi, \pi) \times [0, 1) \to \mathbb{B}^2$$
$$\widetilde{G}(\theta, r) := (1 - r)(\cos\theta, \sin\theta)$$

and also the inclusion map $\iota : \{u \in \mathbb{R}^2 : |u| < 1\} \to \mathbb{B}^2$. We need to show that the transition maps are smooth. There are six possible transition maps:

$$\widetilde{G}^{-1} \circ G, \quad G^{-1} \circ \widetilde{G}, \quad \iota^{-1} \circ G, \quad \iota^{-1} \circ \widetilde{G}, \quad G^{-1} \circ \iota, \quad \text{and} \quad \widetilde{G}^{-1} \circ \iota.$$

The first one is given by (we leave it as an exercise for computing these transition maps):

$$\widetilde{G}^{-1} \circ G : ((0, \pi) \cup (\pi, 2\pi)) \times [0, 1) \to ((-\pi, 0) \cup (0, \pi)) \times [0, 1)$$

$$\widetilde{G}^{-1} \circ G(\theta, r) = \begin{cases} (\theta, r) & \text{if } \theta \in (0, \pi) \\ (\theta - 2\pi, r) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

which can be smoothly extended to the domain $((0, \pi) \cup (\pi, 2\pi)) \times (-1, 1)$. Therefore, $\widetilde{G}^{-1} \circ G$ is smooth. The second transition map $G^{-1} \circ \widetilde{G}$ can be computed and verified to be smooth in a similar way.

For $\iota^{-1} \circ G$, by examining the overlap part of ι and G on $\mathbb{B}^2$, we see that the domain of the transition map is an open set $(0, 2\pi) \times (0, 1)$ in $\mathbb{R}^2$. On this domain, $\iota^{-1} \circ G$ is essentially G, which is clearly smooth. Similar for $\iota^{-1} \circ \widetilde{G}$.

To show $G^{-1} \circ \iota$ is smooth, we use the Inverse Function Theorem. The domain of $\iota^{-1} \circ G$ is $(0, 2\pi) \times (0, 1)$. By writing $(x, y) = \iota^{-1} \circ G(\theta, r) = (1 - r)(\cos\theta, \sin\theta)$, we check that on the domain of $\iota^{-1} \circ G$, we have:

$$\det \frac{\partial(x, y)}{\partial(\theta, r)} = 1 - r \neq 0.$$

Therefore, the inverse $G^{-1} \circ \iota$ is smooth. Similar for $\widetilde{G}^{-1} \circ \iota$.

Combining all of the above verifications, we conclude that $\mathbb{B}^2$ is a 2-dimensional manifold with boundary. The boundary $\partial\mathbb{B}^2$ is given by points with zero r-coordinates, namely the unit circle $\{|x| = 1\}$. $\square$

Exercise 4.2. Compute all transition maps

$$\widetilde{G}^{-1} \circ G, \quad G^{-1} \circ \widetilde{G}, \quad \iota^{-1} \circ G, \quad \iota^{-1} \circ \widetilde{G}, \quad G^{-1} \circ \iota, \quad \text{and} \quad \widetilde{G}^{-1} \circ \iota$$

in Example 4.5. Indicate clearly their domains, and verify that they are smooth on their domains.

Exercise 4.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar function. The region in $\mathbb{R}^{n+1}$ above the graph of f is given by:

$$\Gamma_f := \{(u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1} : u_{n+1} \geq f(u_1, \ldots, u_n)\}.$$

Show that Γ_f is an n-dimensional manifold with boundary, and the boundary $\partial\Gamma_f$ is the graph of f in $\mathbb{R}^{n+1}$.

Exercise 4.4. Show that ∂M (assumed non-empty) of any n-dimensional manifold M is an $(n-1)$-dimensional manifold without boundary.

From the above example and exercise, we see that verifying a set is a manifold with boundary may be cumbersome. The following proposition provides us with a very efficient way to do so.

Proposition 4.6. *Let $f : M^m \to \mathbb{R}$ be a smooth function from a smooth manifold M. Suppose $c \in \mathbb{R}$ such that the set $\Sigma := f^{-1}([c, \infty))$ is non-empty and that f is a submersion at any $p \in f^{-1}(c)$, then the set Σ is an m-dimensional manifold with boundary. The boundary $\partial\Sigma$ is given by $f^{-1}(c)$.*

Proof. We need to construct local parametrizations for the set Σ. Given any point $p \in \Sigma$, then by the definition of Σ, we have $f(p) > c$ or $f(p) = c$.

For the former case $f(p) > c$, we are going to show that near p there is a local parametrization of Σ of interior type. Regarding p as a point in the manifold M, there exists a smooth local parametrization $F : \mathcal{U} \subset \mathbb{R}^n \to M$ of M covering p. We argue that such a local parametrization of M induces naturally a local parametrization of Σ near p. Note that f is continuous and so $f^{-1}(c, \infty)$ is an open set of M containing p. Denote $\mathcal{O} = f^{-1}(c, \infty)$, then F restricted to $\mathcal{U} \cap F^{-1}(\mathcal{O})$ will have its image in $\mathcal{O} \subset \Sigma$, and so is a local parametrization of Σ near p.

For the later case $f(p) = c$, we are going to show that near p there is a local parametrization of Σ of boundary type. Since f is a submersion at p, by the Submersion Theorem (Theorem 2.48) there exist a local parametrization $G : \widetilde{\mathcal{U}} \to M$ of M near p, and a local parametrization H of $\mathbb{R}$ near c such that $G(0) = p$ and $H(0) = c$, and:

$$H^{-1} \circ f \circ G(u_1, \ldots, u_m) = u_m.$$

Without loss of generality, we assume that H is an increasing function near 0. We argue that by restricting the domain of G to $\mathcal{U} \cap \{u_m \geq 0\}$, which is an open set in $\mathbb{R}^m_+$, the restricted G is a boundary-type local parametrization of Σ near p. To argue this, we note that:

$$f(G(u_1, \ldots, u_m)) = H(u_m) \geq H(0) = c \quad \text{ whenever } u_m \geq 0.$$

Therefore, $G(u_1, \ldots, u_m) \in f^{-1}([c, \infty)) = \Sigma$ whenever $u_m \geq 0$, and so G (when restricted to $\mathcal{U} \cap \{u_m \geq 0\}$) is a local parametrization of Σ.

Since all local parametrizations F and G of Σ constructed above are induced from local parametrizations of M (whether it is of interior or boundary type), their transition maps are all smooth. This shows Σ is an m-dimensional manifold with boundary. To identify the boundary, we note that for any boundary-type local parametrization G constructed above, we have:

$$H^{-1} \circ f \circ G(u_1, \ldots, u_{m-1}, 0) = 0$$

and so $f(G(u_1, \ldots, u_{m-1})) = H(0) = c$, and therefore:

$$G(u_1, \ldots, u_{m-1}, 0) \in f^{-1}(c).$$

This show $\partial\Sigma \subset f^{-1}(c)$. The other inclusion $f^{-1}(c) \subset \partial\Sigma$ follows from the fact that for any $p \in f^{-1}(c)$, the boundary-type local parametrization G has the property that $G(0) = p$ (and hence $p = G(0, \ldots, 0, 0) \in \partial\Sigma$). □

Remark 4.7. It is worthwhile to note that the above proof only requires that f is a submersion at any $p \in f^{-1}(c)$, and we do not require that it is a submersion at any $p \in \Sigma = f^{-1}([c, \infty))$. Furthermore, the codomain of f is $\mathbb{R}$ which has dimension 1, hence f is a submersion at p if and only if the tangent map $(f_*)_p$ at p is non-zero – and so it is very easy to verify this condition. □

With the help of Proposition 4.6, one can show many sets are manifolds with boundary by picking a suitable submersion f.

Example 4.8. The n-dimensional ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is an n-manifold with boundary. To argue this, let $f : \mathbb{R}^n \to \mathbb{R}$ be the function:

$$f(x) = 1 - |x|^2.$$

Then $\mathbb{B}^n = f^{-1}([0, \infty))$.

The tangent map f_* is represented by the matrix:

$$[f_*] = \left[\frac{\partial f}{\partial x_1}, \ \cdots \ , \ \frac{\partial f}{\partial x_n}\right] = -2\,[x_1, \ \cdots \ , \ x_n]$$

which is surjective if and only if $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$. For any $x \in f^{-1}(0)$, we have $|x|^2 = 1$ and so in particular $x \neq 0$. Therefore, f is a submersion at every $x \in f^{-1}(0)$. By Proposition 4.6, we proved $\mathbb{B}^n = f^{-1}([0, \infty))$ is an n-dimensional manifold with boundary, and the boundary is $f^{-1}(0) = \{x \in \mathbb{R}^n : |x| = 1\}$, i.e. the unit circle. □

Exercise 4.5. Suppose $f : M^m \to \mathbb{R}$ is a smooth function defined on a smooth manifold M. Suppose $a, b \in \mathbb{R}$ such that $\Sigma := f^{-1}([a, b])$ is non-empty, and that f is a submersion at any $p \in f^{-1}(a)$ and any $q \in f^{-1}(b)$. Show that Σ is an m-manifold with boundary, and $\partial\Sigma = f^{-1}(a) \cup f^{-1}(b)$.

4.1.3. Tangent Spaces at Boundary Points. On a manifold M^n without boundary, the tangent space T_pM at p is the span of partial differential operators $\left\{ \dfrac{\partial}{\partial u_i}\bigg|_p \right\}_{i=1}^{n}$, where $(u_1, \ldots, u_n)$ are local coordinates of a parametrization $F(u_1, \ldots, u_n)$ near p.

Now on a manifold M^n with boundary, near any boundary point $p \in \partial M^n$ there exists a local parametrization $G(u_1, \ldots, u_n) : \mathcal{V} \subset \mathbb{R}^n_+ \to M$ of boundary type. Although G is only defined when $u_n \geq 0$, we *still* define T_pM to be the span of $\left\{ \dfrac{\partial}{\partial u_i}\bigg|_p \right\}_{i=1}^{n}$. Although such a definition of T_pM (when $p \in \partial M$) is a bit counter-intuitive, the perk is that T_pM is still a vector space. Given a vector $V \in T_pM$ with coefficients:

$$V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial u_i}\bigg|_p .$$

We say that V is *inward-pointing* if $V^n > 0$; and *outward-pointing* if $V^n < 0$.

Furthermore, the tangent space $T_p(\partial M)$ of the boundary manifold ∂M at p can be regarded as a subspace of T_pM:

$$T_p(\partial M) = \text{span} \left\{ \frac{\partial}{\partial u_i}\bigg|_p \right\}_{i=1}^{n-1} \subset T_pM.$$

4.2. Orientability

In Multivariable Calculus, we learned (or was told) that Stokes' Theorem requires the surface to be orientable, meaning that the unit normal vector ν varies continuously on the surface. The Möbius strip is an example of *non-orientable* surface.

Now we are talking about abstract manifolds which may not sit inside any Euclidean space, and so it does not make sense to define *normal vectors* to the manifold. Even when the manifold M is a subset of $\mathbb{R}^n$, if the dimension of the manifold is $\dim M \leq n-2$, the manifold does not have a unique normal vector direction. As such, in order to generalize the notion of orientability of abstract manifolds, we need to seek a reasonable definition without using normal vectors.

In this section, we first show that for hypersurfaces M^n in $\mathbb{R}^{n+1}$, the notion of orientability using normal vectors is equivalent to another notion using transition maps. Then, we extend the notion of orientability to abstract manifolds using transition maps.

4.2.1. Orientable Hypersurfaces. To begin, we first state the definition of orientable hypersurfaces in $\mathbb{R}^{n+1}$:

> **Definition 4.9** (Orientable Hypersurfaces). A regular hypersurface M^n in $\mathbb{R}^{n+1}$ is said to be *orientable* if there exists a continuous unit normal vector ν defined on the whole M^n

Let's explore the above definition a bit in the easy case $n = 2$. Given a regular surface M^2 in $\mathbb{R}^3$ with a local parametrization $(x, y, z) = F(u_1, u_2) : \mathcal{U} \to M$, one can find a normal vector to the surface by taking cross product:

$$\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} = \det \frac{\partial(y,z)}{\partial(u_1,u_2)}\hat{i} + \det \frac{\partial(z,x)}{\partial(u_1,u_2)}\hat{j} + \det \frac{\partial(x,y)}{\partial(u_1,u_2)}\hat{k}$$

and hence the unit normal along this direction is given by:

$$\nu_F = \frac{\det \frac{\partial(y,z)}{\partial(u_1,u_2)}\hat{i} + \det \frac{\partial(z,x)}{\partial(u_1,u_2)}\hat{j} + \det \frac{\partial(x,y)}{\partial(u_1,u_2)}\hat{k}}{\left|\det \frac{\partial(y,z)}{\partial(u_1,u_2)}\hat{i} + \det \frac{\partial(z,x)}{\partial(u_1,u_2)}\hat{j} + \det \frac{\partial(x,y)}{\partial(u_1,u_2)}\hat{k}\right|} \quad \text{on } F(\mathcal{U}).$$

Note that the above ν is defined locally on the domain $F(\mathcal{U})$.

Now given another local parametrization $(x, y, z) = G(v_1, v_2) : \mathcal{V} \to M$, one can find a unit normal using G as well:

$$\nu_G = \frac{\det \frac{\partial(y,z)}{\partial(v_1,v_2)}\hat{i} + \det \frac{\partial(z,x)}{\partial(v_1,v_2)}\hat{j} + \det \frac{\partial(x,y)}{\partial(v_1,v_2)}\hat{k}}{\left|\det \frac{\partial(y,z)}{\partial(v_1,v_2)}\hat{i} + \det \frac{\partial(z,x)}{\partial(v_1,v_2)}\hat{j} + \det \frac{\partial(x,y)}{\partial(v_1,v_2)}\hat{k}\right|} \quad \text{on } G(\mathcal{V}).$$

Using the chain rule, we have the following relation between the Jacobian determinants:

$$\det \frac{\partial(*,**)}{\partial(v_1,v_2)} = \det \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)} \det \frac{\partial(*,**)}{\partial(u_1,u_2)}$$

(here $*$ and $**$ mean any of the x, y and z) and therefore ν_F and ν_G are related by:

$$\nu_G = \frac{\det \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}}{\left|\det \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\right|}\nu_F.$$

Therefore, if there is an overlap between local coordinates (u_1, u_2) and (v_1, v_2), the unit normal vectors ν_F and ν_G agree with each other on the overlap $F(\mathcal{U}) \cap G(\mathcal{V})$ if and only if $\det \dfrac{\partial(u_1,u_2)}{\partial(v_1,v_2)} > 0$ (equivalently, $\det D(F^{-1} \circ G) > 0$).

From above, we see that consistency of unit normal vector on different local coordinate charts is closely related to the positivity of the determinants of transition maps. A consistence choice of unit normal vector ν exists if and only if it is possible to pick a family of local parametrizations $F_\alpha : \mathcal{U}_\alpha \to M^2$ covering the whole M such that $\det D(F_\beta^{-1} \circ F_\alpha) > 0$ on $F_\alpha^{-1}\left(F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta)\right)$ for any α and β in the family. The notion of normal vectors makes sense only for hypersurfaces in $\mathbb{R}^n$, while the notion of transition maps can extend to any abstract manifold.

Note that given two local parametrizations $F(u_1, u_2)$ and $G(v_1, v_2)$, it is *not* always possible to make sure $\det \dfrac{\partial(u_1, u_2)}{\partial(v_1, v_2)} > 0$ on the overlap even by switching v_1 and v_2. It is because it sometimes happens that the overlap $F(\mathcal{U}) \cap G(\mathcal{V})$ is a disjoint union of two open sets. If on one open set the determinant is positive, and on another one the determinant is negative, then switching v_1 and v_2 cannot make the determinant positive on both open sets. Let's illustrate this issue through two contrasting examples: the cylinder and the Möbius strip:

Example 4.10. The unit cylinder Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$F : (0, 2\pi) \times \mathbb{R} \to \Sigma^2 \qquad \widetilde{F} : (-\pi, \pi) \times \mathbb{R} \to \Sigma^2$$
$$F(\theta, z) := (\cos\theta, \sin\theta, z) \qquad \widetilde{F}(\widetilde{\theta}, \widetilde{z}) := (\cos\widetilde{\theta}, \sin\widetilde{\theta}, \widetilde{z})$$

Then, the transition map $\widetilde{F}^{-1} \circ F$ is defined on a disconnected domain $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $z \in \mathbb{R}$, and it is given by:

$$\widetilde{F}^{-1} \circ F(\theta, z) = \begin{cases} (\theta, z) & \text{if } \theta \in (0, \pi) \\ (\theta - 2\pi, z) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

By direct computations, the Jacobian of this transition map is given by:

$$D(\widetilde{F}^{-1} \circ F)(\theta, z) = I$$

in either case $\theta \in (0, \pi)$ or $\theta \in (\pi, 2\pi)$. Therefore, $\det D(\widetilde{F}^{-1} \circ F) > 0$ on the overlap.

The unit normal vectors defined using these F and $\widetilde{F}$:

$$\nu_F = \frac{\frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta}}{\left|\frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta}\right|} \quad \text{on } F((0, 2\pi) \times \mathbb{R})$$
$$\nu_{\widetilde{F}} = \frac{\frac{\partial \widetilde{F}}{\partial \widetilde{r}} \times \frac{\partial \widetilde{F}}{\partial \widetilde{\theta}}}{\left|\frac{\partial \widetilde{F}}{\partial \widetilde{r}} \times \frac{\partial \widetilde{F}}{\partial \widetilde{\theta}}\right|} \quad \text{on } \widetilde{F}((-\pi, \pi) \times \mathbb{R})$$

will agree with each other on the overlap. Therefore, it defines a *global* continuous unit normal vector across the whole cylinder. □

Example 4.11. The Möbius strip Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$F : (-1, 1) \times (0, 2\pi) \to \Sigma^2 \qquad \widetilde{F} : (-1, 1) \times (-\pi, \pi) \to \Sigma^2$$
$$F(u, \theta) = \begin{bmatrix} \left(3 + u\cos\frac{\theta}{2}\right)\cos\theta \\ \left(3 + u\cos\frac{\theta}{2}\right)\sin\theta \\ u\sin\frac{\theta}{2} \end{bmatrix} \qquad \widetilde{F}(\widetilde{u}, \widetilde{\theta}) = \begin{bmatrix} \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \\ \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \\ \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \end{bmatrix}$$

In order to compute the transition map $\widetilde{F}^{-1} \circ F(u, \theta)$, we need to solve the system of equations, i.e. find $(\widetilde{u}, \widetilde{\theta})$ in terms of (u, θ):

$$\left(3 + u \cos \frac{\theta}{2}\right) \cos \theta = \left(3 + \widetilde{u} \cos \frac{\widetilde{\theta}}{2}\right) \cos \widetilde{\theta} \tag{4.1}$$

$$\left(3 + u \cos \frac{\theta}{2}\right) \sin \theta = \left(3 + \widetilde{u} \cos \frac{\widetilde{\theta}}{2}\right) \sin \widetilde{\theta} \tag{4.2}$$

$$u \sin \frac{\theta}{2} = \widetilde{u} \sin \frac{\widetilde{\theta}}{2} \tag{4.3}$$

By considering $(4.1)^2 + (4.2)^2$, we get:

$$u \cos \frac{\theta}{2} = \widetilde{u} \cos \frac{\widetilde{\theta}}{2} \tag{4.4}$$

We leave it as an exercise for readers to check that $\theta \neq \pi$ in order for the system to be solvable. Therefore, $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and so the domain of overlap is a disjoint union of two open sets.

When $\theta \in (0, \pi)$, from (4.3) and (4.4) we can conclude that $\widetilde{u} = u$ and $\widetilde{\theta} = \theta$.

When $\theta \in (\pi, 2\pi)$, we *cannot* have $\widetilde{\theta} = \theta$ since $\widetilde{\theta} \in (-\pi, \pi)$. However, one can have $\widetilde{u} = -u$ so that (4.3) and (4.4) become:

$$\sin \frac{\theta}{2} = -\sin \frac{\widetilde{\theta}}{2} \quad \text{and} \quad \cos \frac{\theta}{2} = -\cos \frac{\widetilde{\theta}}{2}$$

which implies $\widetilde{\theta} = \theta - 2\pi$.

To conclude, we have:

$$\widetilde{F}^{-1} \circ F(u, \theta) = \begin{cases} (u, \theta) & \text{if } \theta \in (0, \pi) \\ (-u, \theta - 2\pi) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

By direct computations, we get:

$$\det D(\widetilde{F}^{-1} \circ F)(u, \theta) = \begin{cases} 1 & \text{if } \theta \in (0, \pi) \\ -1 & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

Therefore, no matter how we switch the order of u and θ, or $\widetilde{u}$ and $\widetilde{\theta}$, we can never allow $\det D(\widetilde{F}^{-1} \circ F) > 0$ everywhere on the overlap. In other words, even if the unit normal vectors ν_F and $\nu_{\widetilde{F}}$ agree with each other when $\theta \in (0, \pi)$, it would point in opposite direction when $\theta \in (\pi, 2\pi)$. □

Next, we are back to hypersurfaces M^n in $\mathbb{R}^{n+1}$ and prove the equivalence between consistency of unit normal and positivity of transition maps. To begin, we need the following result about normal vectors (which is left as an exercise for readers):

Exercise 4.6. Let M^n be a smooth hypersurface in $\mathbb{R}^{n+1}$ whose coordinates are denoted by $(x_1, \ldots, x_{n+1})$, and the unit vector along the x_i-direction is denoted by $\hat{e}_i$. Let $F(u_1, \ldots, u_n) : \mathcal{U} \to M^n$ be a local parametrization of M. Show that the following vector defined on $F(\mathcal{U})$ is normal to the hypersurface M^n:

$$\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1}, \ldots, x_{n+1}, x_1, \ldots, x_{i-1})}{\partial(u_1, \ldots, u_n)} \hat{e}_i.$$

Proposition 4.12. *Given a smooth hypersurface M^n in $\mathbb{R}^{n+1}$, the following are equivalent:*

(i) M^n is orientable;

(ii) There exists a family of local parametrizations $F_\alpha : \mathcal{U}_\alpha \to M$ covering M such that for any F_α, F_β in the family with $F_\beta(\mathcal{U}_\beta) \cap F_\alpha(\mathcal{U}_\alpha) \neq \emptyset$, we have:

$$\det D(F_\alpha^{-1} \circ F_\beta) > 0 \quad \text{on } F_\beta^{-1}\left(F_\beta(\mathcal{U}_\beta) \cap F_\alpha(\mathcal{U}_\alpha)\right).$$

Proof. We first prove (ii) $\Longrightarrow$ (i). Denote $(u_1^\alpha, \ldots, u_n^\alpha)$ to be the local coordinates of M under the parametrization F_α. On every $F_\alpha(\mathcal{U}_\alpha)$, using the result from Exercise 4.6, one can construct a unit normal vector locally defined on $F_\alpha(\mathcal{U}_\alpha)$:

$$\nu_\alpha = \frac{\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \hat{e}_i}{\left|\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \hat{e}_i\right|}$$

Similarly, on $F_\beta(\mathcal{U}_\beta)$, we have another locally defined unit normal vectors:

$$\nu_\beta = \frac{\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\beta,\ldots,u_n^\beta)} \hat{e}_i}{\left|\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\beta,\ldots,u_n^\beta)} \hat{e}_i\right|}$$

Then on the overlap $F_\beta^{-1}\left(F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta)\right)$, the chain rule asserts that:

$$\begin{aligned}
&\det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\beta,\ldots,u_n^\beta)} \\
&= \det \frac{\partial(u_1^\alpha,\ldots,u_n^\alpha)}{\partial(u_1^\beta,\ldots,u_n^\beta)} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \\
&= \det D(F_\alpha^{-1} \circ F_\beta) \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)}
\end{aligned}$$

and so the two unit normal vectors are related by:

$$\nu_\beta = \frac{\det D(F_\alpha^{-1} \circ F_\beta)}{\left|\det D(F_\alpha^{-1} \circ F_\beta)\right|} \nu_\alpha.$$

By the condition that $\det D(F_\alpha^{-1} \circ F_\beta) > 0$, we have $\nu_\beta = \nu_\alpha$ on the overlap. Define $\nu := \nu_\alpha$ on every $F_\alpha(\mathcal{U}_\alpha)$, it is then a continuous unit normal vector globally defined on M. This proves (i).

Now we show (i) $\Longrightarrow$ (ii). Suppose ν is a continuous unit normal vector defined on the whole M. Suppose $F_\alpha(u_1^\alpha, \ldots, u_n^\alpha) : \mathcal{U}_\alpha \to M$ is *any* family of local parametrizations that cover the whole M. On every $F_\alpha(\mathcal{U}_\alpha)$, we consider the locally defined unit normal vector:

$$\nu_\alpha = \frac{\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \hat{e}_i}{\left|\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \hat{e}_i\right|}.$$

As a hypersurface M^n in $\mathbb{R}^{n+1}$, there is only one direction of normal vectors, and so we have either $\nu_\alpha = \nu$ or $\nu_\alpha = -\nu$ on $F_\alpha(\mathcal{U}_\alpha)$. For the latter case, one can modify the parametrization F_α by switching any pair of u_i^α's such that $\nu_\alpha = \nu$.

After making suitable modification on every F_α, we can assume without loss of generality that F_α's are local parametrizations such that $\nu_\alpha = \nu$ on every $F_\alpha(\mathcal{U}_\alpha)$. In particular, on the overlap $F_\beta^{-1}\left(F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta)\right)$, we have $\nu_\alpha = \nu_\beta$.

By $\nu_\beta = \dfrac{\det D(F_\alpha^{-1} \circ F_\beta)}{\left|\det D(F_\alpha^{-1} \circ F_\beta)\right|}\nu_\alpha$, we conclude that $\det D(F_\alpha^{-1} \circ F_\beta) > 0$, proving (ii).

□

Remark 4.13. According to Proposition 4.12, the cylinder in Example 4.10 is orientable, while the Möbius strip in Example 4.11 is not orientable. □

Exercise 4.7. Show that the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is orientable.

Exercise 4.8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose $c \in \mathbb{R}$ such that $f^{-1}(c)$ is non-empty and f is a submersion at every $p \in f^{-1}(c)$. Show that $f^{-1}(c)$ is an orientable hypersurface in $\mathbb{R}^3$.

4.2.2. Orientable Manifolds. On an abstract manifold M, it is not possible to define normal vectors on M, and so the notion of orientability cannot be defined using normal vectors. However, thanks to Proposition 4.12, the notion of orientability of hypersurfaces is equivalent to positivity of Jacobians of transition maps, which we can also talk about on abstract manifolds. Therefore, motivated by Proposition 4.12, we define:

Definition 4.14 (Orientable Manifolds)**.** A smooth manifold M is said to be *orientable* if there exists a family of local parametrizations $F_\alpha : \mathcal{U}_\alpha \to M$ covering M such that for any F_α and F_β in the family with $F_\beta(\mathcal{U}_\beta) \cap F_\alpha(\mathcal{U}_\alpha) \neq \emptyset$, we have:

$$\det D(F_\alpha^{-1} \circ F_\beta) > 0 \quad \text{on } F_\beta^{-1}\left(F_\beta(\mathcal{U}_\beta) \cap F_\alpha(\mathcal{U}_\alpha)\right).$$

In this case, we call the family $\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\}$ of local parametrizations to be an *oriented atlas* of M.

Example 4.15. Recall that the real projective space $\mathbb{RP}^2$ consists of homogeneous triples $[x_0 : x_1 : x_2]$ where $(x_0, x_1, x_2) \neq (0, 0, 0)$. The standard parametrizations are given by:

$$\begin{aligned} F_0(x_1, x_2) &= [1 : x_1 : x_2] \\ F_1(y_0, y_2) &= [y_0 : 1 : y_2] \\ F_2(z_0, z_1) &= [z_0 : z_1 : 1] \end{aligned}$$

By the fact that $[y_0 : 1 : y_2] = [1 : y_0^{-1} : y_2 y_0^{-1}]$, the transition map $F_0^{-1} \circ F_1$ is defined on $\{(y_0, y_2) \in \mathbb{R}^2 : y_0 \neq 0\}$, and is given by: $(x_1, x_2) = (y_0^{-1}, y_2 y_0^{-1})$. Hence,

$$\begin{aligned} D(F_0^{-1} \circ F_1) &= \frac{\partial(x_1, x_2)}{\partial(y_0, y_2)} = \begin{bmatrix} -y_0^{-2} & 0 \\ -y_2 y_0^{-2} & y_0^{-1} \end{bmatrix} \\ \det D(F_0^{-1} \circ F_1) &= -\frac{1}{y_0^3} \end{aligned}$$

Therefore, it is impossible for $\det D(F_0^{-1} \circ F_1) > 0$ on the overlap domain $\{(y_0, y_2) \in \mathbb{R}^2 : y_0 \neq 0\}$.

At this stage, we have shown that this altas is not an oriented one. In order to prove $\mathbb{RP}^2$ is non-orientable, we need to show *any* altas of $\mathbb{RP}^2$ is not oriented. We will prove this using Proposition 4.25 later. □

Exercise 4.9. Show that $\mathbb{RP}^3$ is orientable. Propose a conjecture about the orientability of $\mathbb{RP}^n$.

Exercise 4.10. Show that for *any* smooth manifold M (whether or not it is orientable), the tangent bundle TM must be orientable.

Exercise 4.11. Show that for a smooth orientable manifold M with boundary, the boundary manifold ∂M must also be orientable.

4.3. Integrations of Differential Forms

Generalized Stokes' Theorem concerns about integrals of differential forms. In this section, we will give a rigorous definition of these integrals.

4.3.1. Single Parametrization. In the *simplest* case if a manifold M can be covered by a single parametrization:

$$F(u_1, \ldots, u_n) : (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n) \to M,$$

then given an n-form $\varphi(u_1, \ldots, u_n)\, du^1 \wedge du^2 \wedge \cdots \wedge du^n$, the integral of ω over the manifold M is given by:

$$\underbrace{\int_M \varphi(u_1, \ldots, u_n)\, du^1 \wedge du^2 \wedge \cdots \wedge du^n}_{\text{integral of differential form}} := \underbrace{\int_{\alpha_n}^{\beta_n} \cdots \int_{\alpha_1}^{\beta_1} \varphi(u_1, \ldots, u_n)\, du^1\, du^2 \cdots du^n}_{\text{ordinary integral in Multivariable Calculus}}$$

From the definition, we see that it only makes sense to integrate an n-form on an n-dimensional manifold.

Very few manifolds can be covered by a single parametrization. Of course, $\mathbb{R}^n$ is an example. One less trivial example is the graph of a smooth function. Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Consider its graph:

$$\Gamma_f := \{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

which can be globally parametrized by $F : \mathbb{R}^2 \to \Gamma_f$ where

$$F(x, y) = (x, y, f(x, y)).$$

Let $\omega = e^{-x^2-y^2}\, dx \wedge dy$ be a 2-form on Γ_f, then its integral over Γ_f is given by:

$$\int_{\Gamma_f} \omega = \int_{\Gamma_f} e^{-x^2-y^2}\, dx \wedge dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \pi.$$

Here we leave the computational detail as an exercise for readers.

It *appears* that integrating a differential form is just like "erasing the wedges", yet there are two subtle (but important) issues:

(1) In the above example, note that ω can also be written as:

$$\omega = -e^{-x^2-y^2}\, dy \wedge dx.$$

It suggests that we also have:

$$\int_{\Gamma_f} \omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -e^{-x^2-y^2}\, dy\, dx = -\pi,$$

which is *not* consistent with the previous result. How shall we fix it?

(2) Even if a manifold can be covered by one single parametrization, such a parametrization may not be unique. If both $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are global coordinates of M, then a differential form ω can be expressed in terms of either u_i's or v_i's. Is the integral independent of the chosen coordinate system?

The first issue can be resolved easily. Whenever we talk about integration of differential forms, we need to first fix the order of the coordinates. Say on $\mathbb{R}^2$ we fix the order to be (x, y), then for any given 2-form we should express it in terms of $dx \wedge dy$ before "erasing the wedges". For the 2-form ω above, we must first express it as:

$$\omega = e^{-x^2-y^2}\, dx \wedge dy$$

before integrating it.

For higher (say $\dim = 4$) dimensional manifolds M^4 covered by a single parametrization $F(u_1, \ldots, u_4) : \mathcal{U} \to M$, if we choose (u_1, u_2, u_3, u_4) to be the order of coordinates and given a 4-form:

$$\Omega = f(u_1, \ldots, u_4)\, du^1 \wedge du^3 \wedge du^2 \wedge du^4 + g(u_1, \ldots, u_4)\, du^4 \wedge du^3 \wedge du^2 \wedge du^1,$$

then we need to re-order the wedge product so that:

$$\Omega = -f(u_1, \ldots, u_4)\, du^1 \wedge du^2 \wedge du^3 \wedge du^4 + g(u_1, \ldots, u_4) du^1 \wedge du^2 \wedge du^3 \wedge du^4.$$

The integral of ω over M^4 with respect to the order (u_1, u_2, u_3, u_4) is given by:

$$\int_M \Omega = \int_{\mathcal{U}} \left(-f(u_1, \ldots, u_4) + g(u_1, \ldots, u_4)\right)\, du^1\, du^2\, du^3\, du^4.$$

Let's examine the second issue. Suppose M is an n-manifold with two different global parametrizations $F(u_1, \ldots, u_n) : \mathcal{U} \to M$ and $G(v_1, \ldots, v_n) : \mathcal{V} \to M$. Given an n-form ω which can be expressed as:

$$\omega = \varphi\, du^1 \wedge \cdots \wedge du^n,$$

then from Proposition 3.56, ω can be expressed in terms of v_i's by:

$$\omega = \varphi \det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)}\, dv^1 \wedge \cdots \wedge dv^n.$$

Recall that the change-of-variable formula in Multivariable Calculus asserts that:

$$\int_{\mathcal{U}} \varphi\, du^1 \cdots du^n = \int_{\mathcal{V}} \varphi \left| \det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)} \right| dv^1 \cdots dv^n.$$

Therefore, in order for $\displaystyle\int_M \omega$ to be well-defined, we need

$$\int_{\mathcal{U}} \varphi\, du^1 \wedge \cdots \wedge du^n \text{ and } \int_{\mathcal{V}} \varphi \det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)}\, dv^1 \wedge \cdots \wedge dv^n$$

to be equal, and so we require:

$$\det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)} > 0.$$

When defining an integral of a differential form, we *not only* need to choose a convention on the order of coordinates, say $(u_1, \ldots, u_n)$, but also we shall only consider those coordinate systems $(v_1, \ldots, v_n)$ such that $\det \dfrac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)} > 0$. Therefore, in order to integrate a differential form, we require the manifold to be *orientable*.

4.3.2. Multiple Parametrizations. A majority of smooth manifolds are covered by more than one parametrizations. Integrating a differential form over such a manifold is not as straight-forward as previously discussed.

In case M can be "almost" covered by a single parametrization $F : \mathcal{U} \to M$ (i.e. the set $M \backslash F(\mathcal{U})$ has measure zero) and the n-form ω is continuous, then it is still possible to compute $\displaystyle\int_M \omega$ by computing $\displaystyle\int_{F(\mathcal{U})} \omega$. Let's consider the example of a sphere:

Example 4.16. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin. Consider the 2-form ω on $\mathbb{R}^3$ defined as:

$$\omega = dx \wedge dy.$$

Let $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion map, then $\iota^*\omega$ is a 2-form on $\mathbb{S}^2$. We are interested in the value of the integral $\displaystyle\int_{\mathbb{S}^2} \iota^*\omega$.

Note that $\mathbb{S}^2$ can be covered almost everywhere by spherical coordinate parametrization $F(\varphi, \theta) : (0, \pi) \times (0, 2\pi) \to \mathbb{S}^2$ given by:

$$F(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Under the local coordinates (φ, θ), we have:

$$\begin{aligned}
\iota^*(dx) &= d(\sin\varphi\cos\theta) = \cos\varphi\cos\theta\, d\varphi - \sin\varphi\sin\theta\, d\theta \\
\iota^*(dy) &= d(\sin\varphi\sin\theta) = \cos\varphi\sin\theta\, d\varphi + \sin\varphi\cos\theta\, d\theta \\
\iota^*\omega &= \iota^*(dx) \wedge \iota^*(dy) \\
&= \sin\varphi\cos\varphi\, d\varphi \wedge d\theta.
\end{aligned}$$

Therefore,

$$\int_M \iota^*\omega = \int_M \sin\varphi\cos\varphi\, d\varphi \wedge d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos\varphi\, d\varphi\, d\theta = 0.$$

Here we pick (φ, θ) as the order of coordinates. □

Exercise 4.12. Let $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$. Compute

$$\int_{\mathbb{S}^2} \iota^*\omega$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centered at the origin, and $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map.

Exercise 4.13. Let $\mathbb{T}^2$ be the torus in $\mathbb{R}^4$ defined as:

$$\mathbb{T}^2 := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}\right\}.$$

Let $\iota : \mathbb{T}^2 \to \mathbb{R}^4$ be the inclusion map. Compute the following integral:

$$\int_{\mathbb{T}^2} \iota^*\left(x_1 x_2 x_3\, dx^4 \wedge dx^3\right).$$

FYI: Clifford Torus

The torus $\mathbb{T}^2$ in Exercise 4.13 is a well-known object in Differential Geometry called the *Clifford Torus*. A famous conjecture called the Hsiang-Lawson's Conjecture concerns about this torus. One of the proposers Wu-Yi Hsiang is a retired faculty of HKUST Math. This conjecture was recently solved by Simon Brendle in 2012.

Next, we will discuss how to define integrals of differential forms when M is covered by multiple parametrizations none of which can almost cover the whole manifold. The key idea is to break down the n-form into small pieces, so that each piece is completely covered by one single parametrization. It will be done using *partition of unity* to be discussed.

We first introduce the notion of *support* which appears often in the rest of the course (as well as in advanced PDE courses).

Definition 4.17 (Support)**.** Let M be a smooth manifold. Given a k-form ω (where $0 \le k \le n$) defined on M, we denote and define the *support of* ω to be:

$$\operatorname{supp}\omega := \overline{\{p \in M : \omega(p) \neq 0\}},$$

i.e. the *closure* of the set $\{p \in M : \omega(p) \neq 0\}$.

Suppose M^n is an oriented manifold with $F(u_1, \ldots, u_n) : \mathcal{U} \to M$ as one of (many) local parametrizations. If an n-form ω on M^n only has "stuff" inside $F(\mathcal{U})$, or precisely:

$$\operatorname{supp} \omega \subset F(\mathcal{U}),$$

then one can define $\displaystyle\int_M \omega$ as in the previous subsection. Namely, if on $F(\mathcal{U})$ we have $\omega = \varphi \, du^1 \wedge \cdots \wedge du^n$, then we define:

$$\int_M \omega = \int_{F(\mathcal{U})} \omega = \int_{\mathcal{U}} \varphi \, du^1 \, \cdots \, du^n.$$

Here we pick the order of coordinates to be $(u_1, \ldots, u_n)$.

The following important tool called *partitions of unity* will "chop" a differential form into "little pieces" such that each piece has support covered by a single parametrization.

Definition 4.18 (Partitions of Unity)**.** Let M be a smooth manifold with an atlas $\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\}$ such that $M = \bigcup_{\text{all } \alpha} F_\alpha(\mathcal{U}_\alpha)$. A *partition of unity subordinate to the atlas* $\mathcal{A}$ is a family of smooth functions $\rho_\alpha : M \to [0, 1]$ with the following properties:

(i) $\operatorname{supp} \rho_\alpha \subset F_\alpha(\mathcal{U}_\alpha)$ for any α.

(ii) For any $p \in M$, there exists an open set $\mathcal{O} \subset M$ containing p such that

$$\operatorname{supp} \rho_\alpha \cap \mathcal{O} \neq \emptyset$$

for finitely many α's only.

(iii) $\displaystyle\sum_{\text{all } \alpha} \rho_\alpha \equiv 1$ on M.

Remark 4.19. It can be shown that given any smooth manifold with any atlas, partitions of unity subordinate to that given atlas must exist. The proof is very technical and is not in the same spirit with other parts of the course, so we omit the proof here. It is more important to know what partitions of unity are for, than to know the proof of existence. □

Remark 4.20. Note that partitions of unity subordinate to a given atlas may not be unique! □

Remark 4.21. Condition (ii) in Definition 4.18 is merely a technical analytic condition to make sure the sum $\sum_{\text{all } \alpha} \rho_\alpha(p)$ is a finite sum for each fixed $p \in M$, so that we do not need to worry about convergence issues. If the manifold can be covered by finitely many local parametrizations, then condition (ii) automatically holds (and we do not need to worry about). □

Now, take an n-form ω defined on an orientable manifold M^n, which is parametrized by an oriented atlas $\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\}$. Let $\{\rho_\alpha : M \to [0, 1]\}$ be a partition of unity subordinate to $\mathcal{A}$, then by condition (iii) in Definition 4.18, we get:

$$\omega = \underbrace{\left(\sum_{\text{all } \alpha} \rho_\alpha\right)}_{=1} \omega = \sum_{\text{all } \alpha} \rho_\alpha \omega.$$

Condition (i) says that $\operatorname{supp} \rho_\alpha \subset F_\alpha(\mathcal{U}_\alpha)$, or heuristically speaking ρ_α vanishes outside $F_\alpha(\mathcal{U}_\alpha)$. Naturally, we have $\operatorname{supp}(\rho_\alpha \omega) \subset F_\alpha(\mathcal{U}_\alpha)$ for each α. Therefore, as previously discussed, we can integrate $\rho_\alpha \omega$ for each *individual* α:

$$\int_M \rho_\alpha \omega := \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega.$$

Given that we can integrate each $\rho_\alpha\omega$, we define the integral of ω as:

$$\int_M \omega := \sum_{\text{all }\alpha} \int_M \rho_\alpha\omega = \sum_{\text{all }\alpha} \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega. \tag{4.5}$$

However, the sum involved in (4.5) is in general an infinite (possible uncountable!) sum. To avoid convergence issue, from now on we will only consider n-forms ω which have *compact support*, i.e.

$$\operatorname{supp}\omega \text{ is a compact set.}$$

Recall that every open cover of a compact set has a finite sub-cover. Together with condition (ii) in Definition 4.18, one can show that $\rho_\alpha\omega$ are identically zero for all except finitely many α's. The argument goes as follows: at each $p \in \operatorname{supp}\omega$, by condition (ii) in Definition 4.18, there exists an open set $\mathcal{O}_p \subset M$ containing p such that the set:

$$S_p := \{\alpha : \operatorname{supp}\rho_\alpha \cap \mathcal{O}_p \neq \emptyset\}$$

is finite. Evidently, we have

$$\operatorname{supp}\omega \subset \bigcup_{p\in\operatorname{supp}\omega} \mathcal{O}_p$$

and by compactness of $\operatorname{supp}\omega$, there exists $p_1, \ldots, p_N \in \operatorname{supp}\omega$ such that

$$\operatorname{supp}\omega \subset \bigcup_{i=1}^N \mathcal{O}_{p_i}.$$

Since $\{q \in M : \rho_\alpha(q)\omega(q) \neq 0\} \subset \{q \in M : \rho_\alpha(q) \neq 0\} \cap \{q \in M : \omega(q) \neq 0\}$, we have:

$$\begin{aligned}
\operatorname{supp}(\rho_\alpha\omega) &= \overline{\{q \in M : \rho_\alpha(q)\omega(q) \neq 0\}} \\
&\subset \overline{\{q \in M : \rho_\alpha(q) \neq 0\} \cap \{q \in M : \omega(q) \neq 0\}} \\
&\subset \overline{\{q \in M : \rho_\alpha(q) \neq 0\}} \cap \overline{\{q \in M : \omega(q) \neq 0\}} \\
&= \operatorname{supp}\rho_\alpha \cap \operatorname{supp}\omega \subset \bigcup_{i=1}^N \left(\operatorname{supp}\rho_\alpha \cap \mathcal{O}_{p_i}\right).
\end{aligned}$$

Therefore, if α is an index such that $\operatorname{supp}(\rho_\alpha\omega) \neq \emptyset$, then there exists $i \in \{1, \ldots, N\}$ such that $\operatorname{supp}\rho_\alpha \cap \mathcal{O}_{p_i} \neq \emptyset$, or in other words, $\alpha \in S_{p_i}$ for some i, and so:

$$\{\alpha : \operatorname{supp}(\rho_\alpha\omega) \neq \emptyset\} \subset \bigcup_{i=1}^N S_{p_i}.$$

Since each S_{p_i} is a finite set, the set $\{\alpha : \operatorname{supp}(\rho_\alpha\omega) \neq \emptyset\}$ is also finite. Therefore, there are only finitely many α's such that $\displaystyle\int_{F_\alpha(\mathcal{U}_\alpha)}$ is non-zero, and so the sum stated in (4.5) is in fact a *finite sum*.

Now we have understood that there is no convergence issue for (4.5) provided that ω has compact support (which is automatically true if the manifold M is itself compact). There are still two well-definedness issues to resolve, namely whether the integral in (4.5) is independent of oriented atlas $\mathcal{A}$, and for each atlas whether the integral is independent of the choice of partitions of unity.

Proposition 4.22. *Let M^n be an orientable smooth manifold with two oriented atlas*
$$\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\} \text{ and } \mathcal{B} = \{G_\beta : \mathcal{V}_\beta \to M\}$$
such that $\det D(F_\alpha^{-1} \circ G_\beta) > 0$ *on the overlap for any pair of α and β. Suppose $\{\rho_\alpha : M \to [0,1]\}$ and $\{\sigma_\beta : M \to [0,1]\}$ are partitions of unity subordinate to $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, given any compactly supported differential n-form ω on M^n, we have:*
$$\sum_{\text{all } \alpha} \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega = \sum_{\text{all } \beta} \int_{G_\beta(\mathcal{V}_\beta)} \sigma_\beta \omega.$$

Proof. By the fact that $\sum_{\text{all } \beta} \sigma_\beta \equiv 1$ on M, we have:
$$\sum_{\text{all } \alpha} \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega = \sum_{\text{all } \alpha} \int_{F_\alpha(\mathcal{U}_\alpha)} \left(\sum_{\text{all } \beta} \sigma_\beta\right) \rho_\alpha \omega = \sum_{\text{all } \alpha} \sum_{\text{all } \beta} \int_{F_\alpha(\mathcal{U}_\alpha) \cap G_\beta(\mathcal{V}_\beta)} \rho_\alpha \sigma_\beta \omega.$$
The last equality follows from the fact that $\operatorname{supp} \sigma_\beta \subset G_\beta(\mathcal{V}_\beta)$.

One can similarly work out that
$$\sum_{\text{all } \beta} \int_{G_\beta(\mathcal{V}_\beta)} \sigma_\beta \omega = \sum_{\text{all } \beta} \sum_{\text{all } \alpha} \int_{F_\alpha(\mathcal{U}_\alpha) \cap G_\beta(\mathcal{V}_\beta)} \rho_\alpha \sigma_\beta \omega.$$
Note that $\sum_\alpha \sum_\beta$ is a finite double sum and so there is no issue of switching them. It completes the proof. □

By Proposition 4.22, we justified that (4.5) is independent of oriented atlas and the choice of partitions of unity. We can now define:

Definition 4.23. Let M^n be an orientable smooth manifold with an oriented atlas $\mathcal{A} = \{F_\alpha(u_\alpha^1, \ldots, u_\alpha^n) : \mathcal{U}_\alpha \to M\}$ where $(u_\alpha^1, \ldots, u_\alpha^n)$ is the chosen order of local coordinates. Pick a partition of unity $\{\rho_\alpha : M \to [0,1]\}$ subordinate to the atlas $\mathcal{A}$. Then, given any n-form ω, we define its *integral over M* as:
$$\int_M \omega := \sum_{\text{all } \alpha} \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega.$$
If $\omega = \varphi_\alpha \, du_\alpha^1 \wedge \cdots \wedge du_\alpha^n$ on each $F_\alpha(\mathcal{U}_\alpha)$, then:
$$\int_M \omega = \sum_{\text{all } \alpha} \int_{\mathcal{U}_\alpha} \rho_\alpha \varphi_\alpha \, du_\alpha^1 \cdots du_\alpha^n.$$

Remark 4.24. It is generally impossible to compute such an integral, as we know only the existence of ρ_α's but typically not the exact expressions. Even if such a partition of unity ρ_α's can be found, it often involves some terms such as e^{-1/x^2}, which is almost impossible to integrate. To conclude, we do not attempt compute such an integral, but we will study the properties of it based on the definition. □

4.3.3. Orientation of Manifolds.

Partition of unity is a powerful tool to construct a smooth *global* item from *local* ones. For integrals of differential forms, we first defines integral of forms with support contained in a single parametrization chart, then we uses a partition of unity to *glue* each chart together. There are some other uses in this spirit. The following beautiful statement can be proved using partitions of unity:

Proposition 4.25. *A smooth n-dimensional manifold M is orientable if and only if there exists a non-vanishing smooth n-form globally defined on M.*

Proof. Suppose M is orientable, then by definition there exists an oriented atlas $\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\}$ such that $\det D(F_\beta^{-1} \circ F_\alpha) > 0$ for any α and β. For each local parametrization F_α, we denote $(u_\alpha^1, \ldots, u_\alpha^n)$ to be its local coordinates, then the n-form:

$$\eta_\alpha := du_\alpha^1 \wedge \cdots \wedge du_\alpha^n$$

is *locally defined* on $F_\alpha(\mathcal{U}_\alpha)$.

Let $\{\rho_\alpha : M \to [0,1]\}$ be a partition of unity subordinate to $\mathcal{A}$. We define:

$$\omega = \sum_{\text{all } \alpha} \rho_\alpha \eta_\alpha = \sum_{\text{all } \alpha} \rho_\alpha \, du_\alpha^1 \wedge \cdots \wedge du_\alpha^n.$$

We claim $\omega(p) \neq 0$ at every point $p \in M$. Suppose $p \in F_\beta(\mathcal{U}_\beta)$ for some β in the atlas. By (3.19), for each α, locally near p we have:

$$du_\alpha^1 \wedge \cdots \wedge du_\alpha^n = \det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} \, du_\beta^1 \wedge \cdots \wedge du_\beta^n,$$

and so:

$$\omega = \left(\sum_{\text{all } \alpha} \rho_\alpha \, \det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} \right) du_\beta^1 \wedge \cdots \wedge du_\beta^n.$$

Since $\rho_\alpha \geq 0$, $\sum_{\text{all } \alpha} \rho_\alpha \equiv 1$ and $\det \dfrac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} > 0$, we must have:

$$\sum_{\text{all } \alpha} \rho_\alpha \, \det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} > 0 \quad \text{near } p.$$

This shows ω is a non-vanishing n-form on M.

Conversely, suppose Ω is a non-vanishing n-form on M. Let $\mathcal{C} = \{G_\alpha : \mathcal{V}_\alpha \to M\}$ be any atlas on M, and for each α we denote $(v_\alpha^1, \ldots, v_\alpha^n)$ to be its local coordinates. Express Ω in terms of local coordinates:

$$\Omega = \varphi_\alpha \, dv_\alpha^1 \wedge \cdots \wedge dv_\alpha^n.$$

Since Ω is non-vanishing, φ_α must be either positive on $\mathcal{V}_\alpha$, or negative on $\mathcal{V}_\alpha$. Re-define the local coordinates by:

$$(\widetilde{v}_\alpha^1, \widetilde{v}_\alpha^2, \ldots, \widetilde{v}_\alpha^n) := \begin{cases} (v_\alpha^1, v_\alpha^2, \ldots, v_\alpha^n) & \text{if } \varphi_\alpha > 0 \\ (-v_\alpha^1, v_\alpha^2, \ldots, v_\alpha^n) & \text{if } \varphi_\alpha < 0 \end{cases}$$

Then, under these new local coordinates, we have:

$$\Omega = |\varphi_\alpha| \, d\widetilde{v}_\alpha^1 \wedge \cdots \wedge d\widetilde{v}_\alpha^n.$$

From (3.19), we can deduce:

$$\Omega = |\varphi_\alpha| \, d\widetilde{v}_\alpha^1 \wedge \cdots \wedge d\widetilde{v}_\alpha^n = |\varphi_\alpha| \, \det \frac{\partial(\widetilde{v}_\alpha^1, \ldots, \widetilde{v}_\alpha^n)}{\partial(\widetilde{v}_\beta^1, \ldots, \widetilde{v}_\beta^n)} \, d\widetilde{v}_\beta^1 \wedge \cdots \wedge d\widetilde{v}_\beta^n$$

on the overlap of any two local coordinates $(\widetilde{v}_\alpha^1, \ldots, \widetilde{v}_\alpha^n)$ and $(\widetilde{v}_\beta^1, \ldots, \widetilde{v}_\beta^n)$. On the other hand, we have:

$$\Omega = |\varphi_\beta| \, d\widetilde{v}_\beta^1 \wedge \cdots \wedge d\widetilde{v}_\beta^n.$$

This shows:

$$\det \frac{\partial(\widetilde{v}_\alpha^1, \ldots, \widetilde{v}_\alpha^n)}{\partial(\widetilde{v}_\beta^1, \ldots, \widetilde{v}_\beta^n)} = \left| \frac{\varphi_\beta}{\varphi_\alpha} \right| > 0 \quad \text{for any } \alpha, \beta.$$

Therefore, M is orientable. $\square$

The significance of Proposition 4.25 is that it relates the orientability of an n-manifold (which was defined in a rather *local* way) with the existence of a non-vanishing n-form (which is a *global* object). For abstract manifolds, unit normal vectors cannot be defined. Here the non-vanishing global n-form plays a similar role as a continuous unit normal does for hypersurfaces. In the rest of the course we will call:

Definition 4.26 (Orientation of Manifolds)**.** Given an orientable manifold M^n, a non-vanishing global n-form Ω is called an *orientation of* M. A basis of tangent vectors $\{T_1, \ldots, T_n\} \in T_pM$ is said to be Ω*-oriented* if $\Omega(T_1, \ldots, T_n) > 0$. A local coordinate system $(u_1, \ldots, u_n)$ is said to be Ω*-oriented* if $\Omega\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}\right) > 0$.

Recall that when we integrate an n-form, we need to first pick an order of local coordinates $(u_1, \ldots, u_n)$, then express the n-form according to this order, and locally define the integral as:

$$\int_{F(\mathcal{U})} \varphi \, du^1 \wedge \cdots \wedge du^n = \int_{\mathcal{U}} \varphi \, du^1 \cdots du^n.$$

Note that picking the *order of coordinates* is a *local* notion. To rephrase it using *global* terms, we can first pick an orientation Ω (which is a global object on M), then we require the order of any local coordinates $(u_1, \ldots, u_n)$ to be Ω-oriented. Any pair of local coordinate systems $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ which are both Ω-oriented will automatically satisfy $\det \dfrac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)} > 0$ on the overlap.

To summarize, given an orientable manifold M^n with a chosen orientation Ω, then for *any* local coordinate system $F(u_1, \ldots, u_n) : \mathcal{U} \to M$, we define:

$$\int_{F(\mathcal{U})} \varphi \, du^1 \wedge \cdots \wedge du^n = \begin{cases} \int_{\mathcal{U}} \varphi \, du^1 \cdots du^n & \text{if } (u_1, \ldots, u_n) \text{ is } \Omega\text{-oriented} \\ -\int_{\mathcal{U}} \varphi \, du^1 \cdots du^n & \text{if } (u_1, \ldots, u_n) \text{ is not } \Omega\text{-oriented} \end{cases}$$

or to put it in a more elegant (yet equivalent) way:

$$\int_{F(\mathcal{U})} \varphi \, du^1 \wedge \cdots \wedge du^n = \operatorname{sgn}\left[\Omega\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}\right)\right] \int_{\mathcal{U}} \varphi \, du^1 \cdots du^n.$$

Exercise 4.14. Let $\Omega := dx \wedge dy \wedge dz$ be the orientation of $\mathbb{R}^3$. Which of the following is Ω-oriented?

(a) local coordiantes (x, y, z)

(b) vectors $\{\hat{i}, \hat{k}, \hat{j}\}$

(c) vectors $\{u, V, u \times V\}$ where u and V are linearly independent vectors in $\mathbb{R}^3$.

Exercise 4.15. Consider three linearly independent vectors $\{u, V, \mathsf{w}\}$ in $\mathbb{R}^3$ such that $u \perp \mathsf{w}$ and $V \perp \mathsf{w}$. Show that $\{u, V, \mathsf{w}\}$ has the same orientation as $\{\hat{i}, \hat{j}, \hat{k}\}$ if and only if $\mathsf{w} = cu \times V$ for some *positive* constant c.

Proposition 4.25 can be used to complete the proof that $\mathbb{RP}^2$ is not orientable in Example 4.15. In that example, we demonstrated that there are two local parametrizations $F_0(u_1, u_2)$ and $F_1(v_1, v_2)$ with the properties that:

- the domain of each of F_i is connected; while
- their overlap, i.e. domain of $F_0^{-1} \circ F_1$, is not connected; and
- $\det D(F_0^{-1} \circ F_1)$ is positive on one component U, but negative on another component V.

To show that $\mathbb{RP}^2$ is not orientable, we argue by contradiction that there exists a global non-vanishing 2-form Ω. Then, if $\Omega(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}) > 0$, then one has $\Omega(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}) > 0$ on U since $\det D(F_0^{-1} \circ F_1) > 0$ on U, and $\Omega(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}) < 0$ on V since $\det D(F_0^{-1} \circ F_1) < 0$. However, since the domain of $F_1(v_1, v_2)$ is connected and $\Omega(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2})$ is a smooth (in particular continuous) function on that domain, there must be a point p in the domain of F_1 such that $\Omega(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}) = 0$ at p. It leads to a contradiction that Ω is non-vanishing. Similar for the case $\Omega(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}) < 0$.

4.4. Generalized Stokes' Theorem

In this section, we (finally) state and give a proof of an elegant theorem, Generalized Stokes' Theorem. It not only unifies Green's, Stokes' and Divergence Theorems which we learned in Multivariable Calculus, but also generalize it to higher dimensional abstract manifolds.

4.4.1. Boundary Orientation.

Since the statement of Generalized Stokes' Theorem involves integration on differential forms, we will assume all manifolds discussed in this section to be *orientable*. Let's fix an orientation Ω of M^n, which is a non-vanishing n-form, and this orientation determines how local coordinates on M are ordered as discussed in the previous section.

Now we deal with the orientation of the boundary manifold ∂M. Given a local parametrization $G(u_1, \ldots, u_n) : \mathcal{V} \subset \mathbb{R}^n_+ \to M$ of boundary type. The tangent space T_pM for points $p \in \partial M$ is defined as the span of $\left\{\dfrac{\partial}{\partial u_i}\right\}_{i=1}^n$. As $\mathcal{V}$ is a subset of the upper half-space $\{u_n \geq 0\}$, the vector $\eta := -\frac{\partial}{\partial u_n}$ in T_pM is often called an *outward-pointing* "normal" vector to ∂M.

An orientation Ω of M^n is a non-vanishing n-form. The boundary manifold ∂M^n is an $(n-1)$-manifold, and so an orientation of ∂M^n should be a non-vanishing $(n-1)$-form. Using the outward-pointing normal vector η, one can produce such an $(n-1)$-form in a natural way. Given any tangent vectors $T_1, \ldots, T_{n-1}$ on $T(\partial M)$, we consider the interior product $i_\eta \Omega$, which is defined as:

$$(i_\eta \Omega)(T_1, \ldots, T_{n-1}) := \Omega(\eta, T_1, \ldots, T_{n-1}).$$

Then $i_\eta \Omega$ is an alternating multilinear map in $\wedge^{n-1} T^*(\partial M)$.

Locally, given a local coordinate system $(u_1, \ldots, u_n)$, by recalling that $\eta = -\frac{\partial}{\partial u_n}$ we can compute:

$$\begin{aligned}(i_\eta \Omega)\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}}\right) &= \Omega\left(\eta, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}}\right) \\ &= \Omega\left(-\frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}}\right) \\ &= (-1)^n \Omega\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}}, \frac{\partial}{\partial u_n}\right)\end{aligned}$$

which is non-zero. Therefore, $i_\eta \Omega$ is a non-vanishing $(n-1)$-form on ∂M, and we can take it as an orientation for ∂M. From now on, whenever we pick an orientation Ω for M^n, we will by-default pick $i_\eta \Omega$ to be the orientation for ∂M.

Given an Ω-oriented local coordinate system $G(u_1, \ldots, u_n) : \mathcal{V} \to M$ of boundary type for M^n, then $(u_1, \ldots, u_{n-1})$ is $i_\eta\Omega$-oriented if n is *even*; and is not $i_\eta\Omega$-oriented if n is *odd*. Therefore, when integrating an $(n-1)$-form $\varphi\, du^1 \wedge \cdots \wedge du^{n-1}$ on ∂M, we need to take into account of the parity of n, i.e.

$$\int_{G(\mathcal{V}) \cap \partial M} \varphi\, du^1 \wedge \cdots \wedge du^{n-1} = (-1)^n \int_{\mathcal{V} \cap \{u_n = 0\}} \varphi\, du^1 \cdots du^{n-1}. \tag{4.6}$$

The "extra" factor of $(-1)^n$ does not look nice at the first glance, but as we will see later, it will make Generalized Stokes' Theorem nicer. We are now ready to state Generalized Stokes' Theorem in a precise way:

Theorem 4.27 (Generalized Stokes' Theorem)**.** *Let M be an orientable smooth n-manifold, and let ω be a compactly supported smooth $(n-1)$-form on M. Then, we have:*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{4.7}$$

Here if Ω is a chosen to be an orientation of M, then we will take $i_\eta \Omega$ to be the orientation of ∂M where η is an outward-point normal vector of ∂M.

In particular, if $\partial M = \emptyset$, then $\int_M d\omega = 0$.

4.4.2. Proof of Generalized Stokes' Theorem. The proof consists of three steps:

Step 1: a special case where $\operatorname{supp}\omega$ is contained inside a single parametrization chart of interior type;

Step 2: another special case where $\operatorname{supp}\omega$ is contained inside a single parametrization chart of boundary type;

Step 3: use partitions of unity to deduce the general case.

Proof of Theorem 4.27. Throughout the proof, we will let Ω be the orientation of M, and $i_\eta\Omega$ be the orientation of ∂M with η being an outward-point normal vector to ∂M. All local coordinate system $(u_1, \ldots, u_n)$ of M is assumed to be Ω-oriented.

Step 1: Suppose $\operatorname{supp}\omega$ *is contained in a single parametrization chart of interior type.*

Let $F(u_1, \ldots, u_n) : \mathcal{U} \subset \mathbb{R}^n \to M$ be a local parametrization of interior type such that $\operatorname{supp}\omega \subset F(\mathcal{U})$. Denote:

$$du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n := du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n,$$

or in other words, it means the form with du^i removed.

In terms of local coordinates, the $(n-1)$-form ω can be expressed as:

$$\omega = \sum_{i=1}^n \omega_i \, du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n.$$

Taking the exterior derivative, we get:

$$d\omega = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \omega_i}{\partial u_j} du^j \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n$$

For each i, the wedge product $du^j \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n$ is zero if $j \neq i$. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^n \frac{\partial \omega_i}{\partial u_i} du^i \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \wedge \cdots \wedge du^i \wedge \cdots \wedge du^n \end{aligned}$$

By definition of integrals of differential forms, we get:

$$\int_M d\omega = \int_{\mathcal{U}} \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n.$$

Since $\operatorname{supp}\omega \subset F(\mathcal{U})$, the functions ω_i's are identically zero near and outside the boundary of $\mathcal{U} \subset \mathbb{R}^n$. Therefore, we can replace the domain of integration $\mathcal{U}$ of the RHS integral by a rectangle $[-R,R]\times\cdots\times[-R,R]$ in $\mathbb{R}^n$ where $R>0$ is a sufficiently large number. The value of the integral is unchanged. Therefore, using the Fubini's Theorem, we get:

$$\begin{aligned}\int_M d\omega &= \int_{-R}^{R}\cdots\int_{-R}^{R}\sum_{i=1}^n(-1)^i\frac{\partial\omega_i}{\partial u_i}du^1\cdots du^n\\ &=\sum_{i=1}^n(-1)^{i-1}\int_{-R}^{R}\cdots\int_{-R}^{R}\frac{\partial\omega_i}{\partial u_i}du^i\,du^1\cdots\widehat{du^i}\cdots du^n\\ &=\sum_{i=1}^n(-1)^{i-1}\int_{-R}^{R}\cdots\int_{-R}^{R}[\omega_i]_{u_i=-R}^{u_i=R}\,du^1\cdots\widehat{du^i}\cdots du^n.\end{aligned}$$

Since ω_i's vanish at the boundary of the rectangle $[-R,R]^n$, we have $\omega_i=0$ when $u_i=\pm R$. As a result, we proved $\int_M d\omega=0$. Since $\operatorname{supp}\omega$ is contained in a single parametrization chart of interior type, we have $\omega=0$ on the boundary ∂M. Evidently, we have $\int_{\partial M}\omega=0$ in this case. Hence, we proved

$$\int_M d\omega=\int_{\partial M}\omega=0$$

in this case.

Step 2: Suppose $\operatorname{supp}\omega$ *is contained inside a single parametrization chart of boundary type.*

Let $G(u_1,\ldots,u_n):\mathcal{V}\subset\mathbb{R}^n_+\to M$ be a local parametrization of boundary type such that $\operatorname{supp}\omega\subset G(\mathcal{V})$. As in Step 1, we express

$$\omega=\sum_{i=1}^n\omega_i\,du^1\wedge\cdots\wedge\widehat{du^i}\wedge\cdots\wedge du^n.$$

Proceed exactly in the same way as before, we arrive at:

$$\int_M d\omega=\int_{\mathcal{V}}\sum_{i=1}^n(-1)^{i-1}\frac{\partial\omega_i}{\partial u_i}du^1\cdots du^n.$$

Now $\mathcal{V}$ is an open set in $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$. Recall that the boundary is the set of points with $u_n=0$. Therefore, this time we replace $\mathcal{V}$ by the half-space rectangle $[-R,R]\times\cdots\times[-R,R]\times[0,R]$ where $R>0$ again is a sufficiently large number.

One key difference from Step 1 is that even though ω_i's has compact support inside $\mathcal{V}$, it may not vanish on the boundary of M. Therefore, we can only guarantee $\omega_i(u_1,\ldots,u_n)=0$ when $u_n=R$, but we cannot claim $\omega_i=0$ when $u_n=0$. Some more work needs to be done:

$$\begin{aligned}\int_M d\omega &= \int_{\mathcal{V}}\sum_{i=1}^n(-1)^{i-1}\frac{\partial\omega_i}{\partial u_i}du^1\cdots du^n\\ &=\int_0^R\int_{-R}^{R}\cdots\int_{-R}^{R}\sum_{i=1}^n(-1)^{i-1}\frac{\partial\omega_i}{\partial u_i}du^1\cdots du^n\\ &=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^{R}\cdots\int_{-R}^{R}(-1)^{i-1}\frac{\partial\omega_i}{\partial u_i}du^1\cdots du^n\\ &\quad+(-1)^{n-1}\int_0^R\int_{-R}^{R}\cdots\int_{-R}^{R}\frac{\partial\omega_n}{\partial u_n}du^1\cdots du^n\end{aligned}$$

One can proceed as in Step 1 to show that the first term:

$$\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R(-1)^{i-1}\frac{\partial\omega_i}{\partial u_i}du^1\cdots du^n=0,$$

which follows from the fact that whenever $1\leq i\leq n-1$, we have $\omega_i=0$ on $u_i=\pm R$.

The second term:

$$(-1)^{n-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\frac{\partial\omega_n}{\partial u_n}du^1\cdots du^n$$

is handled in a different way:

$$\begin{aligned}&(-1)^{n-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\frac{\partial\omega_n}{\partial u_n}du^1\cdots du^n\\&=(-1)^{n-1}\int_{-R}^R\cdots\int_{-R}^R\int_0^R\frac{\partial\omega_n}{\partial u_n}\,du^n\,du^1\cdots du^{n-1}\\&=(-1)^{n-1}\int_{-R}^R\cdots\int_{-R}^R[\omega_n]_{u_n=0}^{u_n=R}\,du^1\cdots du^{n-1}\\&=(-1)^n\int_{-R}^R\cdots\int_{-R}^R\omega_n(u_1,\ldots,u_{n-1},0)\,du^1\cdots du^{n-1}\end{aligned}$$

where we have used the following fact:

$$\begin{aligned}[\omega_n(u_1,\ldots,u_n)]_{u_n=0}^{u_n=R}&=\omega_n(u_1,\ldots,u_{n-1},R)-\omega_i(u_1,\ldots,u_{n-1},0)\\&=0-\omega_n(u_1,\ldots,u_{n-1},0).\end{aligned}$$

Combining all results proved so far, we have:

$$\int_M d\omega=(-1)^n\int_{-R}^R\cdots\int_{-R}^R\omega_n(u_1,\ldots,u_{n-1},0)\,du^1\cdots du^{n-1}$$

On the other hand, we compute $\int_{\partial M}\omega$ and then compare it with $\int_M d\omega$. Note that the boundary ∂M are points with $u_n=0$. Therefore, across the boundary ∂M, we have $du^n\equiv 0$, and so on ∂M we have:

$$\begin{aligned}\omega&=\sum_{i=1}^n\omega_i(u_1,\ldots,u_{n-1},0)\,du^1\wedge\cdots\wedge\widehat{du^i}\wedge\cdots\wedge\underbrace{du^n}_{=0}\\&=\omega_n(u_1,\ldots,u_{n-1},0)\,du^1\wedge\cdots\wedge du^{n-1}\\\int_{\partial M}\omega&=\int_{G(\mathcal{V})\cap\partial M}\omega_n(u_1,\ldots,u_{n-1},0)\,du^1\wedge\cdots\wedge du^{n-1}\\&=(-1)^n\int_{\mathcal{V}\cap\{u_n=0\}}\omega_n(u_1,\ldots,u_{n-1},0)\,du^1\cdots du^{n-1}\\&=(-1)^n\int_{-R}^R\cdots\int_{-R}^R\omega_n(u_1,\ldots,u_{n-1},0)\,du^1\cdots du^{n-1}\end{aligned}$$

Recall that we have a factor of $(-1)^n$ because the local coordinate system $(u_1,\ldots,u_{n-1})$ for ∂M is $i_\eta\Omega$ if and only if n is even, as discussed in the previous subsection.

Consequently, we have proved

$$\int_M d\omega=\int_{\partial M}\omega$$

in this case.

Step 3: Use partitions of unity to deduce the general case

Finally, we "glue" the previous two steps together and deduce the general case. Let $\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\}$ be an atlas of M where all local coordinates are Ω-oriented. Here $\mathcal{A}$ contain both interior and boundary types of local parametrizations. Suppose $\{\rho_\alpha : M \to [0,1]\}$ is a partition of unity subordinate to $\mathcal{A}$. Then, we have:

$$\omega = \underbrace{\left(\sum_\alpha \rho_\alpha\right)}_{\equiv 1} \omega = \sum_\alpha \rho_\alpha \omega$$

$$\int_{\partial M} \omega = \int_{\partial M} \sum_\alpha \rho_\alpha \omega = \sum_\alpha \int_{\partial M} \rho_\alpha \omega.$$

For each α, the $(n-1)$-form $\rho_\alpha \omega$ is compactly supported in a single parametrization chart (either of interior or boundary type). From Step 1 and Step 2, we have already proved that Generalized Stokes' Theorem is true for each $\rho_\alpha \omega$. Therefore, we have:

$$\begin{aligned}\sum_\alpha \int_{\partial M} \rho_\alpha \omega &= \sum_\alpha \int_M d(\rho_\alpha \omega) \\ &= \sum_\alpha \int_M (d\rho_\alpha \wedge \omega + \rho_\alpha \, d\omega) \\ &= \int_M d\left(\sum_\alpha \rho_\alpha\right) \wedge \omega + \left(\sum_\alpha \rho_\alpha\right) d\omega.\end{aligned}$$

Since $\sum_\alpha \rho_\alpha \equiv 1$ and hence $d\left(\sum_\alpha \rho_\alpha\right) \equiv 0$, we have proved:

$$\int_{\partial M} \omega = \sum_\alpha \int_{\partial M} \rho_\alpha \omega = \int_M 0 \wedge \omega + 1 \, d\omega = \int_M d\omega.$$

It completes the proof of Generalized Stokes' Theorem. □

Remark 4.28. As we can see from that the proof (Step 2), if we simply choose an orientation for ∂M such that $(u_1, \ldots, u_{n-1})$ becomes the order of local coordinates for ∂M, then (4.7) would have a factor of $(-1)^n$ on the RHS, which does not look nice. Moreover, if we pick $i_{-\eta}\Omega$ to be the orientation of ∂M (here $-\eta$ is then an inward-pointing normal to ∂M), then the RHS of (4.7) would have a minus sign, which is not nice either. □

4.4.3. Fundamental Theorems in Vector Calculus. We briefly discussed at the end of Chapter 3 how the three fundamental theorems in Vector Calculus, namely Green's, Stokes' and Divergence Theorems, can be formulated using differential forms. Given that we now have proved Generalized Stokes' Theorem (Theorem 4.27), we are going to give a formal proof of the three Vector Calculus theorems in MATH 2023 using the Theorem 4.27.

Corollary 4.29 (Green's Theorem). *Let R be a closed and bounded smooth 2-submanifold in $\mathbb{R}^2$ with boundary ∂R. Given any smooth vector field $V = (P(x,y), Q(x,y))$ defined in R, then we have:*

$$\oint_{\partial R} V \cdot dl = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy,$$

The line integral on the LHS is oriented such that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ has the same orientation as $\{\eta, \mathsf{T}\}$ where η is the outward-pointing normal of R, and T is the velocity vector of the curve ∂R. See Figure 4.3.

Proof. Consider the 1-form $\omega := P\,dx + Q\,dy$ defined on R, then we have:

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Suppose we fix an orientation $\Omega = dx \wedge dy$ for R so that the order of coordinates is (x, y), then by generalized Stokes' Theorem we get:

$$\underbrace{\oint_{\partial R} P\,dx + Q\,dy}_{\oint_{\partial R}\omega} = \underbrace{\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy}_{\int_R d\omega} = \underbrace{\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{(x,y)\text{ is the orientation}}.$$

The only thing left to figure out is the orientation of the line integral. Locally parametrize R by local coordinates (s, t) so that $\{t = 0\}$ is the boundary ∂R and $\{t > 0\}$ is the interior of R (see Figure 4.3). By convention, the local coordinate s for ∂R must be chosen so that $\Omega(\eta, \frac{\partial}{\partial s}) > 0$ where η is a outward-pointing normal vector to ∂R. In other words, the pair $\{\eta, \frac{\partial}{\partial s}\}$ should have the same orientation as $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. According to Figure 4.3, we must choose the local coordinate s for ∂R such that for the outer boundary, s goes counter-clockwisely as it increases; whereas for each inner boundary, s goes clockwisely as it increases. $\square$

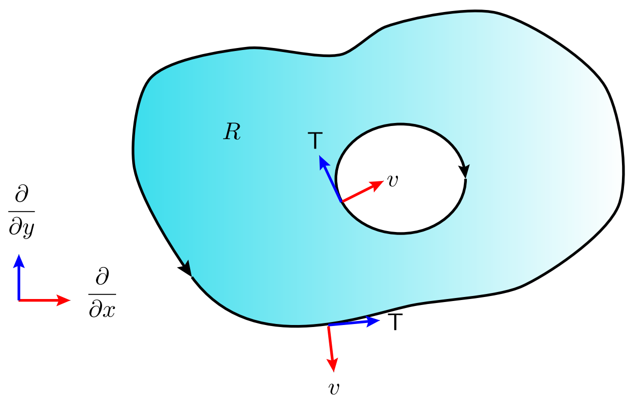


Figure 4.3. Orientation of Green's Theorem

Next we show that Stokes' Theorem in Multivariable Calculus is also a consequence of Generalized Stokes' Theorem. Recall that in MATH 2023 we learned about surface integrals. If $F(u, v) : \mathcal{U} \to \Sigma \subset \mathbb{R}^3$ is a parametrization of the whole surface Σ, then we define the surface element as:

$$dS = \left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right| du\,dv,$$

and the surface integral of a scalar function φ is defined as:

$$\int_\Sigma \varphi\,dS = \int_{\mathcal{U}} \varphi(u, v) \left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right| du\,dv.$$

However, not every surface can be covered (or almost covered) by a single parametrization chart. Generally, if $\mathcal{A} = \{F_\alpha(u_\alpha, v_\alpha) : \mathcal{U}_\alpha \to \mathbb{R}^3\}$ is an oriented atlas of Σ with a partition of unity $\{\rho_\alpha : \Sigma \to [0, 1]\}$ subordinate to $\mathcal{A}$, we then define:

$$dS := \sum_\alpha \rho_\alpha \left|\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}\right| du_\alpha\,dv_\alpha.$$

Corollary 4.30 (Stokes' Theorem)**.** *Let Σ be a closed and bounded smooth 2-submanifold in $\mathbb{R}^3$ with boundary $\partial\Sigma$, and $V = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field which is smooth on Σ, then we have:*

$$\oint_{\partial\Sigma} V \cdot dl = \int_{\Sigma} (\nabla \times V) \cdot \nu \, dS.$$

Here $\{\hat{i}, \hat{j}, \hat{k}\}$ has the same orientation as $\{\eta, \mathsf{T}, \nu\}$, where η is the outward-point normal vector of Σ at points of $\partial\Sigma$, T is the velocity vector of $\partial\Sigma$, and ν is the unit normal vector to Σ in $\mathbb{R}^3$. See Figure 4.4.

Proof. Define:

$$\omega = P\,dx + Q\,dy + R\,dz$$

which is viewed as a 1-form on Σ. Then,

$$\oint_{\partial\Sigma} \omega = \oint_{\partial\Sigma} V \cdot dl. \tag{4.8}$$

By direct computation, the 2-form $d\omega$ is given by:

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz.$$

Now consider an oriented atlas $\mathcal{A} = \{F_\alpha(u_\alpha, v_\alpha) : \mathcal{U}_\alpha \to \mathbb{R}^3\}$ of Σ with a partition of unity $\{\rho_\alpha : \Sigma \to [0,1]\}$, then according to the discussion near the end of Chapter 3, we can express each of $dx \wedge dy$, $dz \wedge dx$ and $dy \wedge dz$ in terms of $du_\alpha \wedge dv_\alpha$, and obtain:

$$\begin{aligned} d\omega &= \sum_\alpha \rho_\alpha \, d\omega \\ &= \sum_\alpha \rho_\alpha \left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz\right] \\ &= \sum_\alpha \rho_\alpha \left\{\left(\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \det \frac{\partial(y,z)}{\partial(u_\alpha, v_\alpha)} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \det \frac{\partial(z,x)}{\partial(u_\alpha, v_\alpha)} \right.\right. \\ &\qquad \left.\left. + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \det \frac{\partial(x,y)}{\partial(u_\alpha, v_\alpha)}\right)\right\} du_\alpha \wedge dv_\alpha. \end{aligned}$$

On each local coordinate chart $F_\alpha(\mathcal{U}_\alpha)$, a normal vector to Σ in $\mathbb{R}^3$ can be found using cross products:

$$\begin{aligned} \frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha} &= \det \frac{\partial(y,z)}{\partial(u_\alpha, v_\alpha)} \hat{i} + \det \frac{\partial(z,x)}{\partial(u_\alpha, v_\alpha)} \hat{j} + \det \frac{\partial(x,y)}{\partial(u_\alpha, v_\alpha)} \hat{k} \\ \nabla \times V &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \hat{j} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \hat{k}. \end{aligned}$$

Hence,

$$d\omega = \sum_\alpha (\nabla \times V) \cdot \left(\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}\right) \rho_\alpha \, du_\alpha \wedge dv_\alpha,$$

and so

$$\int_\Sigma d\omega = \sum_\alpha \int_{\mathcal{U}_\alpha} (\nabla \times V) \cdot \left(\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}\right) \rho_\alpha \, du_\alpha \, dv_\alpha.$$

Denote $\nu = \dfrac{\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}}{\left|\frac{\partial F}{\partial u_\alpha} \times \frac{\partial F}{\partial v_\alpha}\right|}$, and recall the fact that $dS := \sum_\alpha \rho_\alpha \left|\dfrac{\partial F_\alpha}{\partial u_\alpha} \times \dfrac{\partial F_\alpha}{\partial v_\alpha}\right| du_\alpha \, dv_\alpha$, we get:

$$\int_\Sigma d\omega = \int_\Sigma (\nabla \times V) \cdot \nu \, dS. \tag{4.9}$$

Combining the results of (4.8) and (4.9), using Generalized Stokes' Theorem (Theorem 4.7, we get:

$$\oint_{\partial\Sigma} V \cdot dl = \int_{\Sigma} (\nabla \times V) \cdot \nu \, dS$$

as desired.

To see the orientation of $\partial\Sigma$, we locally parametrize Σ by coordinates (s,t) such that $\{t=0\}$ are points on $\partial\Sigma$, and so $\partial\Sigma$ is locally parametrized by s. The outward-pointing normal of $\partial\Sigma$ in Σ is given by $\eta := -\frac{\partial}{\partial t}$. By convention, the orientation of $\{\eta, \frac{\partial}{\partial s}\}$ is the same as $\left\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}\right\}$, and hence:

$$\left\{\eta, \frac{\partial}{\partial s}, \nu\right\} \text{ has the same orientation as } \left\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, \nu\right\}.$$

As $\nu = \dfrac{\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}}{\left|\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}\right|}$, the set $\left\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, \nu\right\}$ has the same orientation as $\{\hat{i}, \hat{j}, \hat{k}\}$. As a result, the set $\{\eta, \frac{\partial}{\partial s}, \nu\}$ is oriented in the way as in Figure 4.4. $\square$

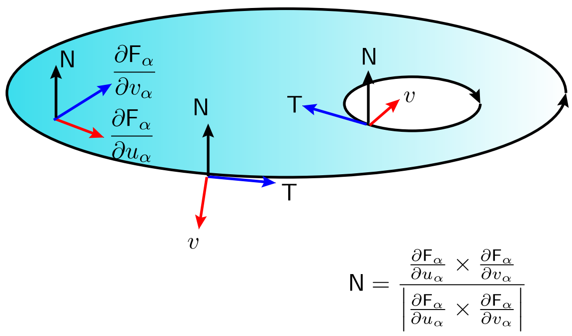


Figure 4.4. Orientation of Stokes' Theorem

Finally, we discuss how to use Generalized Stokes' Theorem to prove Divergence Theorem in Multivariable Calculus.

Corollary 4.31 (Divergence Theorem). *Let D be a closed and bounded 3-submanifold of $\mathbb{R}^3$ with boundary ∂D, and $V = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a smooth vector field defined on D. Then, we have:*

$$\oint_{\partial D} V \cdot \nu \, dS = \int_D \nabla \cdot V \, dx \, dy \, dz.$$

Here ν is the unit normal vector of ∂D in $\mathbb{R}^3$ which points away from D.

Proof. Let $\omega := P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$. Then by direct computations, we get:

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz = \nabla \cdot V \, dx \wedge dy \wedge dz.$$

Using $\{\hat{i}, \hat{j}, \hat{k}\}$ as the orientation for D, then it is clear that:

$$\int_D d\omega = \int_D \nabla \cdot V \, dx \, dy \, dz. \tag{4.10}$$

Consider an atlas $\mathcal{A} = \{F_\alpha(u_\alpha, v_\alpha, w_\alpha) : \mathcal{U}_\alpha \to \mathbb{R}^3\}$ of D such that for the local parametrization of boundary type, the boundary points are given by $\{w_\alpha = 0\}$, and interior points are $\{w_\alpha > 0\}$. Then, ∂D is locally parametrized by (u_α, v_α).

As a convention, the orientation of (u_α, v_α) is chosen such that $\{-\frac{\partial}{\partial w_\alpha}, \frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}\}$ has the same orientation as $\{\hat{i}, \hat{j}, \hat{k}\}$, or equivalently, $\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, -\frac{\partial}{\partial w_\alpha}\}$ has the same orientation as $\{\hat{i}, \hat{j}, \hat{k}\}$.

Furthermore, let ν be the unit normal of ∂D given by:

$$\nu = \frac{\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}}{\left|\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha}\right|}.$$

By the convention of cross products, $\{\frac{\partial F_\alpha}{\partial u_\alpha}, \frac{\partial F_\alpha}{\partial v_\alpha}, \nu\}$ must have the same orientation as $\{\hat{i}, \hat{j}, \hat{k}\}$. Now that $\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, -\frac{\partial}{\partial w_\alpha}\}$ and $\{\frac{\partial F_\alpha}{\partial u_\alpha}, \frac{\partial F_\alpha}{\partial v_\alpha}, \nu\}$ have the same orientation, so ν and $-\frac{\partial}{\partial w_\alpha}$ are both pointing in the same direction. In other words, ν is the outward-point normal.

The rest of the proof goes by writing ω in terms of $du_\alpha \wedge dv_\alpha$ on each local coordinate chart:

$$\begin{aligned}
\omega &= \sum_\alpha \rho_\alpha \omega \\
&= \sum_\alpha \rho_\alpha \left(P \det \frac{\partial(y,z)}{\partial(u_\alpha, v_\alpha)} + Q \det \frac{\partial(z,x)}{\partial(u_\alpha, v_\alpha)} + R \det \frac{\partial(x,y)}{\partial(u_\alpha, v_\alpha)} \right) du_\alpha \wedge dv_\alpha \\
&= \sum_\alpha V \cdot \left(\frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha} \right) \rho_\alpha \, du_\alpha \wedge dv_\alpha \\
&= \sum_\alpha V \cdot \nu \, \rho_\alpha \left| \frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha} \right| du_\alpha \wedge dv_\alpha
\end{aligned}$$

Therefore, we get:

$$\oint_{\partial D} \omega = \oint_{\partial D} \sum_\alpha V \cdot \nu \, \rho_\alpha \left| \frac{\partial F_\alpha}{\partial u_\alpha} \times \frac{\partial F_\alpha}{\partial v_\alpha} \right| du_\alpha \, dv_\alpha = \oint_{\partial D} V \cdot \nu \, dS. \tag{4.11}$$

Combining with (4.10), (4.11) and Generalized Stokes' Theorem, the proof of this corollary is completed. $\square$

Chapter 5

De Rham Cohomology

"'Should you just be an algebraist or a geometer?' is like saying 'Would you rather be deaf or blind?'"

Michael Atiyah

In Chapter 3, we discussed closed and exact forms. As a reminder, a smooth k-form ω on a smooth manifold M is *closed* if $d\omega = 0$ on M, and is *exact* if $\omega = d\eta$ for some smooth $(k-1)$-form η defined on the whole M.

By the fact that $d^2 = 0$, an exact form must be closed. It is then natural to ask whether every closed form is exact. The answer is no in general. Here is a counter-example. Let $M = \mathbb{R}^2\backslash\{(0,0)\}$, and define

$$\omega := -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

It can be computed easily that $d\omega = 0$ on M, and so ω is closed.

However, we can show that ω is *not* exact. Consider the unit circle C parametrized by $(x,y) = (\cos t, \sin t)$ where $0 < t < 2\pi$, and also the induced 1-form $\iota^*\omega$ (where $\iota : C \to M$). By direct computation, we get:

$$\oint_C \iota^*\omega = \int_0^{2\pi} -\frac{\sin t}{\cos^2 t + \sin^2 t}\,d(\cos t) + \frac{\cos t}{\cos^2 t + \sin^2 t}\,d(\sin t) = 2\pi.$$

If ω *were* exact, then $\omega = df$ for some smooth function $f : M \to \mathbb{R}$. Then, we would have:

$$\oint_C \iota^*\omega = \oint_C \iota^*(df) = \oint_C d(\iota^* f) = \int_0^{2\pi} \frac{d(\iota^* f)}{dt}\,dt.$$

Since $t = 0$ and $t = 2\pi$ represent the same point on C, by Fundamental Theorem of Calculus, we finally get:

$$\oint_C \iota^*\omega = 0$$

which is a contradiction! Therefore, ω is *not* exact on $\mathbb{R}^2\backslash\{(0,0)\}$.

Heuristically, de Rham cohomology studies "how many" smooth k-forms defined on a given manifold M are closed but not exact. We should refine the meaning of "how many". Certainly, if η is any $(k-1)$-form on M, then $\omega + d\eta$ is also closed but not exact. Therefore, when we "count" how many smooth k-forms on M which are closed but not exact, it is fair to group ω and $\omega + d\eta$'s together, and count them as one. In formal mathematical language, equivalence classes are used as we will discuss in detail. It turns out that the "number" of closed, not exact k-forms on a given M is a related to the *topology* of M!

In this chapter, we will learn the basics of de Rham cohomology, which is a beautiful topic to end the course MATH 4033.

5.1. De Rham Cohomology

Let M be a smooth manifold (with or without boundary). Recall that the exterior derivative d is a linear map that takes a k-form to a $(k+1)$-form, i.e. $d : \wedge^k T^*M \to \wedge^{k+1} T^*M$. We can then talk about the kernel and image of these maps. We define:

$$\begin{aligned}
\ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right) &= \{\omega \in \wedge^k T^*M : d\omega = 0\} \\
&= \{\text{closed } k\text{-forms on } M\} \\
\text{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right) &= \{\omega \in \wedge^k T^*M : \omega = d\eta \text{ for some } \eta \in \wedge^{k-1} T^*M\} \\
&= \{\text{exact } k\text{-forms on } M\}
\end{aligned}$$

In many occasions, we may simply denote the above kernel and image by $\ker(d)$ and $\text{Im}\,(d)$ whenever the value of k is clear from the context.

By $d^2 = 0$, it is easy to see that:

$$\text{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right) \subset \ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right).$$

If all closed k-forms on a certain manifold are exact, then we have $\text{Im}\,(d) = \ker(d)$. How "many" closed k-forms are exact is then measured by how $\text{Im}\,(d)$ is "smaller" than $\ker(d)$, which is precisely measured by the size of the quotient vector space $\ker(d)/\text{Im}\,(d)$. We call this quotient the de Rham cohomology group[1].

Definition 5.1 (de Rham Cohomology Group)**.** Let M be a smooth manifold. For any positive integer k, we define the *k-th de Rham cohomology group of M* to be the quotient vector space:

$$H^k_{\text{dR}}(M) := \frac{\ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right)}{\text{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right)}.$$

Remark 5.2. When $k = 0$, then $\wedge^k T^*M = \wedge^0 T^*M = C^\infty(M, \mathbb{R})$ and $\wedge^{k-1} T^*M$ is not defined. Instead, we *define*

$$H^0_{\text{dR}}(M) := \ker\left(d : C^\infty(M, \mathbb{R}) \to \wedge^1 T^*M\right) = \{f \in C^\infty(M, \mathbb{R}) : df = 0\},$$

which is the vector space of all locally constant functions on M. If M has N connected components, then a locally constant function f is determined by its value on each of the components. The space of functions $\{f : df = 0\}$ is in one-to-one correspondence an N-tuple $(k_1, \ldots, k_N) \in \mathbb{R}^N$, where k_i is the value of f on the i-th component of M. Therefore, $H^0_{\text{dR}}(M) \simeq \mathbb{R}^N$ where N is the number of connected components of M. □

5.1.1. Quotient Vector Spaces.

Let's first review the basics about quotient vector spaces in Linear Algebra. Given a subspace W of a vector space V, we can define an equivalence relation $\sim$ by declaring that $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$. For example, if W is the x-axis and V is the xy-plane, then two vector v_1 and v_2 are equivalent under this relation if and only if they have the same $\hat{j}$-component.

[1] A vector space is also a group whose addition is the vector addition. Although it is more appropriate or precise to call the quotient the "de Rham cohomology space", we will follow the history to call it a group.

For each element $v \in V$ (the bigger space), one can define an equivalence class:

$$[v] := \{u \in V : u \sim v\} = \{u \in V : u - v \in W\}$$

which is the set of all vectors in V that are equivalent to v. For example, if W is the x-axis and V is $\mathbb{R}^2$, then the class $[(2,3)]$ is given by:

$$[(2,3)] = \{(x,3) : x \in \mathbb{R}\}$$

which is the horizontal line $\{y = 3\}$. Similarly, one can figure out $[(1,3)] = [(2,3)] = [(3,3)] = \ldots$ as well, but $[(2,3)] \neq [(2,2)]$, and the latter is the line $\{y = 2\}$.

The quotient space V/W is defined to be the set of all equivalence classes, i.e.

$$V/W := \{[v] : v \in V\}.$$

For example, if V is $\mathbb{R}^2$ and W is the x-axis, then V/W is the set of all horizontal lines in $\mathbb{R}^2$. For finite dimensional vector spaces, one can show (see Exercise 5.1) that

$$\dim(V/W) = \dim V - \dim W,$$

and so the "size" (precisely, the dimension) of the quotient V/W measures how small W is when compared to V. In fact, if the bases of V and W are suitably chosen, we can describe the basis of V/W in a precise way (see Exercise 5.1).

Exercise 5.1. Let W be a subspace of a finite dimensional vector space V. Suppose $\{w_1, \ldots, w_k\}$ is a basis for W, and $\{w_1, \ldots, w_k, v_1, \ldots, v_l\}$ is a basis for V (Remark: given any basis $\{w_1, \ldots, w_k\}$ for the subspace W, one can always complete it to form a basis for V).

(a) Show that given any vector $\sum_{i=1}^k \alpha_i w_i + \sum_{j=1}^l \beta_j v_j \in V$, the equivalence class represented by this vector is given by:

$$\left[\sum_{i=1}^k \alpha_i w_i + \sum_{j=1}^l \beta_j v_j\right] = \left\{\sum_{i=1}^k \gamma_i w_i + \sum_{j=1}^l \beta_j v_j : \gamma_i \in \mathbb{R}\right\} = \left[\sum_{j=1}^l \beta_j v_j\right].$$

(b) Hence, show that $\{[v_1], \ldots, [v_l]\}$ is a basis for V/W, and so

$$\dim V/W = l = \dim V - \dim W.$$

Exercise 5.2. Given a subspace W of a vector space V, and define an equivalence relation $\sim$ by declaring that $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$. Show that the following are equivalent:

(1) $u \in [v]$

(2) $u - v \in W$

(3) $[u] = [v]$

5.1.2. Cohomology Classes and Betti numbers. Recall that the k-th de Rham cohomology group $H^k_{\mathrm{dR}}(M)$, where $k \geq 1$, of a smooth manifold M is defined to be the quotient vector space:

$$H^k_{\mathrm{dR}}(M) := \frac{\ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right)}{\operatorname{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right)}.$$

Given a closed k-form ω, we then define its equivalence class to be:

$$\begin{aligned}
[\omega] &:= \{\omega' : \omega' - \omega \text{ is exact}\} \\
&= \{\omega' : \omega' = \omega + d\eta \text{ for some } \eta \in \wedge^{k-1} T^*M\} \\
&= \{\omega + d\eta : \eta \in \wedge^{k-1} T^*M\}.
\end{aligned}$$

An equivalence class $[\omega]$ is called the *de Rham cohomology class* represented by (or containing) ω, and ω is said to be a representative of this de Rham cohomology class.

By Exercise 5.1, its dimension is given by

$$\begin{aligned}&\dim H^k_{\text{dR}}(M)\\&= \dim\ker\left(d : \wedge^k T^*M \to \wedge^{k+1}T^*M\right) - \dim \text{Im}\left(d : \wedge^{k-1}T^*M \to \wedge^k T^*M\right)\end{aligned}$$

provided that both kernel and image are finite-dimensional.

Therefore, the dimension of $H^k_{\text{dR}}(M)$ is a measure of "how many" closed k-forms on M are not exact. Due to the importance of this dimension, we have a special name for it:

Definition 5.3 (Betti Numbers). Let M be a smooth manifold. The *k-th Betti number* of M is defined to be:

$$b_k(M) := \dim H^k_{\text{dR}}(M).$$

In particular, $b_0(M) = \dim H^0_{\text{dR}}(M)$ is the number of connected components of M.

In case when $M = \mathbb{R}^2 \backslash \{(0,0)\}$, we discussed that there is a closed 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

defined on M which is not exact. Therefore, $\omega \in \ker\left(d : \wedge^1 T^*M \to \wedge^2 T^*M\right)$ yet $\omega \notin \text{Im}\left(d : \wedge^0 T^*M \to \wedge^1 T^*M\right)$, and so in $H^1_{\text{dR}}(M)$ we have $[\omega] \neq [0]$. From here we can conclude that $H^1_{\text{dR}}(M) \neq \{[0]\}$ and $b_1(M) \geq 1$. We will later show that in fact $b_1(M) = 1$ using some tools in later sections.

Exercise 5.3. If $k > \dim M$, what can you say about $b_k(M)$?

5.1.3. Poincaré Lemma.

A *star-shaped* open set U in $\mathbb{R}^n$ is a region containing a point $p \in U$ (call it a base point) such that any line segment connecting a point $x \in U$ and the base point p must be contained inside U. Examples of star-shaped open sets include convex open sets such an open ball $\{x \in \mathbb{R}^n : |x| < 1\}$, and all of $\mathbb{R}^n$. The following Poincaré Lemma asserts that $H^1_{\text{dR}}(U) = \{[0]\}$.

Theorem 5.4 (Poincaré Lemma for H^1_{dR}). *For any star-shaped open set U in $\mathbb{R}^n$, we have $H^1_{\text{dR}}(U) = \{[0]\}$. In other words, any closed 1-form defined on a star-shaped open set is exact on that open set.*

Proof. Given a closed 1-form ω defined on U, given by $\omega = \sum_i \omega_i\,dx^i$, we need to find a smooth function $f : U \to \mathbb{R}$ such that $\omega = df$. In other words, we need $\dfrac{\partial f}{\partial x_i} = \omega_i$ for any i.

Let p be the base point of U, then given any $x \in U$, we define:

$$f(x) := \int_{L_x} \omega$$

where L_x is the line segment joining p and x, which can be parametrized by:

$$\gamma(t) = (1-t)p + tx, \quad t \in [0,1].$$

Write $p = (p_1, \ldots, p_n)$, $x = (x_1, \ldots, x_n)$, then $f(x)$ can be expressed in terms of t by:

$$f(x) = \int_0^1 \sum_{i=1}^n \omega_i(\gamma(t)) \cdot (x_i - p_i)\,dt.$$

Using the chain rule, we can directly verify that:

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x) &= \frac{\partial}{\partial x_j}\int_0^1 \sum_{i=1}^n \omega_i(\gamma(t))\cdot(x_i-p_i)\,dt \\
&= \sum_{i=1}^n \int_0^1 \left(\frac{\partial}{\partial x_j}\omega_i(\gamma(t))\cdot(x_i-p_i)+\omega_i(\gamma(t))\cdot\frac{\partial}{\partial x_j}(x_i-p_i)\right)\,dt \\
&= \sum_{i=1}^n \int_0^1 \left(\sum_{k=1}^n \frac{\partial \omega_i}{\partial x_k}\frac{\partial\overbrace{((1-t)p_k+tx_k)}^{x_k\circ\gamma(t)}}{\partial x_j}\cdot(x_i-p_i)+\omega_i(\gamma(t))\cdot\delta_{ij}\right)\,dt \\
&= \sum_{i=1}^n \int_0^1 \left(\sum_{k=1}^n t\frac{\partial \omega_i}{\partial x_k}\,\delta_{jk}\cdot(x_i-p_i)+\omega_j(\gamma(t))\right)\,dt \\
&= \sum_{i=1}^n \int_0^1 \left(t\frac{\partial \omega_i}{\partial x_j}\cdot(x_i-p_i)+\omega_j(\gamma(t))\right)\,dt
\end{aligned}$$

Since ω is closed, we have:

$$0 = d\omega = \sum_{i<j}^n \left(\frac{\partial \omega_i}{\partial x_j}-\frac{\partial \omega_j}{\partial x_i}\right)\,dx^j\wedge dx^i$$

and hence $\dfrac{\partial \omega_i}{\partial x_j}=\dfrac{\partial \omega_j}{\partial x_i}$ for any i,j. Using this to proceed our calculation:

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x) &= \int_0^1 \left(t\frac{\partial \omega_j}{\partial x_i}\cdot(x_i-p_i)+\omega_j(\gamma(t))\right)\,dt \\
&= \int_0^1 \frac{d}{dt}\left(t\omega_j(\gamma(t))\right)\,dt \\
&= \left[t\omega_j(\gamma(t))\right]_{t=0}^{t=1} = \omega_j(\gamma(1)) = \omega_j(x).
\end{aligned}$$

In the second equality above, we have used the chain rule backward:

$$\frac{d}{dt}\left(t\omega_j(\gamma(t))\right) = t\frac{\partial \omega_j}{\partial x_i}\cdot(x_i-p_i)+\omega_j(\gamma(t)).$$

From this, we conclude that $\omega = df$ on U, and hence $[\omega]=[0]$ in $H^1_{\mathrm{dR}}(U)$. Since ω is an arbitrary closed 1-form on U, we have $H^1_{\mathrm{dR}}(U)=\{[0]\}$. □

Remark 5.5. Poincaré Lemma also holds for H^k_{dR}, meaning that if U is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{\mathrm{dR}}(U)=\{[0]\}$ for any $k\geq 1$. However, the proof involves the use of Lie derivatives and a formula by Cartan, both of which are beyond the scope of this course. Note also that $H^0_{\mathrm{dR}}(U)\simeq\mathbb{R}$ since a star-shaped open set must be connected. □

Remark 5.6. We have discussed that the 1-form

$$\omega = \frac{-y\,dx+x\,dy}{x^2+y^2}$$

is closed but not exact. To be precise, it is not exact *on* $\mathbb{R}^2\backslash\{(0,0)\}$. However, if we regard the domain to be the first quadrant $U:=\{(x,y):x>0 \text{ and } y>0\}$, which is a star-shaped open set in $\mathbb{R}^2$, then by Poinaré Lemma (Theorem 5.4), ω is indeed an exact 1-form on U. In fact, it is not difficult to verify that

$$\omega = d\left(\tan^{-1}\frac{y}{x}\right) \quad \text{on } U.$$

Note that the scalar function $\tan^{-1}\frac{y}{x}$ is smoothly defined on U. Whether a form is exact or not depends on the choice of its domain! $\square$

5.1.4. Diffeomorphic Invariance. By Proposition 3.57, we learned that the exterior derivative d commutes with the pull-back of a smooth map between two manifolds. An important consequence is that the de Rham cohomology group is invariant under diffeomorphism.

Let $\Phi : M \to N$ be any smooth map between two smooth manifolds. The pull-back map $\Phi^* : \wedge^k T^*N \to \wedge^k T^*M$ induces a well-defined pull-back map (which is also denoted by Φ^*) from $H^k_{\mathrm{dR}}(N)$ to $H^k_{\mathrm{dR}}(M)$. Precisely, given any closed k-form ω on N, we define:

$$\Phi^*[\omega] := [\Phi^*\omega].$$

$\Phi^*\omega$ is a k-form on M. It is closed since $d(\Phi^*\omega) = \Phi^*(d\omega) = \Phi^*(0) = 0$. To show it is well-defined, we take another k-form ω' on N such that $[\omega'] = [\omega]$ in $H^k_{\mathrm{dR}}(N)$. Then, there exists a $(k-1)$-form η on N such that:

$$\omega' - \omega = d\eta \quad \text{on } N.$$

Using again $d \circ \Phi^* = \Phi^* \circ d$, we get:

$$\Phi^*\omega' - \Phi^*\omega = \Phi^*(d\eta) = d(\Phi^*\eta) \quad \text{on } M$$

We conclude $\Phi^*\omega' - \Phi^*\omega$ is exact and so

$$[\Phi^*\omega'] = [\Phi^*\omega] \text{ in } H^k_{\mathrm{dR}}(M).$$

This shows $\Phi^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ is a well-defined map.

Theorem 5.7 (Diffeomorphism Invariance of H^k_{dR}). *If two smooth manifolds M and N are diffeomorphic, then $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(N)$ are isomorphic for any $k \geq 0$.*

Proof. Let $\Phi : M \to N$ be a diffeomorphism, then $\Phi^{-1} : N \to M$ exists and we have $\Phi \circ \Phi^{-1} = \mathrm{id}_N$ and $\Phi^{-1} \circ \Phi = \mathrm{id}_M$. By the chain rule for tensors (Theorem 3.54), we have:

$$(\Phi^{-1})^* \circ \Phi^* = \mathrm{id}_{\wedge^k T^*N} \quad \text{and} \quad \Phi^* \circ (\Phi^{-1})^* = \mathrm{id}_{\wedge^k T^*M}.$$

Given any closed k-form ω on M, then in $H^k_{\mathrm{dR}}(M)$ we have:

$$\Phi^* \circ (\Phi^{-1})^*[\omega] = \Phi^*[(\Phi^{-1})^*\omega] = [\Phi^* \circ (\Phi^{-1})^*\omega] = [\omega].$$

In other words, $\Phi^* \circ (\Phi^{-1})^*$ is also the identity map of $H^k_{\mathrm{dR}}(M)$. Similarly, one can also show $(\Phi^{-1})^* \circ \Phi^*$ is the identity map of $H^k_{\mathrm{dR}}(N)$. Therefore, $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(N)$ are isomorphic (as vector spaces). $\square$

Corollary 5.8. *Given any smooth manifold M which is diffeomorphic to a star-shaped open set in $\mathbb{R}^n$, we have $H^1_{\mathrm{dR}}(M) \simeq \{[0]\}$, or in other words, every closed 1-form ω on such a manifold M is exact.*

Proof. Combine the results of the Poincaré Lemma (Theorem 5.4) and the diffeomorphism invariance of H^1_{dR} (Theorem 5.7). $\square$

Consequently, a large class of open sets in $\mathbb{R}^n$ has trivial H^1_{dR} as long as it is diffeomorphic to a star-shaped manifold. For open sets in $\mathbb{R}^2$, there is a celebrated result called Riemann Mapping Theorem, which says any (non-empty) simply-connected open bounded subset U in $\mathbb{R}^2$ is diffeomorphic to the unit open ball in $\mathbb{R}^2$. In fact, the diffeomorphism can be chosen so that angles are preserved, but we don't need this when dealing with de Rham cohomology.

Under the assumption of Riemann Mapping Theorem (whose proof can be found in advanced Complex Analysis textbooks), we can establish that $H^1_{\text{dR}}(U) = \{[0]\}$ for any (non-empty) simply-connected subset U in $\mathbb{R}^2$. Consequently, any closed 1-form on such a domain U is exact on U. Using the language in Multivariable Calculus (or Physics), this means any curl-zero vector field defined on a (non-empty) simply-connected domain U in $\mathbb{R}^2$ must be conservative on U. You might have learned this fact without proof in MATH 2023.

5.2. Deformation Retracts

In the previous section, we learned that two diffeomorphic manifolds have isomorphic de Rham cohomology groups. In short, we say de Rham cohomology is a diffeomorphic invariance. In this section, we will discuss another type of invariance: *deformation retracts*.

Let M be a smooth manifold (with or without boundary), and Σ is a submanifold of M. Note that Σ can have lower dimension than M. Roughly speaking, we say Σ is a *deformation retract* of M if one can continuously contract M onto Σ. Let's make it more precise:

Definition 5.9 (Deformation Retract)**.** Let M be a smooth manifold, and Σ is a submanifold of M. If there exists a C^1 family of smooth maps $\{\Psi_t : M \to M\}_{t\in[0,1]}$ satisfying all three conditions below:

- $\Psi_0(x) = x$ for any $x \in M$, i.e. $\Psi_0 = \mathrm{id}_M$;
- $\Psi_1(x) \in \Sigma$ for any $x \in M$, i.e. $\Psi_1 : M \to \Sigma$;
- $\Psi_t(p) = p$ for any $p \in \Sigma$, $t \in [0,1]$, i.e. $\Psi_t|_\Sigma = \mathrm{id}_\Sigma$ for any $t \in [0,1]$,

then we say Σ is a deformation retract of M. Equivalently, we can also say M deformation retracts onto Σ.

One good way to think of a deformation retract is to regard t as the time, and Ψ_t is a "movie" that demonstates how M collapses onto Σ. The condition $\Psi_0 = \mathrm{id}_M$ says initially (at $t = 0$), the "movie" starts with the image M. At the final scene (at $t = 1$), the condition $\Psi_1 : M \to \Sigma$ says that the image eventually becomes Σ . The last condition $\Psi_t(p) = p$ for any $p \in \Sigma$ means the points on Σ do not move throughout the movie. Before we talk about the relation between cohomology and deformation retract, let's first look at some examples:

Example 5.10. The unit circle $\mathbb{S}^1$ defined by $\{(x,y) : x^2 + y^2 = 1\}$ is a deformation retract of the annulus $\{(x,y) : \frac{1}{4} < x^2 + y^2 < 4\}$. To describe such a retract, it's best to use polar coordinates:

$$\Psi_t(re^{i\theta}) = (r + t(1-r))\, e^{i\theta}$$

For each $t \in [0,1]$, the map Ψ_t has image inside the annulus since $r + t(1-r) \in (\frac{1}{2}, 2)$ whenever $r \in (\frac{1}{2}, 2)$ and $t \in [0,1]$. One can easily check that $\Psi_0(re^{i\theta}) = re^{i\theta}$, $\Psi_1(re^{i\theta}) = e^{i\theta}$ and $\Psi_t(e^{i\theta}) = e^{i\theta}$ for any (r,θ) and $t \in [0,1]$. Hence Ψ_t fulfills all three conditions stated in Definition 5.9. □

Example 5.11. Intuitively, we can see the letters E, F, H, K, L, M and N all deformation retract onto the letter I. Also, the letter Q deformation retracts onto the letter O. The explicit Ψ_t for each deformation retract is not easy to write down. □

Example 5.12. A two-dimensional torus with a point removed can deformation retract onto two circles joined at one point. Try to visualize it! □

Exercise 5.4. Show that the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a deformation retract of $\mathbb{R}^2\backslash\{(0,0)\}$.

Exercise 5.5. Show that any star-shaped open set U in $\mathbb{R}^n$ deformation retracts onto its base point.

Exercise 5.6. Let M be a smooth manifold, and Σ_0 be the zero section of the tangent bundle, i.e. Σ_0 consists of all pairs $(p, 0_p)$ in TM where $p \in M$ and 0_p is the zero vector in T_pM. Show that the zero section Σ_0 is a deformation retract of the tangent bundle TM.

Exercise 5.7. Define a relation $\sim$ of manifolds by declaring that $M_1 \sim M_2$ if and only if M_1 is a deformation retract of M_2. Is $\sim$ an equivalence relation?

We next show an important result in de Rham theory, which asserts that deformation retracts preserve the first de Rham cohomology group.

Theorem 5.13 (Invariance under Deformation Retracts). *Let M be a smooth manifold, and Σ be a submanifold of M. If Σ is a deformation retract of M, then $H^1_{\mathrm{dR}}(M)$ and $H^1_{\mathrm{dR}}(\Sigma)$ are isomorphic.*

Proof. Let $\iota : \Sigma \to M$ be the inclusion map, and $\{\Psi_t : M \to M\}_{t\in[0,1]}$ be the family of maps satisfying all conditions stated in Definition 5.9. Then, the pull-back map $\iota^* : \wedge^1 T^*M \to \wedge^1 T^*\Sigma$ induces a map $\iota^* : H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(\Sigma)$. Also, the map $\Psi_1 : M \to \Sigma$ induces a pull-back map $\Psi_1^* : H^1_{\mathrm{dR}}(\Sigma) \to H^1_{\mathrm{dR}}(M)$. The key idea of the proof is to show that Ψ_1^* and ι^* are inverses of each other as maps between $H^1_{\mathrm{dR}}(M)$ and $H^1_{\mathrm{dR}}(\Sigma)$.

Let ω be an arbitrary closed 1-form defined on M. Similar to the proof of Poincaré Lemma (Theorem 5.4), we consider the scalar function $f : M \to \mathbb{R}$ defined by:

$$f(x) = \int_{\Psi_t(x)} \omega$$

Here, $\Psi_t(x)$ is regarded as a curve with parameter t joining $\Psi_0(x) = x$ and $\Psi_1(x) \in \Sigma$. We will show the following result:

$$\Psi_1^*\iota^*\omega - \omega = df \tag{5.1}$$

which will imply $[\omega] = \Psi_1^*\iota^*[\omega]$, or in other words, $\Psi_t^* \circ \iota^* = \mathrm{id}$ on $H^1_{\mathrm{dR}}(M)$.

To prove (5.1), we use local coordinates $(u_1, \ldots, u_n)$, and express ω in terms of local coordinates $\omega = \sum_i \omega_i du^i$. For simplicity, let's assume that such a local coordinate chart can cover the whole curve $\Psi_t(x)$ for $t \in [0,1]$. We will fix this issue later. For each $t \in [0,1]$, we write $\Psi_t^i(x)$ to be the u_i-coordinate of $\Psi_t(x)$, i.e. $\Psi_t^i = u_i \circ \Psi_t$. Then, one can calculate df using local coordinates. The calculation is similar to the one we did in the proof of Poincaré Lemma (Theorem 5.4):

$$\begin{aligned}
f(x) &= \int_{\Psi_t(x)} \omega = \int_0^1 \sum_i \omega_i(\Psi_t(x)) \frac{\partial \Psi_t^i}{\partial t}\, dt \\
(df)(x) &= \sum_j \frac{\partial f}{\partial u_j}\, du^j = \sum_j \left\{ \int_0^1 \frac{\partial}{\partial u_j} \left(\sum_i \omega_i(\Psi_t(x)) \frac{\partial \Psi_t^i}{\partial t} \right) dt \right\} du^j \\
&= \sum_j \left\{ \int_0^1 \left[\sum_{i,k} \left.\frac{\partial \omega_i}{\partial u_k}\right|_{\Psi_t(x)} \frac{\partial \Psi_t^k}{\partial u_j} \frac{\partial \Psi_t^i}{\partial t} + \sum_i \omega_i(\Psi_t(x)) \frac{\partial}{\partial t} \left(\frac{\partial \Psi_t^i}{\partial u_j} \right) \right] dt \right\} du^j
\end{aligned}$$

Next, recall that ω is a closed 1-form, so we have $\dfrac{\partial \omega_i}{\partial u_k} = \dfrac{\partial \omega_k}{\partial u_i}$ for any i, k. Using this on the first term, and by switching indices of the second term in the integrand, we get:

$$\begin{aligned}(df)(x) &= \sum_j \left\{ \int_0^1 \left[\sum_{i,k} \frac{\partial \omega_k}{\partial u_i}\bigg|_{\Psi_t(x)} \frac{\partial \Psi_t^k}{\partial u_j} \frac{\partial \Psi_t^i}{\partial t} + \sum_k \omega_k(\Psi_t(x)) \frac{\partial}{\partial t}\left(\frac{\partial \Psi_t^k}{\partial u_j}\right)\right] dt \right\} du^j \\ &= \sum_j \left\{ \int_0^1 \frac{\partial}{\partial t} \left(\sum_k \omega_k(\Psi_t(x)) \frac{\partial \Psi_t^k}{\partial u_j} \right) dt \right\} du^j = \sum_{j,k} \left[\omega_k(\Psi_t(x)) \frac{\partial \Psi_t^k}{\partial u_j} \right]_{t=0}^{t=1} du^j\end{aligned}$$

where the last equality follows from the (backward) chain rule.

Denote $\iota_t : \Psi_t(M) \to M$ the inclusion map at time t, then one can check that

$$\begin{aligned}\Psi_t^* \iota_t^* \omega(x) &= (\iota_t \circ \Psi_t)^* \omega(x) = (\iota_t \circ \Psi_t)^* \sum_k \omega_k du^k \\ &= \sum_k \omega_k(\iota_t \circ \Psi_t(x))\, d(u_k \circ \iota_t \circ \Psi_t(x)) \\ &= \sum_k \omega_k(\iota_t \circ \Psi_t(x))\, d\Psi_t^k \\ &= \sum_{j,k} \omega_k(\Psi_t(x))\, \frac{\partial \Psi_t^k}{\partial u_j}\, du^j.\end{aligned}$$

Therefore, we get:

$$df = \sum_{j,k} \left[\omega_k(\Psi_t(x)) \frac{\partial \Psi_t^k}{\partial u_j} \right]_{t=0}^{t=1} du^j = [\Psi_t^* \iota_t^* \omega]_{t=0}^{t=1} = \Psi_1^* \iota_1^* \omega - \Psi_0^* \iota_0^* \omega.$$

Since $\Psi_0 = \mathrm{id}_M$ and $\iota_0 = \mathrm{id}_M$, we have proved (5.1). In case $\Psi_t(x)$ cannot be covered by one single local coordinate chart, one can then modify the above proof a bit by covering the curve $\Psi_t(x)$ by finitely many local coordinate charts. It can be done because $\Psi_t(x)$ is compact. Suppose $0 = t_0 < t_1 < \ldots < t_N = 1$ is a partition of $[0, 1]$ such that for each α, the curve $\Psi_t(x)$ restricted to $t \in [t_{\alpha-1}, t_\alpha]$ can be covered by a single local coordinate chart, then we have:

$$f(x) = \sum_{\alpha=1}^N \int_{t_{\alpha-1}}^{t_\alpha} \sum_i \omega_i(\Psi_t(x)) \frac{\partial \Psi_t^i}{\partial t}\, dt.$$

Proceed as in the above proof, we can get:

$$df = \sum_{\alpha=1}^N \left(\Psi_{t_\alpha}^* \iota_{t_\alpha}^* \omega - \Psi_{t_\alpha - 1}^* \iota_{t_\alpha - 1}^* \omega \right) = \Psi_1^* \iota_1^* \omega - \Psi_0^* \iota_0^* \omega,$$

which completes the proof of (5.1) in the general case.

To complete the proof of the theorem, we consider an arbitrary 1-form η on Σ. We claim that

$$\iota^* \Psi_1^* \eta = \eta. \tag{5.2}$$

We prove by direct verification using local coordinates $(u_1, \ldots, u_n)$ on M such that:

$$(u_1, \ldots, u_k, 0, \ldots, 0) \in \Sigma.$$

Such a local coordinate system always exists near Σ by Immersion Theorem (Theorem 2.42). Locally, denote $\eta = \sum_{i=1}^k \eta_i \, du^i$, then

$$\begin{aligned}(\Psi_1^*\eta)(x) &= \sum_{i=1}^k \Psi_1^*(\eta_i(x)\, du^i) = \sum_{i=1}^k \eta_i(\Psi_1(x))\, d(u^i \circ \Psi_1) \\ &= \sum_{i=1}^k \sum_{j=1}^k \eta_i(\Psi_1(x)) \frac{\partial \Psi_1^i(x)}{\partial u_j}\, du^j.\end{aligned}$$

Since $\Psi_1(x) = x$ whenever $x \in \Sigma$, we have $\Psi_1^i(x) = u_i(x)$ where $u_i(x)$ is the i-th coordinate of x. Therefore, we get $\dfrac{\partial \Psi_1^i(x)}{\partial u_j} = \dfrac{\partial u_i}{\partial u_j} = \delta_{ij}$ and so:

$$(\Psi_1^*\eta)(x) = \sum_{i,j=1}^k \eta_i(x)\delta_{ij}\, du^j = \sum_{i=1}^k \eta_i(x)\, du^i = \eta(x)$$

for any $x \in \Sigma$. In other words, $\iota^*\Psi_1^*\eta = \eta$ on Σ. This proves (5.2).

Combining (5.1) and (5.2), we get $\iota^* \circ \Psi_1^* = \text{id}$ on $H^1_{\text{dR}}(\Sigma)$, and $\Psi_1^* \circ \iota^* = \text{id}$ on $H^1_{\text{dR}}(M)$. As a result, Ψ_1^* and ι^* are inverses of each other in H^1_{dR}. It completes the proof that $H^1_{\text{dR}}(M)$ and $H^1_{\text{dR}}(\Sigma)$ are isomorphic. □

Using Theorem 5.13, we see that $H^1_{\text{dR}}(\mathbb{R}^2 \backslash \{(0,0)\})$ and $H^1_{\text{dR}}(\mathbb{S}^1)$ are isomorphic, and hence $b_1(\mathbb{R}^2 \backslash \{(0,0)\}) = b_1(\mathbb{S}^1)$. At this moment, we still don't know the exact value of $b_1(\mathbb{S}^1)$, but we will figure it out in the next section.

Note that Theorem 5.13 holds for H^k_{dR} for any $k \geq 2$ as well, but the proof again uses some Lie derivatives and Cartan's formula, which are beyond the scope of this course.

Another nice consequence of Theorem 5.13 is the 2-dimensional case of the following celebrated theorem in topology:

Theorem 5.14 (Brouwer's Fixed-Point Theorem on $\mathbb{R}^2$). *Let $B_1(0)$ be the closed ball with radius 1 centered at origin in $\mathbb{R}^2$. Suppose $\Phi : B_1(0) \to B_1(0)$ is a smooth map between $B_1(0)$. Then, there exists a point $x \in B_1(0)$ such that $\Phi(x) = x$.*

Proof. We prove by contradiction. Suppose $\Phi(x) \neq x$ for any $x \in B_1(0)$. Then, we let $\Psi_t(x)$ be a point in $B_1(0)$ defined in the following way:

(1) Consider the vector $x - \Phi(x)$ which is non-zero.
(2) Consider the straight ray emanating from x in the direction of $x - \Phi(x)$. This ray will intersect the unit circle $\mathbb{S}^1$ at a unique point p_x.
(3) We then define $\Psi_t(x) := (1-t)x + tp_x$

We leave it as an exercise for readers to write down the explicit formula for $\Psi_t(x)$, and show that it is smooth for each $t \in [0,1]$.

Clearly, we have $\Psi_0(x) = x$ for any $x \in B_1(0)$; $\Psi_1(x) = p_x \in \mathbb{S}^1$; and if $|x| = 1$, then $p_x = x$ and so $\Psi_t(x) = x$.

Therefore, it shows $\mathbb{S}^1$ is a deformation retract of $B_1(0)$, and by Theorem 5.13, their H^1_{dR}'s are isomorphic. However, we know $H^1_{\text{dR}}(B_1(0)) \simeq \{[0]\}$, while $H^1_{\text{dR}}(\mathbb{S}^1) \simeq H^1_{\text{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) \neq \{[0]\}$. It is a contradiction! It completes the proof that there is at least a point $x \in B_1(0)$ such that $\Phi(x) = x$. □

Exercise 5.8. Write down an explicit expression of p_x in the above proof, and hence show that Ψ_t is smooth for each fixed t.

Exercise 5.9. Generalize the Brouwer's Fixed-Point Theorem in the following way: given a manifold Ω which is diffeomorphic to $B_1(0)$, and a smooth map $\Phi : \Omega \to \Omega$. Using Theorem 5.14, show that there exists a point $p \in \Omega$ such that $\Phi(p) = p$.

Exercise 5.10. What fact(s) are needed to be established in order to prove the Brouwer's Fixed-Point Theorem for general $\mathbb{R}^n$ using a similar way as in the proof of Theorem 5.14?

5.3. Mayer-Vietoris Theorem

In the previous section, we showed that if Σ is a deformation retract of M, then $H^1_{\text{dR}}(\Sigma)$ and $H^1_{\text{dR}}(M)$ are isomorphic. For instance, this shows $H^1_{\text{dR}}(\mathbb{S}^1)$ is isomorphic to $H^1_{\text{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$. Although we have discussed that $H^2_{\text{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$ is non-trivial, we still haven't figured out what this group is. In this section, we introduce a useful tool, called Mayer-Vietoris sequence, that we can use to compute the de Rham cohomology groups of $\mathbb{R}^2\backslash\{(0,0)\}$, as well as many other spaces.

5.3.1. Exact Sequences.

Consider a sequence of homomorphism between abelian groups:

$$\cdots \xrightarrow{T_{k-1}} G_{k-1} \xrightarrow{T_k} G_k \xrightarrow{T_{k+1}} G_{k+1} \xrightarrow{G_{k+1}} \cdots$$

We say it is an *exact sequence* if the image of each homomorphism is equal to the kernel of the next one, i.e.

$$\operatorname{Im} T_{i-1} = \ker T_i \quad \text{for each } i.$$

One can also talk about exact-ness for a finite sequence, say:

$$G_0 \xrightarrow{T_1} G_1 \xrightarrow{T_2} G_2 \xrightarrow{T_3} \cdots \xrightarrow{T_{n-1}} G_{n-1} \xrightarrow{T_n} G_n$$

However, such a T_1 would not have a previous map, and such an T_n would not have the next map. Therefore, whenever we talk about the exact-ness of a finite sequence of maps, we will add two trivial maps at both ends, i.e.

$$0 \xrightarrow{0} G_0 \xrightarrow{T_1} G_1 \xrightarrow{T_2} G_2 \xrightarrow{T_3} \cdots G_{n-1} \xrightarrow{T_n} G_n \xrightarrow{0} 0. \tag{5.3}$$

The first map $0 \xrightarrow{0} G_0$ is the homomorphism taking the zero in the trivial group to the zero in G_0. The last map $G_n \xrightarrow{0} 0$ is the linear map that takes every element in G_n to the zero in the trivial group. We say the finite sequence (5.3) an *exact sequence* if

$$\operatorname{Im}(0 \xrightarrow{0} G_0) = \ker T_1, \quad \operatorname{Im} T_n = \ker(G_n \xrightarrow{0} 0), \quad \text{and} \quad \operatorname{Im} T_i = \ker T_{i+1} \quad \text{for any } i.$$

Note that $\operatorname{Im}(0 \xrightarrow{0} G_0) = \{0\}$ and $\ker(G_n \xrightarrow{0} 0) = G_n$, so if (5.3) is an exact sequence, it is necessary that

$$\ker T_1 = \{0\} \quad \text{and} \quad \operatorname{Im} T_n = G_n$$

or equivalently, T_1 is injective and T_n is surjective.

One classic example of a finite exact sequence is:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{f} \mathbb{C}\backslash\{0\} \to 0$$

where $\iota : \mathbb{Z} \to \mathbb{C}$ is the inclusion map taking $n \in \mathbb{Z}$ to itself $n \in \mathbb{C}$. The map $f : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ is the map taking $z \in \mathbb{C}$ to $e^{2\pi i z} \in \mathbb{C}\backslash\{0\}$.

It is clear that ι is injective and f is surjective (from Complex Analysis). Also, we have $\operatorname{Im}\iota = \mathbb{Z}$ and $\ker f = \mathbb{Z}$ as well (note that the identity of $\mathbb{C}\backslash\{0\}$ is 1, not 0). Therefore, this is an exact sequence.

Exercise 5.11. Given an exact sequence of group homomorphisms:

$$0 \to A \xrightarrow{T} B \xrightarrow{S} C \to 0,$$

(a) If it is given that $C = \{0\}$, what can you say about A and B?

(b) If it is given that $A = \{0\}$, what can you say about B and C?

5.3.2. Mayer-Vietoris Sequences. We talk about exact sequences because there is such a sequence concerning de Rham cohomology groups. This exact sequence, called the Mayer-Vietoris sequence, is particularly useful for computing H^k_{dR} for many manifolds.

The basic setup of a Mayer-Vietoris sequence is a smooth manifold (with or without boundary) which can be expressed a union of two open sets U and V, i.e. $M = U \cup V$. Note that we do not require U and V are disjoint. The intersection $U \cap V$ is a subset of both U and V; and each of U and V is in turn a subset of M. To summarize, we have the following relations of sets:

$$\begin{tikzcd}
 & U \arrow[dr, "j_U"] & \\
U \cap V \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & M \\
 & V \arrow[ur, "j_V"'] &
\end{tikzcd}$$

where i_U, i_V, j_U and j_V are inclusion maps. Each inclusion map, say $j_U : U \to M$, induces a pull-back map $j_U^* : \wedge^k T^*M \to \wedge^k T^*U$ which takes any k-form ω on M, to the k-form $\omega|_U$ restricted on U, i.e. $j_U^*(\omega) = \omega|_U$ for any $\omega \in \wedge^k T^*M$. In terms of local expressions, there is essentially no difference between ω and $\omega|_U$ since U is open. If locally $\omega = \sum_i \omega_i \, du^i$ on M, then $\omega|_U = \sum_i \omega_i \, du^i$ as well. The only difference is the domain: $\omega(p)$ is defined for every $p \in M$, while $\omega|_U(p)$ is defined only when $p \in U$.

To summarize, we have the following diagram:

$$\begin{tikzcd}
 & U \arrow[dl, "i_U^*"'] & \\
U \cap V & & M \arrow[ul, "j_U^*"'] \arrow[dl, "j_V^*"] \\
 & V \arrow[ul, "i_V^*"] &
\end{tikzcd}$$

Using the pull-backs of these four inclusions i_U, i_V, j_U and j_V, one can form a sequence of linear maps for each integer k:

$$0 \to \wedge^k T^*M \xrightarrow{j_U^* \oplus j_V^*} \wedge^k T^*U \oplus \wedge^k T^*V \xrightarrow{i_U^* - i_V^*} \wedge^k T^*(U \cap V) \to 0 \tag{5.4}$$

Here, $\wedge^k T^*U \oplus \wedge^k T^*V$ is the direct sum of the vector spaces $\wedge^k T^*U$ and $\wedge^k T^*V$, meaning that:

$$\wedge^k T^*U \oplus \wedge^k T^*V = \{(\omega, \eta) : \omega \in \wedge^k T^*U \text{ and } \eta \in \wedge^k T^*V\}.$$

The map $j_U^* \oplus j_V^* : \wedge^k T^*M \to \wedge^k T^*U \oplus \wedge^k T^*V$ is defined by:

$$(j_U^* \oplus j_V^*)(\omega) = (j_U^*\omega, j_V^*\omega) = (\omega|_U, \omega|_V).$$

The map $\wedge^k T^*U \oplus \wedge^k T^*V \xrightarrow{i_U^* - i_V^*} \wedge^k T^*(U \cap V)$ is given by:

$$(i_U^* - i_V^*)(\omega, \eta) = i_U^*\omega - i_V^*\eta = \omega|_{U\cap V} - \eta|_{U \cap V}.$$

We next show that the sequence (5.4) is exact. Let's first try to understand the image and kernel of each map involved.

Given $(\omega, \eta) \in \ker(i_U^* - i_V^*)$, we will have $\omega|_{U\cap V} = \eta|_{U\cap V}$. Therefore, $\ker(i_U^* - i_V^*)$ consists of pairs (ω, η) where ω and η agree on the intersection $U \cap V$.

Now consider $\mathrm{Im}\,(j_U^* \oplus j_V^*)$, which consists of pairs of the form $(\omega|_U, \omega|_V)$. Certainly, the restrictions of both $\omega|_U$ and $\omega|_V$ on the intersection $U \cap V$ are the same, and hence the pair is inside $\ker(i_U^* - i_V^*)$. Therefore, we have $\mathrm{Im}\,(j_U^* \oplus j_V^*) \subset \ker(i_U^* - i_V^*)$.

In order to show (5.4) is exact, we need further that:

(1) $j_U^* \oplus j_V^*$ is injective;

(2) $i_U^* - i_V^*$ is surjective; and

(3) $\ker(i_U^* - i_V^*) \subset \operatorname{Im}(j_U^* \oplus j_V^*)$

We leave (1) as an exercises, and will give the proofs of (2) and (3).

Exercise 5.12. Show that $j_U^* \oplus j_V^*$ is injective in the sequence (5.4).

Proposition 5.15. *Let M be a smooth manifold. Suppose there are two open subsets U and V of M such that $M = U \cup V$, and $U \cap V$ is non-empty, then the sequence of maps* (5.4) *is exact.*

Proof. So far we have proved that $j_U^* \oplus j_V^*$ is injective, and $\operatorname{Im}(j_U^* \oplus j_V^*) \subset \ker(i_U^* - i_V^*)$. We next claim that $\ker(i_U^* - i_V^*) \subset \operatorname{Im}(j_U^* \oplus j_V^*)$:

Let $(\omega, \eta) \in \ker(i_U^* - i_V^*)$, meaning that ω is a k-form on U, η is a k-form on V, and that $\omega|_{U\cap V} = \eta|_{U\cap V}$. Define a k-form σ on $M = U \cup V$ by:

$$\sigma = \begin{cases} \omega & \text{on } U \\ \eta & \text{on } V \end{cases}$$

Note that σ is well-defined on $U \cap V$ since ω and η agree on $U \cap V$. Then, we have:

$$(\omega, \eta) = (\sigma|_U, \sigma|_V) = (j_U^*\sigma, j_V^*\sigma) = (j_U^* \oplus j_V^*)\sigma \in \operatorname{Im}(j_U^* \oplus j_V^*).$$

Since (ω, η) is arbitrary in $\ker(i_U^* - i_V^*)$, this shows:

$$\ker(i_U^* - i_V^*) \subset \operatorname{Im}(j_U^* \oplus j_V^*).$$

Finally, we show $i_U^* - i_V^*$ is surjective. Given any k-form $\theta \in \wedge^k T^*(U \cap V)$, we need to find a k-form ω' on U, and a k-form η' on V such that $\omega' - \eta' = \theta$ on $U \cap V$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. We define:

$$\omega' = \begin{cases} \rho_V\theta & \text{on } U \cap V \\ 0 & \text{on } U\backslash V \end{cases}$$

Note that ω' is smooth: If $p \in \operatorname{supp}\rho_V \subset V$, then $p \in V$ (which is open) and so $\omega' = \rho_V\theta$ in an open neighborhood of p. Note that ρ_V and θ are smooth at p, so ω' is also smooth at p. On the other hand, if $p \notin \operatorname{supp}\rho_V$, then $\omega' = 0$ in an open neighborhood of p. In particular, ω' is smooth at p.

Similarly, we define:

$$\eta' = \begin{cases} -\rho_U\theta & \text{on } U \cap V \\ 0 & \text{on } V\backslash U \end{cases}$$

which can be shown to be smooth in a similar way.

Then, when restricted to $U \cap V$, we get:

$$\omega'|_{U\cap V} - \eta'|_{U\cap V} = \rho_V\theta + \rho_U\theta = (\rho_V + \rho_U)\,\theta = \theta.$$

In other words, we have $(i_U^* - i_V^*)(\omega', \eta') = \theta$. Since θ is arbitrary, we proved $i_U^* - i_V^*$ is surjective. $\square$

Recall that a pull-back map on k-forms induces a well-defined pull-back map on H^k_{dR}. The sequence of maps (5.4) between space of wedge products induces a sequence of maps between de Rham cohomology groups:

$$0 \to H^k_{\text{dR}}(M) \xrightarrow{j_U^*\oplus j_V^*} H^k_{\text{dR}}(U) \oplus H^k_{\text{dR}}(V) \xrightarrow{i_U^*-i_V^*} H^k_{\text{dR}}(U \cap V) \to 0. \tag{5.5}$$

Here, $j_U^* \oplus j_V^*$ and $i_U^* - i_V^*$ are defined by:

$$(j_U^* \oplus j_V^*)[\omega] = (j_U^*[\omega], j_V^*[\omega]) = ([j_U^*\omega], [j_V^*\omega])$$
$$(i_U^* - i_V^*)\,([\omega], [\eta]) = i_U^*[\omega] - i_V^*[\eta] = [i_U^*\omega] - [i_V^*\eta].$$

However, the sequence (5.5) is *not* exact because $j_U^* \oplus j_V^*$ may not be injective, and $i_U^* - i_V^*$ may not be surjective. For example, take $M = \mathbb{R}^2 \backslash \{(0,0)\}$, and define using polar coordinates the open sets $U = \{re^{i\theta} : r > 0, \theta \in (0, 2\pi)\}$ and $V = \{re^{i\theta} : r > 0, \theta \in (-\pi, \pi)\}$. Then, both U and V are star-shaped and hence both $H^1_{\text{dR}}(U)$ and $H^1_{\text{dR}}(V)$ are trivial. Nonetheless we have exhibited that $H^1_{\text{dR}}(M)$ is non-trivial. The map $j_U^* \oplus j_V^*$ from a non-trivial group to the trivial group can never be injective!

Exercise 5.13. Find an example of M, U and V such that the map $i_U^* - i_V^*$ in (5.5) is not surjective.

Nonetheless, it is still true that $\ker(i_U^* - i_V^*) = \text{Im}\,(j_U^* \oplus j_V^*)$, and we will verify it in the proof of Mayer-Vietoris Theorem (Theorem 5.16). Mayer-Vietoris Theorem asserts that although (5.5) is not exact in general, but we can connect each short sequence below:

$$H^0_{\text{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^0_{\text{dR}}(U) \oplus H^0_{\text{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^0_{\text{dR}}(U \cap V)$$
$$H^1_{\text{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^1_{\text{dR}}(U \cap V)$$
$$H^2_{\text{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^2_{\text{dR}}(U) \oplus H^2_{\text{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^2_{\text{dR}}(U \cap V)$$
$$\vdots$$

to produce a long exact sequence.

Theorem 5.16 (Mayer-Vietoris Theorem)**.** *Let M be a smooth manifold, and U and V be open sets of M such that $M = U \cup V$. Then, for each $k \geq 0$ there is a homomorphism $\delta : H^k_{\text{dR}}(U \cap V) \to H^{k+1}_{\text{dR}}(M)$ such that the following sequence is exact:*

$$\cdots \xrightarrow{\delta} H^k_{\text{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^k_{\text{dR}}(U) \oplus H^k_{\text{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^k_{\text{dR}}(U \cap V) \xrightarrow{\delta} H^{k+1}_{\text{dR}}(M) \to \cdots$$

This long exact sequence is called the Mayer-Vietoris sequence.

The proof of Theorem 5.16 is purely algebraic. We will learn the proof after looking at some examples.

5.3.3. Using Mayer-Vietoris Sequences. The Mayer-Vietoris sequence is particularly useful for computing de Rham cohomology groups and Betti numbers using linear algebraic methods. Suppose M can be expressed as a union $U \cup V$ of two open sets, such that the H^k_{dR}'s of U, V and $U \cap V$ can be computed easily, then $H^k_{\text{dR}}(M)$ can be deduced by "playing around" the kernels and images in the Mayer-Vietoris sequence. One useful result in Linear (or Abstract) Algebra is the following:

Theorem 5.17 (First Isomorphism Theorem)**.** *Let $T : V \to W$ be a linear map between two vector spaces V and W. Then, we have:*

$$\text{Im}\,T \cong V/\ker T.$$

In particular, if V and W are finite dimensional, we have:

$$\dim \ker T + \dim \text{Im}\,T = \dim V.$$

Proof. Let $\Phi : \operatorname{Im} T \to V/\ker T$ be the map defined by:

$$\Phi(T(v)) = [v]$$

for any $T(v) \in \operatorname{Im} T$. This map is well-defined since if $T(v) = T(w)$ in $\operatorname{Im} T$, then $v - w \in \ker T$, which implies $[v] = [w]$ in the quotient vector space $V/\ker T$. It is easy (hence omitted) to verify that Φ is linear.

Φ is injective since whenever $T(v) \in \ker \Phi$, we have $\Phi(T(v)) = [0]$ which implies $[v] = [0]$ and hence $v \in \ker T$ (i.e. $T(v) = 0$). Also, Φ is surjective since given any $[v] \in V/\ker T$, we have $\Phi(T(v)) = [v]$ by the definition of Φ.

These show Φ is an isomorphism, hence completing the proof. $\square$

Example 5.18. In this example, we use the Mayer-Vietoris sequence to compute $H^1_{\text{dR}}(\mathbb{S}^1)$. Let:

$$M = \mathbb{S}^1, \quad U = M\backslash\{\text{north pole}\}, \quad V = M\backslash\{\text{south pole}\}.$$

Then clearly $M = U \cup V$, and $U \cap V$ consists of two disjoint arcs (each of which deformation retracts to a point). Here are facts which we know and which we haven't yet known:

$$\begin{array}{llll} H^0_{\text{dR}}(M) \cong \mathbb{R} & H^0_{\text{dR}}(U) \cong \mathbb{R} & H^0_{\text{dR}}(V) \cong \mathbb{R} & H^0_{\text{dR}}(U \cap V) \cong \mathbb{R} \oplus \mathbb{R} \\ H^1_{\text{dR}}(M) \text{ unknown} & H^1_{\text{dR}}(U) \cong 0 & H^1_{\text{dR}}(V) \cong 0 & H^1_{\text{dR}}(U \cap V) \cong 0 \end{array}$$

By Theorem 5.16, we know that the following sequence is exact:

$$\cdots \to \underbrace{H^0_{\text{dR}}(U) \oplus H^0_{\text{dR}}(V)}_{\mathbb{R}\oplus\mathbb{R}} \xrightarrow{i_U^* - i_V^*} \underbrace{H^0_{\text{dR}}(U \cap V)}_{\mathbb{R}\oplus\mathbb{R}} \xrightarrow{\delta} \underbrace{H^1_{\text{dR}}(M)}_{?} \xrightarrow{j_U^* \oplus j_V^*} \underbrace{H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V)}_{0}$$

Therefore, δ is surjective.

By First Isomorphism Theorem (Theorem 5.17), we know:

$$H^1_{\text{dR}}(M) = \operatorname{Im} \delta \cong \frac{H^0_{\text{dR}}(U \cap V)}{\ker \delta}.$$

Elements of $H^0_{\text{dR}}(U \cap V)$ are locally constant functions of the form:

$$f_{a,b} = \begin{cases} a & \text{on left arc} \\ b & \text{on right arc} \end{cases}$$

Since the Mayer-Vietoris sequence is exact, we have $\ker \delta = \operatorname{Im}(i_U^* - i_V^*)$. The space $H^0_{\text{dR}}(U)$, $H^0_{\text{dR}}(V)$ and $H^0_{\text{dR}}(U \cap V)$ consist of locally constant functions on U, V and $U \cap V$ respectively, and the maps $i_U^* - i_V^*$ takes constant functions $(k_1, k_2) \in H^0_{\text{dR}}(U) \oplus H^0_{\text{dR}}(V)$ to the constant function $f_{k_1 - k_2, k_1 - k_2}$ on $U \cap V$. Therefore, the first de Rham cohomology group of M is given by:

$$H^1_{\text{dR}}(M) \cong \frac{\{f_{a,b} : a, b \in \mathbb{R}\}}{\{f_{a-b,a-b} : a, b \in \mathbb{R}\}} \cong \frac{\mathbb{R}^2}{\{(x, y) : x = y\}},$$

and hence $b_1(M) = \dim H^1_{\text{dR}}(M) = 2 - 1 = 1$. $\square$

Example 5.19. Let's discuss some consequences of the result proved in the previous example. Recall that $\mathbb{R}^2\backslash\{(0,0)\}$ deformation retracts to $\mathbb{S}^1$. By Theorem 5.13, we know $H^1_{\text{dR}}(\mathbb{R}^2\backslash\{(0,0)\}) \cong H^1_{\text{dR}}(\mathbb{S}^1)$.

This tells us $b_1(\mathbb{R}^2\backslash\{(0,0)\}) = 1$ as well. Recall that the following 1-form:

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

is closed but not exact. The class $[\omega]$ is then trivial in $H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$. In an one-dimensional vector space, any non-zero vector spans that space. Therefore, we conclude:

$$H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\} = \{c[\omega] : c \in \mathbb{R}\}.$$

where ω is defined as in above.

As a result, if ω' is a closed 1-form on $\mathbb{R}^2\backslash\{(0,0)\}$, then we must have

$$[\omega'] = c[\omega]$$

for some $c \in \mathbb{R}$, and so $\omega' = c\omega + df$ for some smooth function $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$.

Using the language of vector fields, if $\mathsf{V}(x,y) : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2$ is a smooth vector field with $\nabla \times \mathsf{V} = 0$, then there is a constant $c \in \mathbb{R}$ and a smooth function $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ such that:

$$\mathsf{V} = c\left(\frac{-y\hat{i} + x\hat{j}}{x^2+y^2}\right) + \nabla f.$$

□

Exercise 5.14. Let $\mathbb{T}^2$ be the two-dimensional torus. Show that $b_1(\mathbb{T}^2) = 2$.

Exercise 5.15. Show that $b_1(\mathbb{S}^2) = 0$. Based on this result, show that any curl-zero vector field defined on $\mathbb{R}^3\backslash\{(0,0,0)\}$ must be conservative.

One good technique of using the Mayer-Vietoris sequence (as demonstrated in the examples and exercises above) is to consider a segment of the sequence that starts and ends with the trivial space, i.e.

$$0 \to V_1 \to V_2 \to \cdots \to V_n \to 0.$$

If all vector spaces V_i's except one of them are known, then the remaining one (at least its dimension) can be deduced using First Isomorphism Theorem. Below is a useful lemma which is particularly useful for finding the Betti number of a manifold:

Lemma 5.20. *Let the following be an exact sequence of finite dimensional vector spaces:*

$$0 \to V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} V_n \to 0.$$

Then, we have:

$$\dim V_1 - \dim V_2 + \dim V_3 - \cdots + (-1)^{n-1} \dim V_n = 0$$

Proof. By exact-ness, the map $T_{n-1} : V_{n-1} \to V_n$ is surjective. By First Isomorphism Theorem (Theorem 5.17), we get:

$$V_n = \mathrm{Im}\, T_{n-1} \cong V_{n-1}/\ker T_{n-1} = V_{n-1}/\mathrm{Im}\, T_{n-2}.$$

As a result, we have:

$$\dim V_n = \dim V_{n-1} - \dim \mathrm{Im}\, T_{n-2}.$$

Similarly, apply First Isomorphism Theorem on $T_{n-2} : V_{n-2} \to V_{n-1}$, we get:

$$\dim \mathrm{Im}\, T_{n-2} = \dim V_{n-2} - \dim \mathrm{Im}\, T_{n-3},$$

and combine with the previous result, we get:

$$\dim V_n = \dim V_{n-1} - \dim V_{n-2} + \dim \mathrm{Im}\, T_{n-3}.$$

Proceed similarly as the above, we finally get:

$$\dim V_n = \dim V_{n-1} - \dim V_{n-2} + \ldots + (-1)^n \dim V_1,$$

as desired. □

In Example 5.18 (about computing $H^1_{\text{dR}}(\mathbb{S}^1)$), the following exact sequence was used:

$$0 \to \underbrace{H^0_{\text{dR}}(\mathbb{S}^1)}_{\mathbb{R}} \to \underbrace{H^0_{\text{dR}}(U) \oplus H^0_{\text{dR}}(V)}_{\mathbb{R}\oplus\mathbb{R}} \to \underbrace{H^0_{\text{dR}}(U \cap V)}_{\mathbb{R}\oplus\mathbb{R}} \to \underbrace{H^1_{\text{dR}}(\mathbb{S}^1)}_{?} \to \underbrace{H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V)}_{0}$$

Using Lemma 5.20, the dimension of $H^1_{\text{dR}}(\mathbb{S}^1)$ can be computed easily:

$$\dim \mathbb{R} - \dim \mathbb{R} \oplus \mathbb{R} + \dim \mathbb{R} \oplus \mathbb{R} - \dim H^1_{\text{dR}}(\mathbb{S}^1) = 0$$

which implies $\dim H^1_{\text{dR}}(\mathbb{S}^1) = 1$ (or equivalently, $b_1(\mathbb{S}^1) = 1$). Although this method does not give a precise description of $H^1_{\text{dR}}(\mathbb{S}^1)$ in terms of inclusion maps, it is no doubt much easier to adopt.

In the forthcoming examples, we will assume the following facts stated below (which we have only proved the case $k = 1$):

- $H^k_{\text{dR}}(U) = 0$, where $k \geq 1$, for any star-shaped region $U \subset \mathbb{R}^n$.
- If Σ is a deformation retract of M, then $H^k_{\text{dR}}(\Sigma) \cong H^k_{\text{dR}}(M)$ for any $k \geq 1$.

Example 5.21. Consider $\mathbb{R}^2\backslash\{p_1, \ldots, p_n\}$ where $p_1, \ldots, p_n$ are n distinct points in $\mathbb{R}^2$. We want to find b_1 of this open set.

Define $U = \mathbb{R}^2\backslash\{p_1, \ldots, p_{n-1}\}$, $V = \mathbb{R}^2\backslash\{p_n\}$, then $U \cup V = \mathbb{R}^2$ and $U \cap V = \mathbb{R}^2\backslash\{p_1, \ldots, p_n\}$. Consider the Mayer-Vietoris sequence:

$$\underbrace{H^1_{\text{dR}}(U \cup V)}_{0} \to H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V) \to H^1_{\text{dR}}(U \cap V) \to \underbrace{H^2_{\text{dR}}(U \cup V)}_{0}.$$

Using Lemma 5.20, we know:

$$\dim H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V) - \dim H^1_{\text{dR}}(U \cap V) = 0$$

We have already figured out that $\dim H^1_{\text{dR}}(V) = 1$. Therefore, we get:

$$\dim H^1_{\text{dR}}(\mathbb{R}^2\backslash\{p_1, \ldots, p_n\}) = \dim H^1_{\text{dR}}(\mathbb{R}^2\backslash\{p_1, \ldots, p_{n-1}\}) + 1.$$

By induction, we conclude:

$$b_1(\mathbb{R}^2\backslash\{p_1, \ldots, p_n\}) = \dim H^1_{\text{dR}}(\mathbb{R}^2\backslash\{p_1, \ldots, p_n\}) = n.$$

□

Example 5.22. Consider the n-sphere S^n (where $n \geq 2$). It can be written as $U \cup V$ where $U := S^n\backslash\{\text{north pole}\}$ and $V := S^n\backslash\{\text{south pole}\}$. Using stereographic projections, one can show both U and V are diffeomorphic to $\mathbb{R}^n$. Furthermore, $U \cap V$ is diffeomorphic to $\mathbb{R}^n\backslash\{0\}$, which deformation retracts to S^{n-1}. Hence $H^k_{\text{dR}}(S^{n-1}) = H^k_{\text{dR}}(U \cap V)$ for any k.

Now consider the Mayer-Vietoris sequence with these U and V, we have for each $k \geq 2$ an exact sequence:

$$\underbrace{H^{k-1}_{\text{dR}}(U) \oplus H^{k-1}_{\text{dR}}(V)}_{0} \to H^{k-1}_{\text{dR}}(U \cap V) \to H^k_{\text{dR}}(S^n) \to \underbrace{H^k_{\text{dR}}(U) \oplus H^k_{\text{dR}}(V)}_{0}.$$

This shows $H^{k-1}_{\text{dR}}(S^{n-1}) \cong H^k_{\text{dR}}(S^n)$ for any $k \geq 2$. By induction, we conclude that $H^n_{\text{dR}}(S^n) \cong H^1_{\text{dR}}(\mathbb{S}^1) \cong \mathbb{R}$ for any $n \geq 2$. □

5.3.4. Proof of Mayer-Vietoris Theorem. To end this chapter (and this course), we present the proof of the Mayer-Vietoris's Theorem (Theorem 5.16). As mentioned before, the proof is purely algebraic. The key ingredient of the proof applies to many other kinds of cohomologies as well (de Rham cohomology is only one kind of many types of cohomology).

For simplicity, we denote:

$$\begin{array}{lll} X^k := \wedge^k T^*M & Y^k := \wedge^k T^*U \oplus \wedge^k T^*V & Z^k := \wedge^k T^*(U \cap V) \\ H^k(X) := H^k_{\text{dR}}(M) & H^k(Y) := H^k_{\text{dR}}(U) \oplus H^k_{\text{dR}}(V) & H^k(Z) := H^k_{\text{dR}}(U \cap V) \end{array}$$

Furthermore, we denote the pull-back maps $i_U^* - i_V^*$ and $j_U^* \oplus j_V^*$ by simply i and j respectively. We then have the following commutative diagram between all these X, Y and Z:

$$\begin{tikzcd} 0 \arrow[r] & X^k \arrow[r, "j"] \arrow[d, "d"] & Y^k \arrow[r, "i"] \arrow[d, "d"] & Z^k \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & X^{k+1} \arrow[r, "j"] \arrow[d, "d"] & Y^{k+1} \arrow[r, "i"] \arrow[d, "d"] & Z^{k+1} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & X^{k+2} \arrow[r, "j"] & Y^{k+2} \arrow[r, "i"] & Z^{k+2} \arrow[r] & 0 \end{tikzcd}$$

The maps in the diagram commute because the exterior derivative d commute with any pull-back map. The map $d : Y^k \to Y^{k+1}$ takes (ω, η) to $(d\omega, d\eta)$.

To give a proof of the Mayer-Vietoris Theorem, we first need to construct a linear map $\delta : H^k_{\text{dR}}(Z) \to H^{k+1}_{\text{dR}}(Z)$. Then, we need to check that the connected sequence:

$$\cdots \xrightarrow{i} H^k(Z) \xrightarrow{\delta} H^{k+1}(X) \xrightarrow{j} H^{k+1}(Y) \xrightarrow{i} H^{k+1}(Z) \xrightarrow{\delta} \cdots$$

is exact. Most arguments involved are done by "chasing the commutative diagram".

Step 1: Construction of $H^k(Z) \xrightarrow{\delta} H^{k+1}(X)$

Let $[\theta] \in H^k(Z)$, where $\theta \in Z^k$ is a closed k-form on $U \cap V$. Recall from Proposition 5.15 that the sequence

$$0 \to X^k \xrightarrow{j} Y^k \xrightarrow{i} Z^k \to 0$$

is exact, and in particular i is surjective. As a result, there exists $\omega \in Y^k$ such that $i(\omega) = \theta$.

From the commutative diagram, we know $id\omega = di\omega = d\theta = 0$, and hence $d\omega \in \ker i$. By exact-ness, $\operatorname{Im} j = \ker i$ and so there exists $\eta \in X^{k+1}$ such that $j(\eta) = d\omega$.

Next we argue that such η must be closed: since $j(d\eta) = d(j\eta) = d(d\omega) = 0$, and j is injective by exact-ness. We must have $d\eta = 0$, and so η represents a class in $H^{k+1}(X)$. To summarize, given $[\theta] \in H^k(Z)$, ω and η are elements such that

$$i(\omega) = \theta \quad \text{and} \quad j(\eta) = d\omega.$$

We then define $\delta[\theta] := [\eta] \in H^{k+1}(X)$.

Step 2: Verify that δ is a well-defined map

Suppose $[\theta] = [\theta']$ in $H^k_{\text{dR}}(Z)$. Let $\omega' \in Y^k$ and $\eta' \in X^{k+1}$ be the corresponding elements associated with θ', i.e.

$$i(\omega') = \theta' \quad \text{and} \quad j(\eta') = d\omega'.$$

We need to show $[\eta] = [\eta']$ in $H^{k+1}(X)$.

From $[\theta] = [\theta']$, there exists a $(k-1)$-form β in Z^{k-1} such that $\theta - \theta' = d\beta$, which implies:

$$i(\omega - \omega') = \theta - \theta' = d\beta.$$

By surjectivity of $i : Y^{k-1} \to Z^{k-1}$, there exists $\alpha \in Y^{k-1}$ such that $i\alpha = \beta$. Then we get:

$$i(\omega - \omega') = d(i\alpha) = id\alpha$$

which implies $(\omega - \omega') - d\alpha \in \ker i$.

By exact-ness, $\ker i = \operatorname{Im} j$ and so there exists $\gamma \in X^k$ such that

$$j\gamma = (\omega - \omega') - d\alpha.$$

Differentiating both sides, we arrive at:

$$dj\gamma = d(\omega - \omega') - d^2\alpha = j(\eta - \eta').$$

Therefore, $jd\gamma = dj\gamma = j(\eta - \eta')$, and by injectivity of j, we get:

$$\eta - \eta' = d\gamma$$

and so $[\eta] = [\eta']$ in $H^{k+1}(X)$.

Step 3: Verify that δ is a linear map

We leave this step as an exercise for readers.

Step 4: Check that $H^k(Y) \xrightarrow{i} H^k(Z) \xrightarrow{\delta} H^{k+1}(X)$ is exact

To prove $\operatorname{Im} i \subset \ker \delta$, we take an arbitrary $[\theta] \in \operatorname{Im} i \subset H^k(Z)$, there is $[\omega] \in H^k(Y)$ such that $[\theta] = i[\omega]$, we will show $\delta[\theta] = 0$. Recall that $\delta[i\omega]$ is the element $[\eta]$ in $H^{k+1}(X)$ such that $j\eta = d\omega$. Now that ω is closed, the injectivity of j implies $\eta = 0$. Therefore, $\delta[\theta] = \delta[i\omega] = [0]$, proving $[\theta] \in \ker \delta$.

Next we show $\ker \delta \subset \operatorname{Im} i$. Suppose $[\theta] \in \ker \delta$, and let ω and η be the forms such that $i(\omega) = \theta$ and $j(\eta) = d\omega$. Then $[\eta] = \delta[\theta] = [0]$, so there exists $\gamma \in X^{k-1}$ such that $\eta = d\gamma$, which implies $j(d\gamma) = d\omega$, and so $\omega - j\gamma$ is closed. By exact-ness, $i(j\gamma) = 0$, and so:

$$\theta = i(\omega) = i(\omega - j\gamma).$$

For $\omega - j\gamma$ being closed, we conclude $[\theta] = i[\omega - j\gamma] \in \operatorname{Im} i$ in $H^k(Z)$.

Step 5: Check that $H^k(Z) \xrightarrow{\delta} H^{k+1}(X) \xrightarrow{j} H^{k+1}(Y)$ is exact

First show $\operatorname{Im} \delta \subset \ker j$. Let $[\theta] \in H^{k+1}(Z)$, then $\delta[\theta] = [\eta]$ where

$$i(\omega) = \theta \quad \text{and } j(\eta) = d\omega.$$

As a result, $j\delta[\theta] = j[\eta] = [d\omega] = [0]$. This shows $\delta[\theta] \in \ker j$.

Next we show $\ker j \subset \operatorname{Im} \delta$. Let $j[\omega] = [0]$, then $j\omega = d\alpha$ for some $\alpha \in Y^k$. Since:

$$i(\alpha) = i\alpha \quad \text{and} \quad j(\omega) = d\alpha$$

We conclude $\delta[i\alpha] = [\omega]$, or in other words, $[\omega] \in \operatorname{Im} \delta$.

Step 6: Check that $H^{k+1}(X) \xrightarrow{j} H^{k+1}(Y) \xrightarrow{i} H^{k+1}(Z)$ is exact

The inclusion $\operatorname{Im} j \subset \ker i$ follows from the fact that $i(j\eta) = 0$ for any closed $\eta \in X^{k+1}$, and hence $ij[\eta] = [0]$. Finally, we show $\ker i \subset \operatorname{Im} j$: suppose $[\omega] \in \ker i$ so that $i\omega = d\beta$ for some $\beta \in Z^k$. By surjectivity of $i : Y^k \to Z^k$, there exists $\alpha \in Y^k$ such that $\beta = i\alpha$. As a result, we get:

$$i\omega = di\alpha = id\alpha \quad \Longrightarrow \quad \omega - d\alpha \in \ker i.$$

Since $\ker i = \operatorname{Im} j$ on the level of $X^{k+1} \to Y^{k+1} \to Z^{k+1}$, there exists $\gamma \in X^{k+1}$ such that $j\gamma = \omega - d\alpha$. One can easily show γ is closed by injectivity of j:

$$jd\gamma = dj\gamma = d(\omega - d\alpha) = 0 \quad \Longrightarrow \quad d\gamma = 0$$

and so $[\gamma] \in H^{k+1}(X)$. Finally, we conclude:

$$j[\gamma] = [\omega - d\alpha] = [\omega]$$

and so $[\omega] \in \operatorname{Im} j$.

*** End of the proof of the Mayer-Vietoris Theorem ***
**** End of MATH 4033 ****
***** I hope you enjoy it. *****

Part 2

Euclidean Hypersurfaces

Chapter 6

Geometry of Curves

"A figure with curves always offers a lot of interesting angles."

Wesley Ruggles

Riemannian geometry is a branch in differential geometry which studies intrinsic geometric structure without referencing to the ambient space. It was first developed by Gauss and Riemann, and was later adopted by Einstein to lay the mathematical foundation of general relativity, which regards our space-time as an intrinsic manifold.

Despite the intrinsic nature of Riemannian geometry, many of its important notions and concepts are stemmed from extrinsic geometry, namely hypersurfaces in Euclidean spaces. In this and the next chapters, we will first explore ourselves to the basic differential geometry of Euclidean hypersurfaces.

6.1. Curvature and Torsion

6.1.1. Regular Curves. A curve in the Euclidean space $\mathbb{R}^n$ is regarded as a function $\gamma(t)$ from an interval I to $\mathbb{R}^n$. The interval I can be finite, infinite, open, closed or half-open. Denote the coordinates of $\mathbb{R}^n$ by $(x_1, x_2, \ldots, x_n)$, then a curve $\gamma(t)$ in $\mathbb{R}^n$ can be written in coordinate form as:

$$\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t)).$$

One easy way to make sense of a curve is to regard it as the trajectory of a particle. At any time t, the functions $\gamma^1(t), \ldots, \gamma^n(t)$ give the coordinates of the particle in $\mathbb{R}^n$. Assuming all $x_i(t)$, where $1 \leq i \leq n$, are at least twice differentiable, then the first derivative $\gamma'(t)$ represents the *velocity* of the particle, its magnitude $|\gamma'(t)|$ is the *speed* of the particle, and the second derivative $\gamma''(t)$ represents the *acceleration* of the particle.

We will mostly study those curves which are infinitely differentiable (i.e. C^∞). For some technical purposes as we will explain later, we only study those curves $\gamma(t)$ whose velocity $\gamma'(t)$ is never zero. We call those curves:

Definition 6.1 (Regular Curves). A *regular curve* is a C^∞ function $\gamma(t) : I \to \mathbb{R}^n$ such that $\gamma'(t) \neq 0$ for any $t \in I$.

Example 6.2. The curve $\gamma(t) = (\cos(e^t), \sin(e^t))$, where $t \in (-\infty, \infty)$, is a regular curve since $\gamma'(t) = (-e^t \sin(e^t), e^t \cos(e^t))$ and $|\gamma'(t)| = e^t \neq 0$ for any t.

However, $\widetilde{\gamma}(t) = (\cos t^2, \sin t^2)$, where $t \in (-\infty, \infty)$, is *not* a regular curve since $\widetilde{\gamma}'(t) = (-2t \sin t^2, 2t \cos t^2)$ and so $\widetilde{\gamma}'(0) = 0$.

Although both curves $\gamma(t)$ and $\widetilde{\gamma}(t)$ represent the unit circle centered at the origin in $\mathbb{R}^2$, one is regular but another is not. Therefore, the term *regular* refers to the parametrization rather than the trajectory. □

6.1.2. Arc-Length Parametrization. From Calculus, the arc-length of a curve $\gamma(t)$ from $t = t_0$ to $t = t_1$ is given by:

$$\int_{t_0}^{t_1} |\gamma'(t)| \, dt.$$

Now suppose the curve $\gamma(t)$ starts at $t = 0$ (call it the initial time). Then the following quantity:

$$s(t) := \int_0^t |\gamma'(\tau)| \, d\tau$$

measures the distance traveled by the particle after t unit time since its initial time.

Given a curve $\gamma(t) = (\cos(e^t - 1), \sin(e^t - 1))$, we have

$$\begin{aligned} \gamma'(t) &= (-e^t \sin(e^t - 1), e^t \cos(e^t - 1)), \\ |\gamma'(t)| &= e^t \neq 0 \quad \text{for any } t \in (-\infty, \infty). \end{aligned}$$

Therefore, $\gamma(t)$ is a regular curve. By an easy computation, one can show $s(t) = e^t - 1$ and so, regarding t as a function of s, we have $t(s) = \log(s+1)$. By substituting $t = \log(s+1)$ into $\gamma(t)$, we get:

$$\gamma(t(s)) = \gamma(\log(s+1)) = \Big(\cos(e^{\log(s+1)} - 1), \sin(e^{\log(s+1)} - 1)\Big) = (\cos s, \sin s).$$

The curve $\gamma(t(s))$ is ultimately a function of s. With abuse of notations, we denote $\gamma(t(s))$ simply by $\gamma(s)$. Then, this $\gamma(s)$ has the same trajectory as $\gamma(t)$ and both curves at C^∞. The difference is that the former travels at a unit speed. The curve $\gamma(s)$ is a *reparametrization* of $\gamma(t)$, and is often called an *arc-length parametrization* of the curve.

However, if we *attempt* to do find a reparametrization on a non-regular curve say $\widetilde{\gamma}(t) = (\cos(t^2), \sin(t^2))$, in a similar way as the above, we can see that such the reparametrization obtained will not be smooth. To see this, we first compute

$$s(t) = \int_0^t |\widetilde{\gamma}'(\tau)| \, d\tau = \int_0^t 2|\tau| d\tau = \begin{cases} t^2 & \text{if } t \geq 0; \\ -t^2 & \text{if } t < 0. \end{cases}$$

Therefore, regarding t as a function of s, we have

$$t(s) = \begin{cases} \sqrt{s} & \text{if } s \geq 0; \\ -\sqrt{-s} & \text{if } s < 0. \end{cases}$$

Then,

$$\widetilde{\gamma}(s) := \widetilde{\gamma}(t(s)) = \begin{cases} (\cos(s), \sin(s)) & \text{if } s \geq 0; \\ (\cos(-s), \sin(-s)) & \text{if } s < 0, \end{cases}$$

or in short, $\widetilde{\gamma}(s) = (\cos(s), \sin|s|)$, which is not differentiable at $s = 0$.

It turns out the reason why the reparametrization by s works well for $\gamma(t)$ but not for $\widetilde{\gamma}(t)$ is that the former is regular but the later is not. In general, one can always reparametrize a regular curve by its arc-length s. Let's state it as a theorem:

Theorem 6.3. *Given any regular curve $\gamma(t) : I \to \mathbb{R}^n$, one can always reparametrize it by arc-length. Precisely, let $t_0 \in I$ be a fixed number and consider the following function of t:*

$$s(t) := \int_{t_0}^{t} |\gamma'(\tau)| \, d\tau.$$

Then, t can be regarded as a C^∞ function of s, and the reparametrized curve $\gamma(s) := \gamma(t(s))$ is a regular curve such that $\left|\frac{d}{ds}\gamma(s)\right| = 1$ for any s.

Proof. The Fundamental Theorem of Calculus shows

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} |\gamma'(\tau)| \, d\tau = |\gamma'(t)| > 0.$$

We have $|\gamma'(t)| > 0$ since $\gamma(t)$ is a regular curve. Now $s(t)$ is a strictly increasing function of t, so one can regard t as a function of s by the Inverse Function Theorem. Since $s(t)$ is C^∞ (because $\gamma(t)$ is C^∞ and $|\gamma'(t)| \neq 0$), by the Inverse Function Theorem $t(s)$ is C^∞ too.

To verify that $\left|\frac{d}{ds}\gamma(s)\right| = 1$, we use the chain rule:

$$\begin{aligned}
\frac{d}{ds}\gamma(s) &= \frac{d\gamma}{dt} \cdot \frac{dt}{ds} \\
&= \gamma'(t) \cdot \frac{1}{\frac{ds}{dt}} \\
\left|\frac{d}{ds}\gamma(s)\right| &= |\gamma'(t)| \cdot \frac{1}{|\gamma'(t)|} = 1.
\end{aligned}$$

□

Exercise 6.1. Determine whether each of the following is a regular curve. If so, reparametrize the curve by arc-length:

(a) $\gamma(t) = (\cos t, \sin t, t), \quad t \in (-\infty, \infty)$

(b) $\gamma(t) = (t - \sin t, 1 - \cos t), \quad t \in (-\infty, \infty)$

6.1.3. Definition of Curvature. Curvature is quantity that measures the sharpness of a curve, and is closely related to the acceleration. Imagine you are driving a car along a curved road. On a sharp turn, the force exerted on your body is proportional to the acceleration according to the Newton's Second Law. Therefore, given a parametric curve $\gamma(t)$, the magnitude of the acceleration $|\gamma''(t)|$ *somewhat* reflects the sharpness of the path – the sharper the turn, the larger the $|\gamma''(t)|$.

However, the magnitude $|\gamma''(t)|$ is not *only* affected by the sharpness of the curve, but also on how *fast* you drive. In order to give a *fair* and *standardized* measurement of sharpness, we need to get an arc-length parametrization $\gamma(s)$ so that the "car" travels at unit speed.

Definition 6.4 (Curvature). Let $\gamma(s) : I \to \mathbb{R}^n$ be an arc-length parametrization of a path γ in $\mathbb{R}^n$. The curvature of γ is a function $\kappa : I \to \mathbb{R}$ defined by:

$$\kappa(s) = |\gamma''(s)|\,.$$

Remark 6.5. Since an arc-length parametrization is required in the definition, we talk about curvature for only for regular curves. □

Another way (which is less *physical*) to understand curvature is to regard $\gamma''(s)$ as $\frac{d}{ds}\mathsf{T}(s)$ where $\mathsf{T}(s) := \gamma'(s)$ is the unit tangent vector at $\gamma(s)$. The curvature $\kappa(s)$ is then given by $\left|\frac{d}{ds}\mathsf{T}(s)\right|$ which measures how fast the unit tangents $\mathsf{T}(s)$ move or turn along the curve (see Figure 6.1).

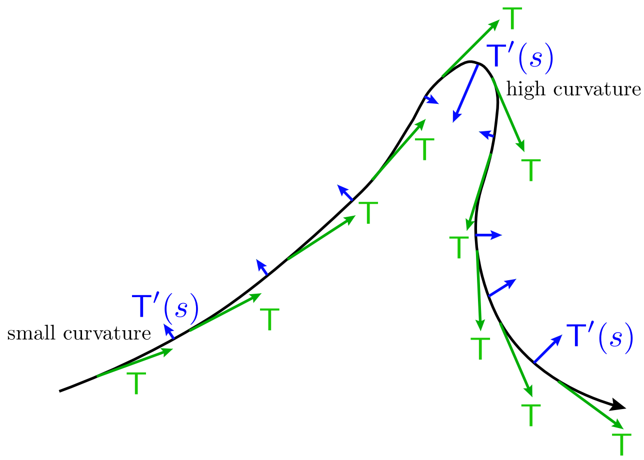


Figure 6.1. curvature measures how fast the unit tangents move

Example 6.6. The circle of radius R centered at the origin $(0,0)$ on the xy-plane can be parametrized by $\gamma(t) = (R\cos t,\ R\sin t)$. It can be easily verified that $|\gamma'(t)| = R$ and so $\gamma(t)$ is not an arc-length parametrization.

To find an arc-length parametrization, we let:

$$s(t) = \int_0^t |\gamma'(\tau)|\ d\tau = \int_0^t R\ d\tau = Rt.$$

Therefore, $t(s) = \frac{s}{R}$ as a function of s and so an arc-length parametrization of the circle is:

$$\gamma(s) := \gamma(t(s)) = \left(R\cos\frac{s}{R},\ R\sin\frac{s}{R}\right).$$

To find its curvature, we compute:

$$\begin{aligned}
\gamma'(s) &= \frac{d}{ds}\left(R\cos\frac{s}{R},\ R\sin\frac{s}{R}\right)\\
&= \left(-\sin\frac{s}{R},\ \cos\frac{s}{R}\right)\\
\gamma''(s) &= \left(-\frac{1}{R}\cos\frac{s}{R},\ -\frac{1}{R}\sin\frac{s}{R}\right)\\
\kappa(s) &= |\gamma''(s)| = \frac{1}{R}.
\end{aligned}$$

Thus the curvature of the circle is given by $\frac{1}{R}$, i.e. the larger the circle, the smaller the curvature. □

Exercise 6.2. Find an arc-length parametrization of the helix:

$$\gamma(t) = (a\cos t,\ a\sin t,\ bt)$$

where a and b are positive constants. Hence compute its curvature.

Exercise 6.3. Prove that a regular curve $\gamma(t)$ is a straight line if and only if its curvature κ is identically zero.

6.1.4. Curvature Formula. Although the curvature is defined as $\kappa(s) = |\gamma''(s)|$ where $\gamma(s)$ is an arc-length parametrization of the curve, it is very often impractical to compute the curvature this way. The main reason is that the arc-length parametrizations of many paths are very difficult to find explicitly. A "notorious" example is the ellipse:

$$\gamma(t) = (a\cos t,\ b\sin t)$$

where a and b are positive constants with $a \neq b$. The arc-length function is given by:

$$s(t) = \int_0^t \sqrt{a^2\sin^2\tau + b^2\cos^2\tau}\, d\tau.$$

While it is very easy to compute the integral when $a = b$, there is no closed form or explicit anti-derivative for the integrand if $a \neq b$. Although the arc-length parametrization *exists* theoretically speaking (Theorem 6.3), it cannot be written down explicitly and so the curvature cannot be computed from the definition.

The purpose of this section is to derive a formula for computing curvature without the need of finding its arc-length parametrization. To begin, we first prove the following important observation:

Lemma 6.7. *Let $\gamma(s) : I \to \mathbb{R}^n$ be a curve parametrized by arc-length, then the velocity $\gamma'(s)$ and the acceleration $\gamma''(s)$ is always orthogonal for any $s \in I$.*

Proof. Since $\gamma(s)$ is parametrized by arc-length, we have $|\gamma'(s)| = 1$ for any s, and so:

$$\begin{aligned}
\frac{d}{ds}\left|\gamma'(s)\right|^2 &= \frac{d}{ds}1 = 0\\
\frac{d}{ds}\left(\gamma'(s)\cdot\gamma'(s)\right) &= 0\\
\gamma''(s)\cdot\gamma'(s) + \gamma'(s)\cdot\gamma''(s) &= 0\\
2\gamma''(s)\cdot\gamma'(s) &= 0\\
\gamma''(s)\cdot\gamma'(s) &= 0
\end{aligned}$$

Therefore, $\gamma'(s)$ is orthogonal to $\gamma''(s)$ for any s. □

Proposition 6.8. *Given any regular curve $\gamma(t)$ in $\mathbb{R}^3$, the curvature as a function of t can be computed by the following formula:*

$$\kappa(t) = \frac{|\gamma'(t)\times\gamma''(t)|}{|\gamma'(t)|^3}.$$

Proof. Since $\gamma(t)$ is a regular curve, there exists an arc-length parametrization $\gamma(t(s))$, which for simplicity we denote it by $\gamma(s)$. From now on, we denote $\gamma'(t)$ as $\frac{d\gamma(t)}{dt}$, regarding t as the parameter of the curve, and $\gamma'(s)$ as $\frac{d\gamma(s)}{ds}$ regarding s as the parameter of the curve.

By the chain rule, we have:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\,\frac{ds}{dt} = \gamma'(s)\,\frac{ds}{dt} \tag{6.1}$$

$$\frac{d^2\gamma}{dt^2} = \frac{d}{dt}\left(\frac{d\gamma}{dt}\right) = \frac{d}{dt}\left(\gamma'(s)\,\frac{ds}{dt}\right) \qquad \text{(from (6.1))} \tag{6.2}$$

$$= \frac{d\gamma'(s)}{dt}\,\frac{ds}{dt} + \gamma'(s)\,\frac{d^2s}{dt^2}$$

By the chain rule again, we get:

$$\frac{d\gamma'(s)}{dt} = \frac{d\gamma'(s)}{ds}\,\frac{ds}{dt} = \gamma''(s)\,\frac{ds}{dt}$$

Substitute this back to (6.2), we obtain:

$$\frac{d^2\gamma}{dt^2} = \gamma''(s)\,\left(\frac{ds}{dt}\right)^2 + \gamma'(s)\,\frac{d^2s}{dt^2} \tag{6.3}$$

Taking the cross product of (6.1) and (6.3) yields:

$$\frac{d\gamma}{dt}\times\frac{d^2\gamma}{dt^2} = \left(\frac{ds}{dt}\right)^3\,\gamma'(s)\times\gamma''(s) + \underbrace{\frac{d^2s}{dt^2}\,\frac{ds}{dt}\,\gamma'(s)\times\gamma'(s)}_{=0} = \left(\frac{ds}{dt}\right)^3\,\gamma'(s)\times\gamma''(s). \tag{6.4}$$

Since $\gamma'(s)$ and $\gamma''(s)$ are two orthogonal vectors by Lemma 6.7, we have $|\gamma'(s)\times\gamma''(s)| = |\gamma'(s)|\,|\gamma''(s)| = \kappa(s)$. Taking the magnitude on both sides of (6.4), we get:

$$\left|\frac{d\gamma}{dt}\times\frac{d^2\gamma}{dt^2}\right| = \kappa\left|\frac{ds}{dt}\right|^3.$$

Therefore, we get:

$$\kappa = \frac{|\gamma'(t)\times\gamma''(t)|}{\left|\frac{ds}{dt}\right|^3}.$$

The proof can be easily completed by the definition of $s(t)$ and the Fundamental Theorem of Calculus:

$$s = \int_0^t |\gamma'(\tau)|\;d\tau$$

$$\frac{ds}{dt} = |\gamma'(t)|$$

□

Remark 6.9. Since the cross product is involved, Proposition 6.8 can only be used for curves in $\mathbb{R}^2$ or $\mathbb{R}^3$. To apply the result for curves in $\mathbb{R}^2$, say $\gamma(t) = (x(t),\ y(t))$, one may regard it as the curve $\gamma(t) = (x(t),\ y(t),\ 0)$ in $\mathbb{R}^3$. □

By Proposition 6.8, the curvature of the ellipse can be computed easily. See the example below:

Example 6.10. Let $\gamma(t) = (a\cos t,\ b\sin t,\ 0)$ be a parametrization of an ellipse on the xy-plane where a and b are positive constants, then we have:

$$\gamma'(t) = (-a\sin t,\ b\cos t,\ 0)$$

$$\gamma''(t) = (-a\cos t,\ -b\sin t,\ 0)$$

$$\gamma'(t)\times\gamma''(t) = (ab\sin^2 t + ab\cos^2 t)\,\hat{k} = ab\,\hat{k}$$

Therefore, by Proposition 6.8, it's curvature function is given by:

$$\kappa(t) = \frac{|\gamma'(t)\times\gamma''(t)|}{|\gamma'(t)|^3} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

□

Exercise 6.4. Consider the graph of a smooth function $y = f(x)$. Regarding the graph as a curve in $\mathbb{R}^3$, it can be parametrized using x as the parameter by $\gamma(x) = (x,\ f(x),\ 0)$. Show that the curvature of the graph is given by:

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}.$$

Exercise 6.5. For each of the following curves: (i) compute the curvature $\kappa(t)$ using Proposition 6.8; (ii) If it is easy to find an explicit arc-length parametrization of the curve, compute also the curvature from the definition; (iii) find the (x, y, z)-coordinates of the point(s) on the curve at which the curvature is the maximum.

(a) $\gamma(t) = (3\cos t,\ 4\cos t,\ 5t)$.

(b) $\gamma(t) = (t^2,\ 0,\ t)$.

(c) $\gamma(t) = \left(2t,\ t^2,\ -\frac{1}{3}t^3\right)$.

6.1.5. Frenet-Serret Frame. For now on, we will concentrate on regular curves in $\mathbb{R}^3$. Furthermore, we consider mostly those curves whose curvature function κ is *nowhere vanishing*. Therefore, straight-lines in $\mathbb{R}^3$, or paths such as the graph of $y = x^3$, are excluded in our discussion.

Definition 6.11 (Non-degenerate Curves). A regular curve $\gamma(t) : I \to \mathbb{R}^3$ is said to be *non-degenerate* if its curvature satisfies $\kappa(t) \neq 0$ for any $t \in I$.

We now introduce an important basis of $\mathbb{R}^3$ in the studies of space curves, the Frenet-Serret Frame, or the TNB-frame. It is an orthonormal basis of $\mathbb{R}^3$ associated to each point of a regular curve in $\mathbb{R}^3$.

Definition 6.12 (Frenet-Serret Frame). Given a non-degenerate curve $\gamma(s) : I \to \mathbb{R}^3$ parametrized by arc-length, we define:

$$\mathsf{T}(s) := \gamma'(s) \qquad \text{(tangent)}$$

$$\mathsf{N}(s) := \frac{\gamma''(s)}{|\gamma''(s)|} \qquad \text{(normal)}$$

$$\mathsf{B}(s) := \mathsf{T}(s) \times \mathsf{N}(s) \qquad \text{(binormal)}$$

The triple $\{\mathsf{T}(s),\ \mathsf{N}(s),\ \mathsf{B}(s)\}$ is called the *Frenet-Serret Frame* of $\mathbb{R}^3$ at the point $\gamma(s)$ of the curve. See Figure 6.2.

Remark 6.13. Note that T is a unit vector since $\gamma(s)$ is arc-length parametrized. Recall that $\kappa(s) := |\gamma''(s)|$ and the curve $\gamma(s)$ is assumed to be non-degenerate. Therefore, N is well-defined for any $s \in I$ and is a unit vector by its definition. From Lemma 6.7, T and N are orthogonal to each other for any $s \in I$. Therefore, by the definition of cross product, B is also a unit vector and is orthogonal to both T and N. To conclude, for each fixed $s \in I$, the Frenet-Serret Frame is an orthonormal basis of $\mathbb{R}^3$. □

Example 6.14. Let $\gamma(s) = \left(\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}}\right)$ where $s \in \mathbb{R}$. It can be verified easily that it is arc-length parametrized, i.e. $|\gamma'(s)| = 1$ for any $s \in \mathbb{R}$. The Frenet-Serret Frame

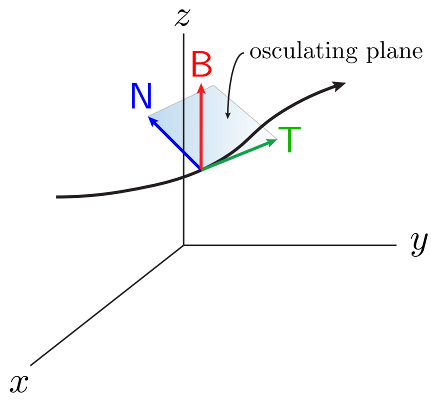


Figure 6.2. Frenet-Serret frame

of this curve is given by:

$$\begin{aligned}
\mathsf{T}(s) &= \gamma'(s) = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right) \\
\gamma''(s) &= \left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}},\ -\frac{1}{2}\sin\frac{s}{\sqrt{2}},\ 0\right) \\
\mathsf{N}(s) &= \frac{\gamma''(s)}{|\gamma''(s)|} = \left(-\cos\frac{s}{\sqrt{2}},\ -\sin\frac{s}{\sqrt{2}},\ 0\right) \\
\mathsf{B}(s) &= \mathsf{T}(s) \times \mathsf{N}(s) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} & -\sin\frac{s}{\sqrt{2}} & 0 \end{vmatrix} \\
&= \left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

□

Definition 6.15 (Osculating Plane)**.** Given a non-degenerate arc-length parametrized curve $\gamma(s) : I \to \mathbb{R}^3$, the *osculating plane* $\Pi(s)$ of the curve is a plane in $\mathbb{R}^3$ containing the point represented by $\gamma(s)$ and parallel to both $\mathsf{T}(s)$ and N, i.e.

$$\Pi(s) := \gamma(s) + \operatorname{span}\{\mathsf{T}(s),\ \mathsf{N}(s)\}.$$

(See Figure 6.2)

Remark 6.16. By the definition of the Frenet-Serret Frame, the binormal vector $\mathsf{B}(s)$ is a unit normal vector to the osculating plane $\Pi(s)$. □

Exercise 6.6. Consider the curve $\gamma(t) = (a\cos t,\ a\sin t,\ bt)$ where a and b positive constants. First find its arc-length parametrization $\gamma(s) := \gamma(t(s))$, and then compute its Frenet-Serret Frame.

Exercise 6.7. Show that if $\gamma(s) : I \to \mathbb{R}^3$ is a non-degenerate arc-length parametrized curve contained in the plane $Ax+By+Cz = D$ where A, B, C and D are constants, then $\mathsf{T}(s)$ and $\mathsf{N}(s)$ are parallel to the plane $Ax + By + Cz = 0$ for any $s \in I$, and $\mathsf{B}(s)$ is a constant vector which is normal to the plane $Ax + By + Cz = 0$.

Exercise 6.8. [**dC76**, P.23] Let $\gamma(s)$ be an arc-length parametrized curve in $\mathbb{R}^3$. The normal line at $\gamma(s)$ is the infinite straight line parallel to $\mathsf{N}(s)$ passing through the point represented by $\gamma(s)$. Suppose the normal line at every $\gamma(s)$ pass through a fixed point $p \in \mathbb{R}^3$. Show that $\gamma(s)$ is a part of a circle.

6.1.6. Torsion. If the curve $\gamma(s) : I \to \mathbb{R}^3$ is contained in a plane Π in $\mathbb{R}^3$, then the osculating plane $\Pi(s)$ coincides the plane Π for any $s \in I$, and hence the binormal vector $\mathsf{B}(s)$ is a unit normal vector to Π for any $s \in I$. By continuity, $\mathsf{B}(s)$ is a constant vector.

On the other hand, the helix considered in Example 6.14 is not planar since $\mathsf{B}(s)$ is changing over s. As s increases, the osculating plane $\Pi(s)$ not only translates but also rotates. The magnitude of $\frac{d\mathsf{B}}{ds}$ is therefore a measurement of how much the osculating plane rotates and how non-planar the curve $\gamma(s)$ looks. It motivates the introduction of *torsion*.

However, instead of defining the torsion of a curve to be $\left|\frac{d\mathsf{B}}{ds}\right|$, we hope to give a *sign* for the torsion. Before we state the definition of torsion, we first prove the following fact:

Lemma 6.17. *Given any non-degenerate, arc-length parametrized curve $\gamma(s) : I \to \mathbb{R}^3$, the vector $\frac{d\mathsf{B}}{ds}$ must be parallel to the normal $\mathsf{N}(s)$ for any $s \in I$.*

Proof. First note that $\{\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)\}$ is an orthonormal basis of $\mathbb{R}^3$ for any $s \in I$. Hence, we have:

$$\frac{d\mathsf{B}(s)}{ds} = a(s)\mathsf{T}(s) + b(s)\mathsf{N}(s) + c(s)\mathsf{B}(s)$$

where $a(s) = \frac{d\mathsf{B}(s)}{ds} \cdot \mathsf{T}(s)$, $b(s) = \frac{d\mathsf{B}(s)}{ds} \cdot \mathsf{N}(s)$ and $c(s) = \frac{d\mathsf{B}(s)}{ds} \cdot \mathsf{B}(s)$. It suffices to show $a(s) = c(s) = 0$ for any $s \in I$.

Since $\mathsf{B}(s)$ is unit, one can easily see that $c(s) \equiv 0$ by considering $\frac{d}{ds}|\mathsf{B}|^2$ (c.f. Lemma 6.7). To show $a(s) \equiv 0$, we consider the fact that:

$$\mathsf{T}(s) \cdot \mathsf{B}(s) = 0 \quad \text{for any } s \in I.$$

Differentiate both sides with respect to s, we get:

$$\frac{d\mathsf{T}}{ds} \cdot \mathsf{B} + \mathsf{T} \cdot \frac{d\mathsf{B}}{ds} = 0. \tag{6.5}$$

Since $\frac{d\mathsf{T}}{ds} = \frac{d}{ds}\gamma'(s) = \gamma''(s) = \kappa\mathsf{N}$, we get $\frac{d\mathsf{T}}{ds} \cdot \mathsf{B} = 0$ by the definition of B.

Combining this result with (6.5), we get $a(s) = \mathsf{T}\cdot\frac{d\mathsf{B}}{ds} = 0$. Hence we have $\frac{d\mathsf{B}}{ds} = b(s)\mathsf{N}$ and it completes the proof. □

Definition 6.18 (Torsion). Let $\gamma(s) : I \to \mathbb{R}^3$ be an arc-length parametrized, non-degenerate curve. The *torsion* of the curve is a function $\tau : I \to \mathbb{R}$ defined by:

$$\tau(s) := -\frac{d\mathsf{B}}{ds} \cdot \mathsf{N}.$$

Remark 6.19. By Lemma 6.17, the vector $\frac{d\mathsf{B}}{ds}$ and N are parallel. Combining with the fact that N is unit, one can see easily that:

$$|\tau(s)| = \left|\frac{d\mathsf{B}}{ds}\right| |\mathsf{N}| \cos 0 = \left|\frac{d\mathsf{B}}{ds}\right|.$$

Therefore, the torsion can be regarded as a *signed* $\left|\frac{d\mathsf{B}}{ds}\right|$ which measures the rate that the osculating plane rotates as s increases (see Figure 6.3). The negative sign appeared in the definition is a historical convention. □

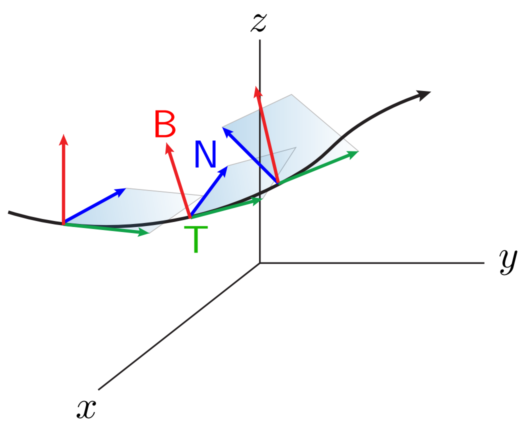


Figure 6.3. Torsion measures how fast the osculating plane changes along a curve

Example 6.20. Consider the curve $\gamma(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ which is the helix appeared in Example 6.14. The normal and binormal were already computed:

$$\mathsf{N}(s) = \frac{\gamma''(s)}{|\gamma''(s)|} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$
$$\mathsf{B}(s) = \left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Taking the derivative, we get:

$$\frac{d\mathsf{B}}{ds} = \left(\frac{1}{2}\cos\frac{s}{\sqrt{2}}, \frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right).$$

Therefore, the torsion of the curve is:

$$\tau(s) = -\frac{d\mathsf{B}}{ds}\cdot\mathsf{N} = \frac{1}{2}.$$

□

Exercise 6.9. Consider the curve $\gamma(t) = (a\cos t,\ a\sin t,\ bt)$ where a and b are positive constants. Find its torsion $\tau(s)$ as a function of the arc-length parameter s.

Exercise 6.10. Let $\gamma(s) : I \to \mathbb{R}^3$ be a non-degenerate, arc-length parametrized curve. Prove that $\tau(s) = 0$ for any $s \in I$ if and only if $\gamma(s)$ is contained in a plane. [Hint: for the "only if" part, consider the dot product $\mathsf{B}\cdot\mathsf{T}$.]

Exercise 6.11. [**dC76**, P.25] Suppose $\gamma(s) : I \to \mathbb{R}^3$ is a non-degenerate, arc-length parametrized curve such that $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for any $s \in I$. Show that the curve lies on a sphere if and only if $\dfrac{1}{\kappa^2} + \left(\dfrac{d}{ds}\dfrac{1}{\kappa}\right)^2 \dfrac{1}{\tau^2}$ is a constant.

The torsion of a non-degenerate curve $\gamma(t)$ can be difficult to compute from the definition since it involves finding an explicit arc-length parametrization. Fortunately, just like the curvature, there is a formula for computing torsion.

Proposition 6.21. *Let $\gamma(t) : I \to \mathbb{R}^3$ be a non-degenerate curve, then the torsion of the curve is given by:*

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{|\gamma'(t) \times \gamma''(t)|^2}.$$

Proof. See Exercise #6.12. $\square$

Exercise 6.12. The purpose of this exercise is to give a proof of Proposition 6.21. As $\gamma(t)$ is a (regular) non-degenerate curve, there exist an arc-length parametrization $\gamma(s) := \gamma(t(s))$ and a Frenet-Serret Frame $\{\mathsf{T}(s),\ \mathsf{N}(s),\ \mathsf{B}(s)\}$ at every point on the curve. With a little abuse of notations, we denote $\kappa(s) := \kappa(t(s))$ and $\tau(s) := \tau(t(s))$.

(a) Show that
$$\tau(s) = \frac{(\gamma'(s) \times \gamma''(s)) \cdot \gamma'''(s)}{\kappa(s)^2}.$$

(b) Using (6.3) in the proof of Proposition 6.8, show that
$$\gamma'''(t) = \left(\frac{ds}{dt}\right)^3 \gamma'''(s) + \mathsf{v}(s)$$
where $\mathsf{v}(s)$ is a linear combination of $\gamma'(s)$ and $\gamma''(s)$ for any s.

(c) Hence, show that
$$(\gamma'(s) \times \gamma''(s)) \cdot \gamma'''(s) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{|\gamma'(t)|^6}.$$
[Hint: use (6.4) in the proof of Proposition 6.8.]

(d) Finally, complete the proof of Proposition 6.21. You may use the curvature formula proved in Proposition 6.8.

Exercise 6.13. Compute the torsion $\tau(t)$ for the ellipsoidal helix:
$$\gamma(t) = (a\cos t,\ b\sin t,\ ct)$$
where a and b are positive and c is non-zero.

6.2. Fundamental Theorem of Space Curves

In this section, we discuss a deep result about non-degenerate curves in $\mathbb{R}^3$. Given an arc-length parametrized, non-degenerate curve $\gamma(s)$, one can define its curvature $\kappa(s)$ and torsion $\tau(s)$ as discussed in the previous section. They are scalar-valued functions of s. The former must be positive-valued, while the latter can take any real value. Both functions are smooth.

Now we ask the following questions:

Existence: If we are *given* a pair of smooth real-valued functions $\alpha(s)$ and $\beta(s)$ defined on $s \in I$ where $\alpha(s) > 0$ for any $s \in I$, does there exist a regular curve $\gamma(s) : I \to \mathbb{R}^3$ such that its curvature $\kappa(s)$ is identically equal to $\alpha(s)$, and its torsion $\tau(s)$ is identically equal to $\beta(s)$?

Uniqueness: Furthermore, if there are two curves $\gamma(s)$ and $\bar{\gamma}(s)$ in $\mathbb{R}^3$ whose curvature are both identical to $\alpha(s)$, and torsion are both identical to $\beta(s)$, then is it necessary that $\gamma(s) \equiv \bar{\gamma}(s)$?

The *Fundamental Theorem of Space Curves* answers both questions above. Using the classic existence and uniqueness theorems in Ordinary Differential Equations (ODEs), one can give an affirmative answer to the above existence question – yes, such a curve exists – and an "almost" affirmative answer to the uniqueness question – that is, although the curves $\gamma(s)$ and $\bar{\gamma}(s)$ may not be identical, one can be transformed from another by a rigid body motion in $\mathbb{R}^3$. The proof of this theorem is a good illustration of how *Differential Equations* interact with *Differential Geometry* – nowadays a field called *Geometric Analysis*.

FYI: Geometric Analysis

Geometric Analysis is a modern field in mathematics which uses Differential Equations to study Differential Geometry. In the past few decades, there are several crowning achievements in this area. Just to name a few, these include Yau's solution to the Calabi Conjecture (1976), and Hamilton–Perelman's solution to the Poincaré Conjecture (2003), and Brendle–Schoen's solution to the Differentiable Sphere Theorem (2007).

6.2.1. Existence and Uniqueness of ODEs.

A system of ODEs (or ODE system) is a set of one or more ODEs. The general form of an ODE system is:

$$\begin{aligned} x_1'(t) &= f_1(x_1,\, x_2,\, \ldots,\, x_n,\, t) \\ x_2'(t) &= f_2(x_1,\, x_2,\, \ldots,\, x_n,\, t) \\ &\vdots \\ x_n'(t) &= f_n(x_1,\, x_2,\, \ldots,\, x_n,\, t) \end{aligned}$$

where t is the independent variable, $x_i(t)$'s are unknown functions, and f_j's are prescribed functions of $(x_1,\, \ldots,\, x_n,\, t)$ from $\mathbb{R}^n \times I \to \mathbb{R}$.

An ODE system with a given initial condition, such as $(x_1(0),\ \ldots,\ x_n(0)) = (a_1,\, \ldots,\, a_n)$ where a_i's are constants, is called an initial-value problem (IVP).

We first state a fundamental existence and uniqueness theorem for ODE systems:

Theorem 6.22 (Existence and Uniqueness Theorem of ODEs). *Given functions f_i's ($1 \leq i \leq n$) defined on $\mathbb{R}^n \times I$, we consider the initial-value problem:*

$$x_i'(t) = f_i(x_1, \ldots, x_n, t) \quad \text{for } 1 \leq i \leq n$$

with initial condition $(x_1(0), \ldots, x_n(0)) = (a_1, \ldots, a_n)$. Suppose for every $1 \leq i, j \leq n$, the first partial derivative $\frac{\partial f_i}{\partial x_j}$ exists and is continuous on $\mathbb{R}^n \times I$, then there exists a unique solution $(x_1(t), \ldots, x_n(t))$, defined at least on a short-time interval $t \in (-\varepsilon, \varepsilon)$, to the initial-value problem. Furthermore, as long as the solution remains bounded, the solution exists for all $t \in I$.

Proof. MATH 4051. □

6.2.2. Frenet-Serret System. Given an arc-length parametrized and non-degenerate curve $\gamma(s) : I \to \mathbb{R}^3$, recall that tangent and binormal satisfy:

$$\begin{aligned} \mathsf{T}'(s) &= \kappa(s)\mathsf{N}(s) \\ \mathsf{B}'(s) &= -\tau(s)\mathsf{N}(s). \end{aligned}$$

Using the fact that $\mathsf{N} = \mathsf{B} \times \mathsf{T}$, one can also compute:

$$\begin{aligned} \mathsf{N}'(s) &= \mathsf{B}'(s) \times \mathsf{T}(s) + \mathsf{B}(s) \times \mathsf{T}'(s) \\ &= -\tau(s)\mathsf{N}(s) \times \mathsf{T}(s) + \mathsf{B}(s) \times \kappa(s)\mathsf{N}(s) \\ &= -\kappa(s)\mathsf{T}(s) + \tau(s)\mathsf{B}(s). \end{aligned}$$

The *Frenet-Serret System* is an ODE system for the Frenet-Serret Frame of a non-degenerate curve $\gamma(s)$:

$$\begin{array}{rcccc} \mathsf{T}' & = & & \kappa\mathsf{N} & \\ \mathsf{N}' & = & -\kappa\mathsf{T} & & +\tau\mathsf{B} \\ \mathsf{B}' & = & & -\tau\mathsf{N} & \end{array}$$

or equivalently in matrix form:

$$\begin{bmatrix} \mathsf{T} \\ \mathsf{N} \\ \mathsf{B} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{N} \\ \mathsf{B} \end{bmatrix} \tag{6.6}$$

Since each vector of the $\{\mathsf{T}, \mathsf{N}, \mathsf{B}\}$ frame has three components, therefore the Frenet-Serret System (6.6) is an ODE system of 9 equations with 9 unknown functions.

6.2.3. Fundamental Theorem. We now state the main theorem of this section:

Theorem 6.23 (Fundamental Theorem of Space Curves). *Given any smooth positive function $\alpha(s) : I \to (0, \infty)$, and a smooth real-valued function $\beta(s) : I \to \mathbb{R}$, there exists an arc-length parametrized, non-degenerate curve $\gamma(s) : I \to \mathbb{R}^3$ such that its curvature $\kappa(s) \equiv \alpha(s)$ and its torsion $\tau(s) \equiv \beta(s)$.*

Moreover, if $\bar{\gamma}(s) : I \to \mathbb{R}^3$ is another arc-length parametrized, non-degenerate curve whose curvature $\bar{\kappa}(s) \equiv \alpha(s)$ and torsion $\bar{\tau}(s) \equiv \beta(s)$, then there exists a 3×3 constant matrix A with $A^T A = I$, and a constant vector p, such that $\bar{\gamma}(s) = A\gamma(s) + p$ for any $s \in I$.

Proof. The existence part consists of three major steps.

Step 1: Use the existence theorem of ODEs (Theorem 6.22) to show there exists a moving orthonormal frame $\{\hat{e}_1(s), \hat{e}_2(s), \hat{e}_3(s)\}$ which satisfies an ODE system (see (6.7) below) *analogous* to the Frenet-Serret System (6.6).

Step 2: Show that there exists a curve $\gamma(s)$ whose Frenet-Serret Frame is given by $\mathsf{T}(s) = \hat{e}_1(s)$, $\mathsf{N}(s) = \hat{e}_2(s)$ and $\mathsf{B}(s) = \hat{e}_3(s)$. Consequently, from the system (6.7), one can claim $\gamma(s)$ is a curve that satisfies the required conditions.

Step 3: Prove the uniqueness part of the theorem.

Step 1: To begin, let's consider the ODE system with unknowns $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$:

$$\begin{bmatrix} \hat{e}_1(s) \\ \hat{e}_2(s) \\ \hat{e}_3(s) \end{bmatrix}' = \begin{bmatrix} 0 & \alpha(s) & 0 \\ -\alpha(s) & 0 & \beta(s) \\ 0 & -\beta(s) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_1(s) \\ \hat{e}_2(s) \\ \hat{e}_3(s) \end{bmatrix} \tag{6.7}$$

which is an analogous system to the Frenet-Serret System (6.6). Impose the initial conditions:

$$\hat{e}_1(0) = \hat{i}, \ \hat{e}_2(0) = \hat{j}, \ \hat{e}_3(0) = \hat{k}.$$

Recall that $\alpha(s)$ and $\beta(s)$ are given to be smooth (in particular, continuously differentiable). By Theorem 6.22, there exists a solution $\{\hat{e}_1(s),\ \hat{e}_2(s),\ \hat{e}_3(s)\}$ defined on a maximal interval $s \in (T_-, T_+)$ that satisfies the system with the above initial conditions.

Note that $\{\hat{e}_1(s),\ \hat{e}_2(s),\ \hat{e}_3(s)\}$ is orthonormal initially at $s = 0$, we claim it remains so as long as solution exists. To prove this, we first derive (see Exercise 6.14):

$$\frac{d}{ds}\begin{bmatrix} \hat{e}_1\cdot\hat{e}_1 \\ \hat{e}_2\cdot\hat{e}_2 \\ \hat{e}_3\cdot\hat{e}_3 \\ \hat{e}_1\cdot\hat{e}_2 \\ \hat{e}_2\cdot\hat{e}_3 \\ \hat{e}_3\cdot\hat{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2\alpha & 0 & 0 \\ 0 & 0 & 0 & -2\alpha & 2\beta & 0 \\ 0 & 0 & 0 & 0 & -2\beta & 0 \\ -\alpha & \alpha & 0 & 0 & 0 & \beta \\ 0 & -\beta & \beta & 0 & 0 & -\alpha \\ 0 & 0 & 0 & -\beta & \alpha & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_1\cdot\hat{e}_1 \\ \hat{e}_2\cdot\hat{e}_2 \\ \hat{e}_3\cdot\hat{e}_3 \\ \hat{e}_1\cdot\hat{e}_2 \\ \hat{e}_2\cdot\hat{e}_3 \\ \hat{e}_3\cdot\hat{e}_1 \end{bmatrix} \tag{6.8}$$

Exercise 6.14. Verify (6.8).

Regarding $\hat{e}_i \cdot \hat{e}_j$'s are unknowns, (6.8) is a linear ODE system of 6 equations with initial conditions:

$$(\hat{e}_1\cdot\hat{e}_1,\ \hat{e}_2\cdot\hat{e}_2,\ \hat{e}_3\cdot\hat{e}_3,\ \hat{e}_1\cdot\hat{e}_2,\ \hat{e}_2\cdot\hat{e}_3,\ \hat{e}_3\cdot\hat{e}_1)_{s=0} = (1,\ 1,\ 1,\ 0,\ 0,\ 0)$$

It can be verified easily that the constant solution $(1,\ 1,\ 1,\ 0,\ 0,\ 0)$ is indeed a solution to (6.8). Therefore, by the uniqueness part of Theorem 6.22, we must have

$$(\hat{e}_1\cdot\hat{e}_1,\ \hat{e}_2\cdot\hat{e}_2,\ \hat{e}_3\cdot\hat{e}_3,\ \hat{e}_1\cdot\hat{e}_2,\ \hat{e}_2\cdot\hat{e}_3,\ \hat{e}_3\cdot\hat{e}_1) = (1,\ 1,\ 1,\ 0,\ 0,\ 0)$$

for any $s \in (T_-, T_+)$. In other words, the frame $\{\hat{e}_1(s),\ \hat{e}_2(s),\ \hat{e}_3(s)\}$ is orthonormal as long as solution exists.

Consequently, each of $\{\hat{e}_1(s),\ \hat{e}_2(s),\ \hat{e}_3(s)\}$ remains bounded, and by last statement of Theorem 6.22, this orthonormal frame can be extended so that it is defined for all $s \in I$.

Step 2: Using the frame $\hat{e}_1(s) : I \to \mathbb{R}^3$ obtained in Step 1, we define:

$$\gamma(s) = \int_0^s \hat{e}_1(s)\, ds.$$

Evidently, $\gamma(s)$ is a curve starting from the origin at $s = 0$. Since $\hat{e}_1(s)$ is continuous, $\gamma(s)$ is well-defined on I and by the Fundamental Theorem of Calculus, we get:

$$\mathsf{T}(s) := \gamma'(s) = \hat{e}_1(s)$$

which is a unit vector for any $s \in I$. Therefore, $\gamma(s)$ is arc-length parametrized.

Next we verify that $\gamma(s)$ is the curve require by computing its curvature and torsion. By (6.8),

$$\gamma''(s) = \hat{e}_1'(s) = \alpha(s)\hat{e}_2(s)$$

By the fact that $\hat{e}_2(s)$ is unit, we conclude that:

$$\kappa(s) = |\gamma''(s)| = \alpha(s)$$

and so $\mathsf{N}(s) = \frac{1}{\kappa(s)}\gamma''(s) = \hat{e}_2(s)$. For the binormal, we observe that $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$ initially at $s = 0$ and that the frame $\{\hat{e}_1(s),\ \hat{e}_2(s),\ \hat{e}_3(s)\}$ remains to be orthonormal for all $s \in I$, we must have $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$ for all $s \in I$ by continuity. Therefore, $\mathsf{B}(s) = \mathsf{T}(s) \times \mathsf{N}(s) = \hat{e}_1(s) \times \hat{e}_2(s) = \hat{e}_3(s)$ for any $s \in I$. By (6.8), we have:

$$\mathsf{B}'(s) = \hat{e}_3'(s) = -\beta(s)\hat{e}_2(s) = -\beta(s)\mathsf{N}.$$

Therefore, $\tau(s) = -\mathsf{B}'(s) \cdot \mathsf{N}(s) = \beta(s)$.

Step 3: Now suppose there exists another curve $\bar{\gamma}(s) : I \to \mathbb{R}^3$ with the same curvature and torsion as $\gamma(s)$. Let $\{\bar{\mathsf{T}}(s),\ \bar{\mathsf{N}}(s),\ \bar{\mathsf{B}}(s)\}$ be the Frenet-Serret Frame of $\bar{\gamma}(s)$. We define the matrix:

$$A = \begin{bmatrix} \bar{\mathsf{T}}(0) & \bar{\mathsf{N}}(0) & \bar{\mathsf{B}}(0) \end{bmatrix}.$$

By orthonormality, one can check that $A^T A = I$. We claim that $\bar{\gamma}(s) = A\gamma(s) + \bar{\gamma}(0)$ for any $s \in I$ using again the uniqueness theorem of ODEs (Theorem 6.22).

First note that A is an orthogonal matrix, so the Frenet-Serret Frame of the transformed curve $A\gamma(s) + \bar{\gamma}(0)$ is given by $\{A\mathsf{T}(s),\ A\mathsf{N}(s),\ A\mathsf{B}(s)\}$ and the frame satisfies the ODE system:

$$\begin{bmatrix} A\mathsf{T}(s) \\ A\mathsf{N}(s) \\ A\mathsf{B}(s) \end{bmatrix}' = \begin{bmatrix} 0 & \alpha(s) & 0 \\ -\alpha(s) & 0 & \beta(s) \\ 0 & -\beta(s) & 0 \end{bmatrix} \begin{bmatrix} A\mathsf{T}(s) \\ A\mathsf{N}(s) \\ A\mathsf{B}(s) \end{bmatrix}$$

since the Frenet-Serret Frame $\{\mathsf{T}(s),\ \mathsf{N}(s),\ \mathsf{B}(s)\}$ does.

Furthermore, the curve $\bar{\gamma}(s)$ also has curvature $\alpha(s)$ and torsion $\beta(s)$, so its Frenet-Serret Frame $\{\bar{\mathsf{T}}(s),\ \bar{\mathsf{N}}(s),\ \bar{\mathsf{B}}(s)\}$ also satisfies the ODE system:

$$\begin{bmatrix} \bar{\mathsf{T}}(s) \\ \bar{\mathsf{N}}(s) \\ \bar{\mathsf{B}}(s) \end{bmatrix}' = \begin{bmatrix} 0 & \alpha(s) & 0 \\ -\alpha(s) & 0 & \beta(s) \\ 0 & -\beta(s) & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathsf{T}}(s) \\ \bar{\mathsf{N}}(s) \\ \bar{\mathsf{B}}(s) \end{bmatrix}.$$

Initially at $s = 0$, the two Frenet-Serret Frames are equal by the definition of A and choice of $\hat{e}_i(0)$'s in Step 1:

$$\begin{aligned} A\mathsf{T}(0) &= \begin{bmatrix} \bar{\mathsf{T}}(0) & \bar{\mathsf{N}}(0) & \bar{\mathsf{B}}(0) \end{bmatrix} \hat{\imath} = \bar{\mathsf{T}}(0) \\ A\mathsf{N}(0) &= \begin{bmatrix} \bar{\mathsf{T}}(0) & \bar{\mathsf{N}}(0) & \bar{\mathsf{B}}(0) \end{bmatrix} \hat{\jmath} = \bar{\mathsf{N}}(0) \\ A\mathsf{B}(0) &= \begin{bmatrix} \bar{\mathsf{T}}(0) & \bar{\mathsf{N}}(0) & \bar{\mathsf{B}}(0) \end{bmatrix} \hat{k} = \bar{\mathsf{B}}(0) \end{aligned}$$

By the uniqueness part of Theorem 6.22, the two frames are equal for all $s \in I$. In particular, we have:

$$A\mathsf{T}(s) \equiv \bar{\mathsf{T}}(s).$$

Finally, to show that $\bar{\gamma}(s) \equiv A\gamma(s) + \bar{\gamma}(0)$, we consider the function

$$f(s) := |\bar{\gamma}(s) - (A\gamma(s) - \bar{\gamma}(0))|^2.$$

Taking its derivative, we get:

$$\begin{aligned} f'(s) &= 2\left(\bar{\gamma}'(s) - A\gamma'(s)\right) \cdot \left(\bar{\gamma}(s) - (A\gamma(s) - \bar{\gamma}(0))\right) \\ &= 2\underbrace{\left(\bar{\mathsf{T}}(s) - A\mathsf{T}(s)\right)}_{=0} \cdot \left(\bar{\gamma}(s) - (A\gamma(s) - \bar{\gamma}(0))\right) \\ &= 0 \end{aligned}$$

for any $s \in I$. Since $f(0) = 0$ initially by the fact that $\gamma(0) = 0$, we have $f(s) \equiv 0$ and so $\bar{\gamma}(s) \equiv A\gamma(s) + \bar{\gamma}(0)$, completing the proof of the theorem. □

The existence part of Theorem 6.23 only shows a curve with prescribed curvature and torsion *exists*, but it is in general difficult to find such a curve explicitly. While the existence part does not have much practical use, the uniqueness part has some nice corollaries.

First recall that a helix is a curve of the form $\gamma_{a,b}(t) = (a\cos t,\ a\sin t,\ bt)$ where $a \neq 0$ and b can be any real number. It's arc-length parametrization is given by:

$$\gamma_{a,b}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right).$$

It can be computed that its curvature and torsion are both constants:

$$\kappa_{a,b}(s) \equiv \frac{a}{a^2+b^2}$$
$$\tau_{a,b}(s) \equiv \frac{b}{a^2+b^2}.$$

Conversely, given two constants $\kappa_0 > 0$ and $\tau_0 \in \mathbb{R}$, by taking $a = \frac{\kappa_0}{\kappa_0^2+\tau_0^2}$ and $b = \frac{\tau_0}{\kappa_0^2+\tau_0^2}$, the helix $\gamma_{a,b}(s)$ with this pair of a and b has curvature κ_0 and torsion τ_0. Hence, the uniqueness part of Theorem 6.23 asserts that:

Corollary 6.24. *A non-degenerate curve $\gamma(s)$ has constant curvature and torsion if and only if $\gamma(s)$ is congruent to one of the helices $\gamma_{a,b}(s)$.*

Remark 6.25. Two space curves $\gamma(s)$ and $\widetilde{\gamma}(s)$ are said to be congruent if there exists a 3×3 orthogonal matrix A and a constant vector $p \in \mathbb{R}^3$ such that $\widetilde{\gamma}(s) = A\gamma(s) + p$. In simpler terms, one can obtain $\widetilde{\gamma}(s)$ by rotating and translating $\gamma(s)$. □

6.3. Plane Curves

A plane curve $\gamma(s)$ is an arc-length parametrized curve in $\mathbb{R}^2$. While it can be considered as a space curve by identifying $\mathbb{R}^2$ and the xy-plane in $\mathbb{R}^3$, there are several aspects of plane curves that make them distinguished from space curves.

6.3.1. Signed Curvature. Given an arc-length parametrized curve $\gamma(s) : I \to \mathbb{R}^2$, we define the tangent frame $\mathsf{T}(s)$ as in space curves, i.e.

$$\mathsf{T}(s) = \gamma'(s).$$

However, instead of defining the normal frame $\mathsf{N}(s) = \frac{1}{\kappa(s)}\mathsf{T}'(s)$, we use the frame $J\mathsf{T}(s)$ where J is the counter-clockwise rotation by $\frac{\pi}{2}$, i.e.

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

One can easily check that $\{\mathsf{T}(s),\ J\mathsf{T}(s)\}$ is an orthonormal frame of $\mathbb{R}^2$ for any $s \in I$. Let's call this the TN-Frame of the curve. We will work with the TN-Frame in place of the Frenet-Serret Frame for plane curves. The reasons for doing so are two-folded. For one thing, the normal frame. $J\mathsf{T}(s)$ is well-defined for any $s \in I$ even though $\kappa(s)$ is zero for some $s \in I$. Hence, one can relax the non-degeneracy assumption here. For another, we can introduce the signed curvature $k(s)$:

Definition 6.26 (Signed Curvature)**.** Given an arc-length parametrized plane curve $\gamma(s) : I \to \mathbb{R}^2$, the signed curvature $k(s) : I \to \mathbb{R}$ is defined as:

$$k(s) := \mathsf{T}'(s) \cdot J\mathsf{T}(s).$$

Note that $\mathsf{T}(s)$ is unit, so by Lemma 6.7 we know $\mathsf{T}(s)$ and $\mathsf{T}'(s)$ are always orthogonal and hence it is either in or against the direction of $J\mathsf{T}(s)$. Therefore, we have

$$|k(s)| = |\mathsf{T}'(s)|\,|J\mathsf{T}(s)| = |\gamma''(s)| = \kappa(s).$$

The sign of $k(s)$ is determined by whether T' and $\hat{\jmath}\mathsf{T}$ are along or against each other (see Figure 6.4).

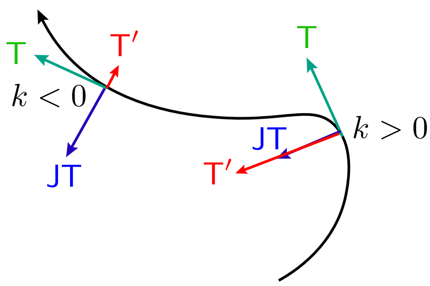


Figure 6.4. Signed curvature

Example 6.27. Let's compute the signed curvature of the unit circle

$$\gamma(s) = (\cos \varepsilon s,\ \sin \varepsilon s)$$

where $\varepsilon = \pm 1$. The curve is counter-clockwise orientable when $\varepsilon = 1$, and is clockwise orientable when $\varepsilon = -1$. Clearly it is arc-length parametrized, and

$$\begin{aligned}
\mathsf{T}(s) &= \gamma'(s) = (-\varepsilon \sin \varepsilon s,\ \varepsilon \cos \varepsilon s) \\
J\mathsf{T}(s) &= (-\varepsilon \cos \varepsilon s,\ -\varepsilon \sin \varepsilon s) \\
k(s) &= \mathsf{T}'(s) \cdot J\mathsf{T}(s) \\
&= (-\varepsilon^2 \cos \varepsilon s,\ -\varepsilon^2 \sin \varepsilon s) \cdot (-\varepsilon \cos \varepsilon s,\ -\varepsilon \sin \varepsilon s) \\
&= \varepsilon^3 = \varepsilon.
\end{aligned}$$

□

Exercise 6.15. Consider a plane curve $\gamma(s)$ parametrized by arc-length. Let $\theta(s)$ be the angle between the x-axis and the unit tangent vector $\mathsf{T}(s)$. Show that:

$$\mathsf{T}'(s) = \theta'(s)\mathsf{J}\mathsf{T}(s) \quad \text{and} \quad k(s) = \theta'(s).$$

Exercise 6.16. [**dC76**, P.25] Consider a plane curve $\gamma(s)$ parametrized by arc-length. Suppose $|\gamma(s)|$ is maximum at $s = s_0$. Show that:

$$|k(s_0)| \geq \frac{1}{|\gamma(s_0)|}.$$

Given a regular plane curve $\gamma(t) : I \to \mathbb{R}^2$, not necessarily arc-length parametrized. Denote the components of the curve by $\gamma(t) = (x_1(t),\ x_2(t))$.

(a) Show that its signed curvature (as a function of t) is given by:

$$k(t) = \frac{\gamma''(t) \cdot J\gamma'(t)}{|\gamma'(t)|^3} = \frac{x_1'(t)x_2''(t) - x_2'(t)x_1''(t)}{(x_1'(t)^2 + x_2'(t)^2)^{3/2}}.$$

(b) Hence, show that the graph of a smooth function $y = f(x)$, when considered as a curve parametrized by x, has signed curvature given by:

$$k(x) = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}.$$

The signed curvature characterizes regular plane curves, as like curvature and torsion characterize non-degenerate space curves.

Theorem 6.28 (Fundamental Theorem of Plane Curves)**.** *Given any smooth real-valued function $\alpha(s) : I \to \mathbb{R}$, there exists a regular plane curve $\gamma(s) : I \to \mathbb{R}^2$ such that its signed curvature $k(s) \equiv \alpha(s)$. Moreover, if $\bar{\gamma}(s) : I \to \mathbb{R}^2$ is another regular plane curve such that its signed curvature $\bar{k}(s) \equiv \alpha(s)$, then there exists a 2×2 orthogonal matrix A and a constant vector $p \in \mathbb{R}^2$ such that $\bar{\gamma}(s) \equiv A\gamma(s) + p$.*

Proof. See Exercise #6.17. □

Exercise 6.17. Prove Theorem 6.28. Although the proof is similar to that of Theorem 6.23 for non-degenerate space curves, please do not use the latter to prove the former in this exercise. Here is a hint on how to begin the proof: Consider the initial-value problem

$$\begin{aligned}
\hat{e}'(s) &= \alpha(s)\ J\hat{e}(s) \\
\hat{e}(0) &= \hat{\imath}
\end{aligned}$$

Exercise 6.18. Using Theorem 6.28, show that a regular plane curve has constant signed curvature if and only if it is a straight line or a circle

Exercise 6.19. [**Küh05**, P.50] Find an explicit plane curve $\gamma(s)$ such that the signed curvature is given by $k(s) = \frac{1}{\sqrt{s}}$.

6.3.2. Total Curvature. In this subsection, we explore an interesting result concerning the signed curvature of a plane curve. We first introduce:

Definition 6.29 (Closed Curves). An arc-length parametrized plane curve $\gamma(s) : [0, L] \to \mathbb{R}^2$ is said to be *closed* if $\gamma(0) = \gamma(L)$. It is said to be *simple closed* if $\gamma(s)$ is closed and if $\gamma(s_1) = \gamma(s_2)$ for some $s_i \in [0, L]$ then one must have $s_1,\ s_2 = 0$ or L.

The following is a celebrated result that relates the local property (i.e. signed curvature) to the global property (topology) of simple closed curves:

Theorem 6.30 (Hopf). *For any arc-length parametrized, simple closed curve $\gamma(s) : [0, L] \to \mathbb{R}^2$ such that $\gamma'(0) = \gamma'(L)$, we must have:*

$$\int_0^L k(s)\ ds = \pm 2\pi.$$

The original proof was due to Hopf. We will not discuss Hopf's original proof in this course, but we will prove a weaker result, under the same assumption as Theorem 6.30, that

$$\int_0^L k(s)\ ds = 2\pi n$$

for some integer n.

Let $\{\mathsf{T}(s),\ J\mathsf{T}(s)\}$ be the TN-frame of $\gamma(s)$. Since $\mathsf{T}(s)$ is unit for any $s \in [0, L]$, one can find a smooth function $\theta(s) : [0, L] \to \mathbb{R}$ such that

$$\mathsf{T}(s) = (\cos\theta(s),\ \sin\theta(s))$$

for any $s \in [0, L]$. Here $\theta(s)$ can be regarded as $2k\pi +$ angle between $\mathsf{T}(s)$ and $\hat{i}$. In order to ensure continuity, we allow $\theta(s)$ to take values beyond $[0, 2\pi]$.

Then $J\mathsf{T}(s) = (-\sin\theta(s),\ \cos\theta(s))$, and so by the definition of $k(s)$, we get:

$$\begin{aligned} k(s) &= \mathsf{T}'(s) \cdot J\mathsf{T}(s) \\ &= (-\theta'(s)\sin\theta(s),\ \theta'(s)\cos\theta(s)) \cdot (-\sin\theta(s),\ \cos\theta(s)) \\ &= \theta'(s). \end{aligned}$$

Therefore, the total curvature is given by:

$$\int_0^L k(s)\ ds = \int_0^L \theta'(s)\ ds = \theta(L) - \theta(0).$$

Since it is assumed that $\mathsf{T}(0) = \mathsf{T}(L)$ in Theorem 6.30, we have

$$\theta(L) \equiv \theta(0) \pmod{2\pi}$$

and so we have:

$$\int_0^L k(s)\ ds = 2\pi n$$

for some integer n.

Chapter 7

Geometry of Euclidean Hypersurfaces

Riemannian geometry is a branch in differential geometry which studies intrinsic geometric structure without referencing to the ambient space. It was first developed by Gauss and Riemann, and was later adopted by Einstein to lay the mathematical foundation of general relativity, which regards our space-time as an intrinsic manifold.

Despite the intrinsic nature of Riemannian geometry, many of its important notions and concepts are stemmed from extrinsic geometry, namely hypersurfaces in Euclidean spaces. In this chapter, we will first explore ourselves to the basic differential geometry of Euclidean hypersurfaces.

7.1. Regular Hypersurfaces in Euclidean Spaces

The prefix *hyper-* in the word "hypersurface" means the manifold is one dimensional lower than the ambient space. A hypersurface in $\mathbb{R}^{n+1}$ is a n-dimensional subset of $\mathbb{R}^{n+1}$. As in the case of regular surfaces in $\mathbb{R}^3$, we want to impose some conditions of a hypersurface so that it becomes *regular* in the way we desire.

Definition 7.1 (Regular Hypersurfaces in $\mathbb{R}^{n+1}$). Let Σ^n be a non-empty subset of $\mathbb{R}^{n+1}$. Suppose Σ^n can be covered by the image of a family of local parametrizations $\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to \Sigma\}$, i.e. $\Sigma = \bigcup_\alpha F_\alpha(\mathcal{U}_\alpha)$, such that each $\mathcal{U}_\alpha$ is an open set in $\mathbb{R}^n$ and each $F_\alpha : \mathcal{U}_\alpha \to \Sigma$ satisfies all three conditions below:

(1) $F_\alpha(u^1_\alpha, \cdots, u^n_\alpha)$ is a C^∞ map from U_α to $\mathbb{R}^{n+1}$;
(2) F_α is a homeomorphism between $\mathcal{U}_\alpha$ and its image $\mathcal{O}_\alpha := F_\alpha(\mathcal{U}_\alpha)$; and
(3) The following vectors in $\mathbb{R}^{n+1}$ are linearly independent

$$\left\{ \frac{\partial F_\alpha}{\partial u^1_\alpha}, \cdots, \frac{\partial F_\alpha}{\partial u^n_\alpha} \right\}$$

for any $(u^1_\alpha, \cdots, u^n_\alpha) \in \mathcal{U}_\alpha$.

Then, we say that Σ^n is a *regular hypersurface* in $\mathbb{R}^{n+1}$.

Each vector $\left.\dfrac{\partial F_\alpha}{\partial u_\alpha^i}\right|_{(u_\alpha^1,\cdots,u_\alpha^n)}$ is a tangent to Σ^n at the based point $p = F_\alpha(u_\alpha^1, \cdots, u_\alpha^n)$. With a bit abuse of notations and for simplicity, we will from now on denote

$$\frac{\partial F_\alpha}{\partial u_\alpha^i}(p) := \left.\frac{\partial F_\alpha}{\partial u_\alpha^i}\right|_{(u_\alpha^1,\cdots,u_\alpha^n)}$$

where $p = F_\alpha(u_\alpha^1, \cdots, u_\alpha^n)$, to emphasize that p is the based point of the tangent vector. By condition (3) in Definition 7.1, we know that the following vector space

$$T_p\Sigma^n := \text{span}\left\{\frac{\partial F_\alpha}{\partial u_\alpha^1}(p), \cdots, \frac{\partial F_\alpha}{\partial u_\alpha^n}(p)\right\}$$

has dimension n. This is called the *tangent space* at $p \in \Sigma^n$.

Example 7.2. The graph Σ_f of any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is a regular hypersurface in $\mathbb{R}^{n+1}$. One can parametrize Σ_f by a single parametrization:

$$\begin{aligned} F : \mathbb{R}^n &\to \Sigma_f \\ (x_1, \cdots, x_n) &\mapsto \big(x_1, \cdots, x_n, f(x_1, \cdots, x_n)\big) \end{aligned}$$

By straight-forward computations, we have:

$$\frac{\partial F}{\partial x_i} = \hat{e}_i + \frac{\partial f}{\partial x_i}\hat{e}_{n+1}$$

where $\{\hat{e}_1, \cdots, \hat{e}_{n+1}\}$ is the standard basis vectors in $\mathbb{R}^{n+1}$. It is clear that $\left\{\frac{\partial F}{\partial x_i}\right\}_{i=1}^n$ are linearly independent. □

Example 7.3. The n-dimensional unit sphere:

$$\mathbb{S}^n := \left\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\right\}$$

is a regular hypersurface in $\mathbb{R}^{n=1}$. It can be parametrized by a pair of (inverse) stereographic projections $F_+ : \mathbb{R}^n \to \mathbb{S}^n \backslash \{(0, \cdots, 0, 1)\}$ and $F_- : \mathbb{R}^n \to \mathbb{S}^n \backslash \{(0, \cdots, 0, -1)\}$ with F_+ given by

$$F_+(u_1, \cdots, u_n) = \sum_{j=1}^n \frac{2u_i\hat{e}_i}{|u|^2 + 1} + \frac{|u|^2 - 1}{|u|^2 + 1}\hat{e}_{n+1}$$

where $|u|^2 = u_1^2 + \cdots + u_n^2$. We leave it as an exercise for readers to verify F_+ satisfies the conditions in Definition 7.1 and write down the south-pole map F_-. □

Exercise 7.1. Fill in the omitted detail in Examples 7.2 and 7.3.

Regular hypersurfaces are the higher dimensional generalization of regular surfaces discussed in Chapter 1. As such many important results about regular surfaces can carry over naturally to regular hypersurfaces. We state the results below and leave the proofs as exercises.

Theorem 7.4 (c.f. Theorem 1.6). *Let $g(x_1, \cdots, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Consider a non-empty level set $g^{-1}(c)$ where c is a constant. If $\nabla g(p) \neq 0$ at all points $p \in g^{-1}(c)$, then the level set $g^{-1}(c)$ is a regular hypersurface in $\mathbb{R}^{n+1}$.*

Proposition 7.5 (c.f. Proposition 1.8). *Assume all given conditions stated in Theorem 7.4. Furthermore, suppose F is a bijective map from an open set $\mathcal{U} \subset \mathbb{R}^n$ to an open set $\mathcal{O} \subset \Sigma := g^{-1}(c)$ which satisfies conditions (1) and (3) in Definition 7.1. Then, F satisfies condition (2) as well and hence is a smooth local parametrization of $g^{-1}(c)$.*

Proposition 7.6 (c.f. Proposition 1.11)**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a regular surface, and $F_\alpha : \mathcal{U}_\alpha \to M$ and $F_\beta : \mathcal{U}_\beta \to M$ be two smooth local parametrizations of Σ with overlapping images, i.e.* $\mathcal{W} := F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta) \neq \emptyset$. *Then, the transition maps defined below are also smooth maps:*

$$(F_\beta^{-1} \circ F_\alpha) : F_\alpha^{-1}(\mathcal{W}) \to F_\beta^{-1}(\mathcal{W})$$
$$(F_\alpha^{-1} \circ F_\beta) : F_\beta^{-1}(\mathcal{W}) \to F_\alpha^{-1}(\mathcal{W})$$

Exercise 7.2. Prove Theorem 7.4, Proposition 7.5, and Proposition 7.6.

Exercise 7.3. Show that a regular hypersurface Σ in $\mathbb{R}^{n+1}$ is a submanifold of $\mathbb{R}^{n+1}$.

7.2. Fundamental Forms and Curvatures

7.2.1. First Fundamental Form. In this subsection, we introduce an important concept in differential geometry – the first fundamental form. Loosely speaking, it is the dot product of the tangent vectors of a regular surface. It captures and encodes intrinsic geometric information (such as curvature) about the hypersurface.

Definition 7.7 (First Fundamental Form)**.** The *first fundamental form* of a regular hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is a 2-tensor g on Σ with local expression $g = g_{ij}\, du^i \otimes du^j$ where g_{ij} is given by:

$$g_{ij} = \left\langle \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j} \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual dot product on $\mathbb{R}^{n+1}$.

Exercise 7.4. Show that $g_{ij}\, du^i \otimes du^j$ is independent of local coordinates, i.e. if $G(v_\alpha)$ is another local parametrization and denote

$$\widetilde{g}_{\alpha\beta} = \left\langle \frac{\partial G}{\partial v_\alpha}, \frac{\partial G}{\partial v_\beta} \right\rangle,$$

then

$$\widetilde{g}_{\alpha\beta}\, dv^\alpha \otimes dv^\beta = g_{ij}\, du^i \otimes du^j.$$

Exercise 7.5. Let $\iota : \Sigma \to \mathbb{R}^{n+1}$ be an inclusion map from a regular hypersurface Σ. Show that the first fundamental form of Σ can be expressed as:

$$g = \iota^* \left(\sum_{\alpha=1}^{n+1} dx^\alpha \otimes dx^\alpha \right)$$

where (x_α) is the standard coordinates of $\mathbb{R}^{n+1}$.

Example 7.8. Let $\mathbb{S}^2$ be the unit 2-sphere and F be the following local parametrization:

$$F(u, v) = (\sin u \cos v,\ \sin u \sin v,\ \cos u), \quad (u, v) \in (0, \pi) \times (0, 2\pi)$$

By direct computations, we have:

$$\frac{\partial F}{\partial u} = (\cos u \cos v,\ \cos u \sin v,\ -\sin u)$$
$$\frac{\partial F}{\partial v} = (-\sin u \sin v,\ \sin u \cos v,\ 0)$$

$$g\left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\right) = \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial u} = 1 \qquad g\left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right) = \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v} = 0$$
$$g\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial u}\right) = \frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial u} = 0 \qquad g\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial v}\right) = \frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial v} = \sin^2 u$$

Therefore, we get its first fundamental equals

$$g = du \otimes du + \sin^2 u\, dv \otimes dv.$$

□

Exercise 7.6. Show that the first fundamental form of the graph Σ_f in Example 7.2 is given by:

$$g = \left(\delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right) dx^i \otimes dx^j = \delta_{ij}\, dx^i \otimes dx^j + df \otimes df.$$

As $g_{ij} = g_{ji}$, the tensor g is symmetric. As such the tensor notation $g_{ij}\, du^i \otimes du^j$ is often written simply as $g_{ij}\, du^i\, du^j$. For instance, the first fundamental form of the unit sphere in Example 7.8 can be expressed as:

$$g = du^2 + \sin^2 u\, dv^2$$

where du^2 is interpreted as $(du)^2 = du\, du$, not $d(u^2)$.

Another way to represent the first fundamental form is by the matrix:

$$[g] := \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

It is a *symmetric* matrix since $g\left(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right) = g\left(\frac{\partial F}{\partial u_j}, \frac{\partial F}{\partial u_i}\right)$.

Given two tangent vectors $Y = Y^i \frac{\partial F}{\partial u_i}$ and $Z = Z^i \frac{\partial F}{\partial u_i}$ in $T_p\Sigma$, the value of $g(Y, Z)$ is related to the entries of $[g]$ in the following way:

$$\begin{aligned} g(Y, Z) = g\left(Y^i \frac{\partial F}{\partial u_i}, Z^j \frac{\partial F}{\partial u_j}\right) &= Y^i Z^j g_{ij} \\ &= \begin{bmatrix} Y^1 & \cdots & Y^n \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} Z^1 \\ \vdots \\ Z^n \end{bmatrix} \end{aligned}$$

Note that the matrix $[g]$ depends on local coordinates although the tensor g does not.

As computed in Example 7.8, the matrix $[g]$ of the unit sphere (with respect the parametrization F used in the example) is given by:

$$[g] := \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix}$$

Evidently, it is a diagonal matrix. Try to think about the geometric meaning of $[g]$ being diagonal.

We will see in subsequent sections that g "encodes" crucial geometric information such as curvatures. There are also some familiar geometric quantities, such as length and area, which are related to the first fundamental form g.

Consider a curve γ on a regular hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ parametrized by $F(u_i)$. Suppose the curve can be parametrized by $\gamma(t)$, $a < t < b$, then from calculus we know the arc-length of the curve is given by:

$$L(\gamma) = \int_a^b |\gamma'(t)|\ dt$$

In fact one can express this above in terms of g. The argument is as follows:

Suppose $\gamma(t)$ has local coordinates coordinates $(\gamma^i(t))$ such that $F(\gamma^i(t)) = \gamma(t)$. Using the chain rule, we then have:

$$\gamma'(t) = \frac{\partial F}{\partial u_i}\frac{d\gamma^i}{dt}.$$

Recall that $|\gamma'(t)| = \sqrt{\langle \gamma'(t), \gamma'(t)\rangle}$ and that $\gamma'(t)$ lies on T_pM, we then have:

$$|\gamma'(t)| = \sqrt{g\left(\gamma'(t), \gamma'(t)\right)}$$

We can then express it in terms of the matrix components g_{ij}'s:

$$g\left(\gamma'(t), \gamma'(t)\right) = g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} \tag{7.1}$$

where g_{ij}'s are evaluated at the point $\gamma(t)$. Therefore, the arc-length can be expressed in terms of the first fundamental form by:

$$L(\gamma) = \int_a^b \sqrt{g\left(\gamma'(t), \gamma'(t)\right)}\, dt = \int_a^b \sqrt{g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}}\, dt$$

Another familiar geometric quantity which is also related to g is the area of a surface. For simplicity, we focus on dimension 2 first. Suppose a regular surface Σ can be almost everywhere parametrized by $F(u,v)$ with $(u,v) \in D \subset \mathbb{R}^2$ where D is a bounded domain, the area of this surface is given by:

$$A(M) = \iint_D \left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right|\, dudv$$

It is also possible to express $\left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right|$ in terms of the first fundamental form g. Let θ be the angle between the two vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, then from elementary vector geometry, we have:

$$\begin{aligned}
\left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right|^2 &= \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 \sin^2\theta \\
&= \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 - \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 \cos^2\theta \\
&= \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 - \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v}\right)^2 \\
&= \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial u}\right)\left(\frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial v}\right) - \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v}\right)^2 \\
&= g_{11}g_{22} - (g_{12})^2 \\
&= \det[g].
\end{aligned}$$

Therefore,

$$A(M) = \iint_D \sqrt{\det[g]}\, dudv \tag{7.2}$$

In fact, for higher dimensional regular hypersurfaces parametrized almost everywhere by $F(u_i) : \mathcal{U} \to \Sigma$, we also have its area equals

$$\int_{\mathcal{U}} \sqrt{\det[g]}\, du^1 \cdots du^n.$$

Example 7.9. Let Σ_f be the graph of a smooth function $f : \mathcal{U} \to \mathbb{R}$ defined on an open subset $\mathcal{U}$ of $\mathbb{R}^n$, then Σ_f has a globally defined smooth parametrization:

$$F(u_1, \cdots, u_n) = (u_1, \cdots, u_n,\, f(u_1, \cdots, u_n)).$$

By straight-forward computations, we can get:

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j},$$

Therefore, the length the curve $\gamma(t) := F(\gamma^1(t), \cdots, \gamma^n(t))$ can be computed by integrating the square root of:

$$\begin{aligned} g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} &= \sum_{i=1}^n \left(\frac{d\gamma^i}{dt}\right)^2 + \sum_{i,j=1}^n \frac{\partial f}{\partial u_i}\frac{d\gamma^i}{dt}\frac{\partial f}{\partial u_j}\frac{d\gamma^j}{dt} \\ &= \sum_{i=1}^n \left(\frac{d\gamma^i}{dt}\right)^2 + \left(\frac{d}{dt}f(\gamma(t))\right)^2. \end{aligned}$$

To compute the surface area of a region $F(\Omega) \subset \Sigma_f$ where Ω is a bounded domain on $\mathcal{U}$, we first compute $\det[g]$. The matrix $[g]$ can be written as

$$[g] = I_{n\times n} + (\nabla f)(\nabla f)^T.$$

By standard linear algebra[1], we know that its eigenvalues are

$$1 + |\nabla f|^2, 1, \cdots, 1,$$

and so $\det[g] = 1 + |\nabla f|^2$. According to (7.2), we have:

$$A(F(\Omega)) = \iint_\Omega \sqrt{1 + |\nabla f|^2}\, du^1 \cdots du^n$$

which is exactly the same as what you have seen in multivariable calculus. □

7.2.2. Second Fundamental Form. In this subsection we introduce another important 2-tensor on $T_p\Sigma$, the second fundamental form h. While the first fundamental form g encodes information about angle, length and area, the second fundamental form encodes information about various curvatures.

We will see in subsequent sections that curvatures of a regular hypersurface are, roughly speaking, determined by rate of changes of tangent and normal vectors just like the case for regular curves. Let's first talk about the normal vector – or in differential geometry jargon – the Gauss Map.

Given a regular hypersurface Σ in $\mathbb{R}^{n+1}$ with $F(u_i) : \mathcal{U} \subset \mathbb{R}^2 \to \Sigma$ as one of its local parametrization. Let $p \in M$, the orthogonal complement of the tangent space $T_p\Sigma$ in $\mathbb{R}^{n+1}$ is a 1-dimensional vector space denoted by $N_p\Sigma$. There are exactly two unit vectors in $N_p\Sigma$. In dimension 2 (i.e. regular surfaces), $\frac{\partial F}{\partial u}(p)$ and $\frac{\partial F}{\partial v}(p)$ are two linearly independent tangents at p, and so the two unit normals are given by

$$\nu(p) = \frac{\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)}{\left|\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)\right|} \quad \text{or} \quad -\frac{\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)}{\left|\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)\right|}$$

See the result from Exercise 4.6 for higher dimensional hypersurfaces.

Example 7.10. Consider the unit sphere $\mathbb{S}^2(1)$ with smooth local parametrization:

$$F(u,v) = (\sin u \cos v,\ \sin u \sin v,\ \cos u), \quad (u,v) \in (0,\pi) \times (0, 2\pi)$$

It is straight-forward to compute that:

$$\begin{aligned} \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} &= \left(\sin^2 u \cos v,\ \sin^2 u \sin v,\ \sin u \cos u\right) \\ \left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right| &= \sin u \\ \nu(u,v) &= (\sin u \cos v,\ \sin u \sin v,\ \cos u) = F(u,v) \end{aligned}$$

This unit normal vector ν is outward-pointing. □

[1] Note that ∇f is an eigenvector with eigenvalue $1 + |\nabla f|^2$, and $(\nabla f)(\nabla f)^T$ has rank 1.

Given a regular hypersurface, there are always two choices of normal vector at each point. For a sphere, once the normal vector direction is chosen, it is always *consistent* with your choice when we move the normal vector across the sphere. That is, when you draw a closed path on the sphere and see how the unit normal vector varies along the path, you will find that the unit normal remains the same when you come back to the original point. We call it an *orientable* hypersurface if there exists a continuous choice of unit normal vector across the whole hypersurface. See Chapter 4 for more thorough discussions on orientability.

When Σ is an orientable regular hypersurface, the chosen unit normal vector ν can then be regarded as a map. The domain of ν is Σ. Since ν is unit, the codomain can be taken to be the *unit sphere* $\mathbb{S}^n$. We call this map as:

Definition 7.11 (Gauss Map)**.** Suppose Σ is an orientable regular hypersurface. The *Gauss map* of Σ is a smooth function $\nu : \Sigma \to \mathbb{S}^n$ such that for any $p \in \Sigma$, the output $\nu(p)$ is a unit normal vector of Σ at p. Here $\mathbb{S}^n$ is the unit n-sphere in $\mathbb{R}^{n+1}$.

As computed in Example 7.10, the Gauss map ν for the unit sphere $\mathbb{S}^2$ is given by F (assuming the outward-pointing convention is observed). It is not difficult to see that the Gauss map ν for a sphere with radius R centered the origin in $\mathbb{R}^3$ is given by $\frac{1}{R}F$. Readers should verify this as an exercise.

For a plane Π, the unit normal vector at each point is the same. Therefore, the Gauss map $\nu(p)$ is a constant vector independent of p.

A unit cylinder with z-axis as its central axis can be parametrized by:

$$F(u,v) = (\cos u,\, \sin u,\, v), \qquad (u,v) \in (0,2\pi) \times \mathbb{R}.$$

By straight-forward computations, one can get:

$$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} = (\cos u,\, \sin u,\, 0)$$

which is already unit. Therefore, the Gauss map of the cylinder is given by:

$$\nu(u,v) = (\cos u,\, \sin u,\, 0).$$

The image of ν in $\mathbb{S}^2$ is the equator.

It is not difficult to see that the *image* of the Gauss map ν, which is a subset of $\mathbb{S}^2$, is related to how "spherical" or "planar" the surface looks. The smaller the image, the more planar it is.

The curvature of a regular curve is a scalar function $\kappa(p)$. Since a curve is one dimensional, we can simply use one single value to measure the curvature at each point. However, a regular hypersurface has arbitrary dimensions and hence has higher degree of freedom than curves. It may bend by a different extent along different direction. As such, there are various notions of curvatures for regular hypersurfaces. We first talk about the normal curvature, which is fundamental to many other notions of curvatures.

Let Σ be an orientable regular hypersurface with its Gauss map denoted by ν. At each point $p \in \Sigma$, we pick a unit tangent vector $\hat{e}$ in $T_p\Sigma$. Heuristically, the normal curvature at p measures the curvature of the surface along a direction $\hat{e}$. Precisely, we define:

Definition 7.12 (Normal Curvature)**.** Let Σ be an orientable regular hypersurface with Gauss map ν. For each point $p \in \Sigma$, and any unit tangent vector $\hat{e} \in T_p\Sigma$, we let $\Pi_p(\nu, \hat{e})$ be the plane in $\mathbb{R}^{n+1}$ passing through p and parallel to both $\nu(p)$ and $\hat{e}$ (see Figure 7.1).

Let γ be the curve of intersection of Σ and $\Pi_p(\nu, \hat{e})$. The *normal curvature at p in the direction of* $\hat{e}$ of Σ, denoted by $k_n(p, \hat{e})$, is defined to be the signed curvature $k(p)$ of the curve γ at p with respect to the Gauss map ν.

Remark 7.13. Since $k_n(p, \hat{e})$ is defined using the Gauss map ν, which always comes with two possible choice for any orientable regular surface, the normal curvature *depends* on the choice of the Gauss map ν. If the opposite unit normal is chosen to be the Gauss map, the normal curvature will differ by a sign.

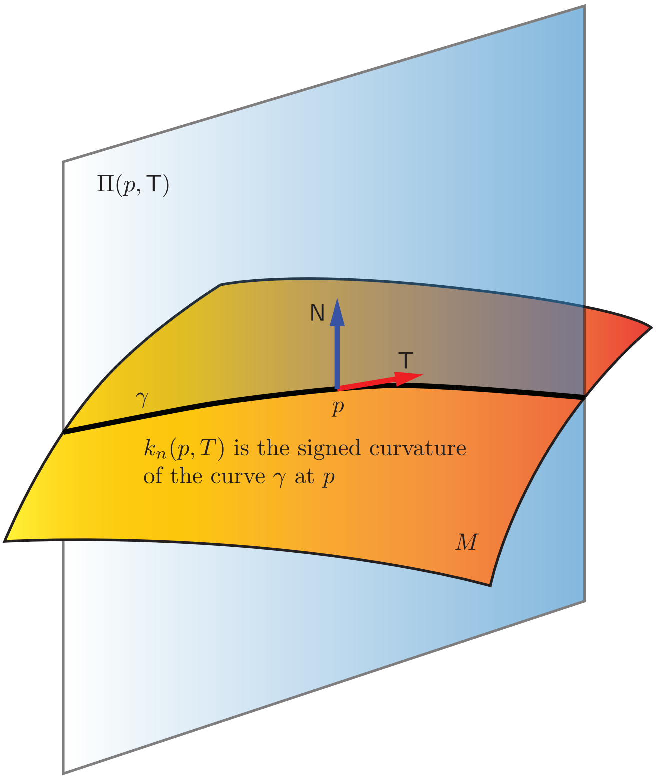


Figure 7.1. normal curvature at p in a given direction $\hat{e}$

We will first make sense of normal curvatures through elementary examples, then we will prove a general formula for computing normal curvatures.

Example 7.14. Let P be any plane in $\mathbb{R}^3$. For any point $p \in P$ and unit tangent $\hat{e} \in T_pP$, the plane $\Pi_p(\nu, \hat{e})$ must cut through P along a straight-line γ. Since γ has curvature 0, we have:

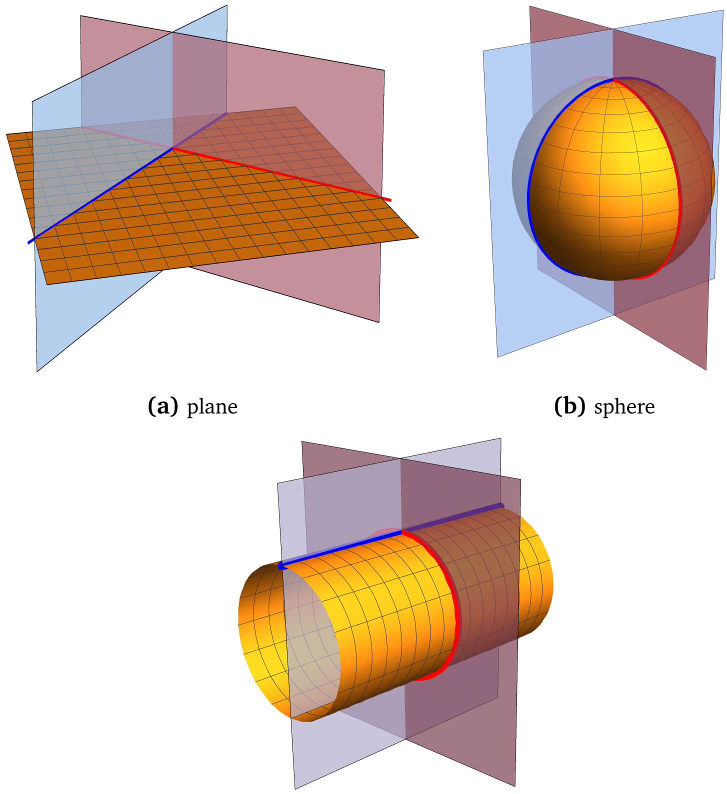
$$k_n(p, \hat{e}) = 0$$

for any $p \in P$ and $\hat{e} \in T_pP$. See Figure 7.2a. □

Example 7.15. Let $\mathbb{S}^2(R)$ be the sphere with radius R centered at the origin in $\mathbb{R}^3$ with Gauss map ν taken to be inward-pointing. For any point $p \in \mathbb{S}^2$ and $\hat{e} \in T_p\mathbb{S}^2(R)$, the plane $\Pi_p(\nu, \hat{e})$ cuts $\mathbb{S}^2(R)$ along a great circle (with radius R). Since a circle with radius R has constant curvature $\frac{1}{R}$, we have:

$$k_n(p, \hat{e}) = \frac{1}{R}$$

for any $p \in \mathbb{S}^2(R)$ and $\hat{e} \in T_p\mathbb{S}^2(R)$. See Figure 7.2b. □



(a) plane **(b)** sphere

(c) cylinder

Figure 7.2. Normal curvatures of various surfaces

Example 7.16. Let M be the (infinite) cylinder of radius R with x-axis as the central axis with outward-pointing Gauss map ν. Given any $p \in M$, if $\hat{e}_x$ is the unit tangent vector at p parallel to the x-axis, then the $\Pi_p(\nu, \hat{e}_x)$ cuts the cylinder M along a straight-line. Therefore, we have:

$$k_n(p, \hat{e}_x) = 0$$

for any $p \in M$. See the blue curve in Figure 7.2c.

On the other hand, if $\hat{e}_{yz}$ is a horizontal unit tangent vector at p, then $\Pi_p(\nu, \hat{e}_{yz})$ cuts M along a circle with radius R. Therefore, we have:

$$k_n(p, \hat{e}_{yz}) = -\frac{1}{R}$$

for any $p \in M$. See the red curve in Figure 7.2c. Note that the tangent vector of the curve is moving away from the outward-pointing ν. It explains the negative sign above.

For any other choice of unit tangent $\hat{e}$ at p, the plane $\Pi_p(\nu, \hat{e})$ cuts the cylinder along an ellipse, so the normal curvature along $\hat{e}$ may vary between 0 and $-\frac{1}{R}$. □

In the above examples, the normal curvatures $k_n(p, \hat{e})$ are easy to find since the curve of intersection between $\Pi_p(\nu, \hat{e})$ and the surface is either a straight line or a circle. Generally speaking, the curve of intersection may be of arbitrary shape such as an ellipse, and sometimes it is not even easy to identify what curve it is. Fortunately, it is possible to compute $k_n(p, \hat{e})$ for any given unit tangent $\hat{e}$ in a systematic way. Next we derive an expression of $k_n(p, \hat{e})$, which will motivate the definition of the second fundamental form and Weingarten's map (also known as the shape operator).

Proposition 7.17. *Let Σ be the regular hypersurface in $\mathbb{R}^{n+1}$ with Gauss map ν. Fix $p \in \Sigma$ and a unit vector $\hat{e} \in T_p\Sigma$, then the normal curvature of Σ at p along $\hat{e}$ is given by:*

$$k_n(p, \hat{e}) = -\langle \hat{e}, D_{\hat{e}}\nu \rangle .$$

Furthermore, suppose $\hat{e}$ can be locally expressed as

$$\hat{e} = X^i \frac{\partial F}{\partial u_i},$$

then we have

$$k_n(p, \hat{e}) = -\left\langle \frac{\partial F}{\partial u_i}, \frac{\partial \nu}{\partial u_j} \right\rangle X^i X^j = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \nu \right\rangle X^i X^j.$$

Proof. Let γ be the intersection curve between the plane $\Pi_p(\nu, \hat{e})$ and Σ. We parametrize γ by arc-length $s \in (-\varepsilon, \varepsilon)$ such that $\gamma(0) = p$ and $|\gamma'(s)| \equiv 1$ on $(-\varepsilon, \varepsilon)$. Then, $\gamma'(s)$ is orthogonal to $\nu(\gamma(s))$, and so

$$\begin{aligned} k_n(p, \hat{e}) &= \langle \gamma''(s), \nu(\gamma(s)) \rangle_{s=0} \\ &= -\left\langle \gamma'(s), \frac{d}{ds}\nu(\gamma(s)) \right\rangle \Big|_{s=0} \\ &= -\langle \hat{e}, D_{\hat{e}}\nu \rangle. \end{aligned}$$

The last step follows from the fact that $\gamma'(0) = \hat{e}$.

The local expression follows immediately from the fact that

$$D_{X^i \frac{\partial F}{\partial u_i}} \nu = X^i D_{\frac{\partial F}{\partial u_i}} \nu = X^i \frac{\partial \nu}{\partial u_i}.$$

□

Proposition 7.17 shows that $k_n(p, \hat{e})$ can be written as a quadratic form on X^i's with coefficients given by $\left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \nu \right\rangle$. This motivates the second fundamental form:

Definition 7.18 (Second Fundamental Form)**.** Given a regular hypersurface in $\mathbb{R}^{n+1}$ with Gauss map ν. We define the *second fundamental form* of Σ at p to be the 2-tensor:

$$h(X, Y) := -\langle X, D_Y \nu \rangle.$$

Under a local coordinate system $(u_1, \cdots, u_n)$, we denote its component as

$$h_{ij} := h\left(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right) = -\left\langle \frac{\partial F}{\partial u_i}, \frac{\partial \nu}{\partial u_j} \right\rangle = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \nu \right\rangle.$$

Remark 7.19. As such, the normal curvature of Σ at p along $\hat{e}$ is given by

$$k_n(p, \hat{e}) = h(\hat{e}, \hat{e}).$$

Remark 7.20. $h(X, Y)$ is tensorial by the fact that $D_{fY}\nu = fD_Y\nu$ and $\langle fX, Y \rangle = f\langle X, Y \rangle$.

Example 7.21. Let Σ_f be the graph of a smooth function $f(u_1, u_2) : \mathcal{U} \to \mathbb{R}$ defined on an open subset $\mathcal{U}$ of $\mathbb{R}^2$, then Σ_f has a globally defined smooth parametrization:

$$F(u_1, u_2) = (u_1,\, u_2,\, f(u_1, u_2)).$$

By straight-forward computations, we can get:

$$\frac{\partial F}{\partial u_1} = \left(1,\, 0,\, \frac{\partial f}{\partial u_1}\right) \qquad \frac{\partial F}{\partial u_2} = \left(0,\, 1,\, \frac{\partial f}{\partial u_2}\right)$$
$$\frac{\partial^2 F}{\partial u_1^2} = \left(0,\, 0,\, \frac{\partial^2 f}{\partial u_1^2}\right) \qquad \frac{\partial^2 F}{\partial u_1 \partial u_2} = \left(0,\, 0,\, \frac{\partial^2 f}{\partial u_1 \partial u_2}\right)$$
$$\frac{\partial^2 F}{\partial u_2 \partial u_1} = \left(0,\, 0,\, \frac{\partial^2 f}{\partial u_2 \partial u_1}\right) \qquad \frac{\partial^2 F}{\partial u_2^2} = \left(0,\, 0,\, \frac{\partial^2 f}{\partial u_2^2}\right)$$

In short, we have

$$\frac{\partial^2 F}{\partial u_i \partial u_j} = \left(0,\, 0,\, \frac{\partial^2 f}{\partial u_i \partial u_j}\right).$$

Let's take the Gauss map ν to be:

$$\nu = \frac{\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}}{\left|\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right|} = \frac{\left(-\frac{\partial f}{\partial u_1},\, \frac{\partial f}{\partial u_2},\, 1\right)}{\sqrt{1 + \left(\frac{\partial f}{\partial u_1}\right)^2 + \left(\frac{\partial f}{\partial u_2}\right)^2}}.$$

Then, the second fundamental form is given by:

$$h_{ij} = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \nu \right\rangle = \frac{\frac{\partial^2 f}{\partial u_i \partial u_j}}{\sqrt{1 + |\nabla f|^2}}.$$

The matrix whose (i, j)-th entry given by $\frac{\partial^2 f}{\partial u_i \partial u_j}$ is commonly called the *Hessian* of f, denoted by $\nabla\nabla f$ or $\text{Hess}(f)$. Using this notation, the matrix of second fundamental form of Σ_f is given by:

$$[h] = \frac{\text{Hess}(f)}{\sqrt{1 + |\nabla f|^2}}.$$

□

Exercise 7.7. Generalize Example 7.21 to higher dimensional graph in $\mathbb{R}^{n+1}$:

$$x_{n+1} = f(x_1, \cdots, x_n).$$

Given that $k_n(p, \hat{e})$ depends on the unit direction $\hat{e} \in T_p\Sigma$, it is then natural to ask when $k_n(p, \hat{e})$ achieves the maximum and minimum among all unit vectors $\hat{e}$ in $T_p\Sigma$. It is a simple optimization problem of critical points of the $h(\hat{e}, \hat{e})$ (as a function of $\hat{e}$) subject to the condition $g(\hat{e}, \hat{e}) = 1$. We call the critical values of $h(\hat{e}, \hat{e})$ subject to $g(\hat{e}, \hat{e}) = 1$ to be *principal curvatures* and we have the following important result:

Proposition 7.22. *Given a regular hypersurface Σ in $\mathbb{R}^{n+1}$ and fix a point $p \in \Sigma$, the principal curvatures are eigenvalues of the linear map:*

$$\begin{aligned} S_p : T_p\Sigma &\to T_p\Sigma \\ \frac{\partial F}{\partial u_i} &\mapsto \left(g^{jk} h_{ki}\right) \frac{\partial F}{\partial u_j}. \end{aligned}$$

The map S_p is called the Weingarten's map, or the shape operator.

Proof. Denote $\hat{e} = X^i \frac{\partial F}{\partial u_i}$, then $k_n(p, \hat{e}) = h(\hat{e}, \hat{e}) = h_{ij} X^i X^j$ and $|\hat{e}|^2 = g_{ij} X^i X^j$. To determine the principal curvatures, we use Lagrange's multiplier to find the critical values of $h_{ij} X^i X^j$ under the constraint $g_{ij} X^i X^j = 1$. Here we treat $(X^1, \cdots, X^n)$ as the variables, while g_{ij} and h_{ij} are regarded as constants since they do not depend on $\hat{e}$.

We need to solve the system:

$$\begin{aligned} \frac{\partial}{\partial X^k}\big(h_{ij}X^iX^j\big) &= \lambda\frac{\partial}{\partial X^k}\big(g_{ij}X^iX^j\big) \qquad\qquad k=1,2,\cdots,n \\ g_{ij}X^iX^j &= 1 \end{aligned}$$

Using $\frac{\partial X^i}{\partial X^k}=\delta_{ik}$, one can easily obtain

$$\begin{aligned} h_{ij}\big(\delta_{ik}X^j+X^i\delta_{jk}\big) &= \lambda g_{ij}\big(\delta_{ik}X^j+\delta_{jk}X^i\big) \\ \Longrightarrow \quad h_{kj}X^j &= \lambda g_{kj}X^j \end{aligned} \tag{7.3}$$

for any $k=1,2,\cdots,n$. Multiplying g^{ik} (which is the (i,k)-entry of g^{-1}) on both sides, we then get:

$$g^{ik}h_{kj}X^j=\lambda g^{ik}g_{kj}X^j \quad\Longrightarrow\quad g^{ik}h_{kj}X^j=\lambda X^i$$

for any $i=1,2,\cdots,n$. In other words, we have

$$[g]^{-1}[h](X^1,\cdots,X^n)^T=\lambda(X^1,\cdots,X^n)^T,$$

and so λ is the eigenvalue of the matrix $[g]^{-1}[h]$. From (7.3), we have:

$$\underbrace{h_{kj}X^jX^k}_{k_n(p,\hat{e})}=\lambda\underbrace{g_{kj}X^jX^k}_{=1}=\lambda.$$

Therefore, if $k_n(p,\hat{e})$ achieves its maximum and minimum among all $\hat{e}\in T_p\Sigma$, then $k_n(p,\hat{e})$ is an eigenvalue of $[g]^{-1}[h]$, which is the matrix representation of S with respect to local coordinates $(u_1,\cdots,u_n)$ as desired □

From now on we denote

$$h_i^j := g^{jk}h_{ki} \quad\Longrightarrow\quad S\left(\frac{\partial F}{\partial u_i}\right)=h_i^j\frac{\partial F}{\partial u_j}.$$

Remark 7.23. It is also interesting to note that locally

$$\frac{\partial \nu}{\partial u_i}=-g^{jk}h_{ki}\frac{\partial F}{\partial u_j}=-S\left(\frac{\partial F}{\partial u_j}\right). \tag{7.4}$$

To show this, we let

$$\frac{\partial \nu}{\partial u_i}=A_i^j\frac{\partial F}{\partial u_j}.$$

Then we consider

$$h_{ik}=-\left\langle \frac{\partial F}{\partial u_i},\frac{\partial \nu}{\partial u_k}\right\rangle=-\left\langle \frac{\partial F}{\partial u_i},A_k^j\frac{\partial F}{\partial u_j}\right\rangle=-g_{ij}A_k^j.$$

Taking g^{li} on both sides, we get $A_k^l=-g^{li}h_{ik}=-h_k^l$ as desired. In other words, the shape operator can be regarded as the minus of the tangent map of ν.

Denote the eigenvalues of S_p, i.e. the principal curvatures, by $\lambda_1(p),\cdots,\lambda_n(p)$. The maximum and minimum possible normal curvatures at p among all unit directions are two of the principal curvatures. We further define:

Definition 7.24 (Mean Curvature and Gauss Curvature)**.**

$$H(p):=\lambda_1(p)+\cdots+\lambda_n(p)=\mathrm{tr}[g]^{-1}[h] \qquad \text{(mean curvature)}$$

$$K(p):=\lambda_1(p)\cdots\lambda_n(p)=\det[g]^{-1}[h]=\frac{\det[h]}{\det[g]} \qquad \text{(Gauss curvature)}$$

It turns out that when $\dim\Sigma=2$, the Gauss curvature K depends only on the first fundamental form even though it is defined using the second fundamental form as well. It is a famous theorem by Gauss commonly known as *Theorema Egregium*.

Exercise 7.8 (Rigid-Body Motion). A map $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is said to be a *rigid-body motion* if there exist an $(n+1) \times (n+1)$ orthogonal matrix A and a constant vector $p \in \mathbb{R}^{n+1}$ such that $\Phi(x) = Ax + p$ for all $x \in \mathbb{R}^{n+1}$. Consider a regular Euclidean hypersurface Σ in $\mathbb{R}^{n+1}$, and its image $\widetilde{\Sigma} := \Phi(\Sigma)$. Show that first and second fundamental forms (and hence all curvatures we have discussed) of Σ and $\widetilde{\Sigma}$ are the same up to a sign, i.e.

$$\Phi^*\widetilde{g} = g \quad \text{and} \quad \Phi^*\widetilde{h} = h.$$

Here $\widetilde{g}$ and $\widetilde{h}$ are first and second fundamental forms of $\widetilde{\Sigma}$ respectively.

7.2.3. Curvatures of Graphs. This subsection assumes Σ is a two dimensional regular surface in $\mathbb{R}^3$. In Examples 7.9 and 7.21, we computed the first and second fundamental forms of the graph Σ_f of a function f. Using these, it is not difficult to compute various curvatures of the graph. In this subsection, we are going to discuss the geometric meaning of each curvature in this context, especially at the point where the tangent plane is horizontal.

Proposition 7.25. *Let Σ_f be the graph of a two-variable function $f(u_1, u_2) : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$. Suppose p is a point on Σ_f such that the tangent plane $T_p\Sigma_f$ is horizontal, i.e. p is a critical point of f, and the Gauss map ν is taken to be upward-pointing, then*

- $K(p) > 0$ *and* $H(p) > 0 \implies p$ *is a local minimum of* f
- $K(p) > 0$ *and* $H(p) < 0 \implies p$ *is a local maximum of* f
- $K(p) < 0 \implies p$ *is a saddle of* f

Proof. At a critical p of f, we have $\nabla f(p) = 0$. From Examples 7.9 and 7.21, we have computed:

$$g_{ij}(p) = \delta_{ij} + \underbrace{\frac{\partial f}{\partial u_i}(p)\frac{\partial f}{\partial u_j}(p)}_{=0} = \delta_{ij}$$

$$h_{ij}(p) = \frac{\frac{\partial^2 f}{\partial u_i \partial u_j}(p)}{\sqrt{1 + |\nabla f(p)|^2}} = \frac{\partial^2 f}{\partial u_i \partial u_j}(p)$$

Note that the Gauss map ν was taken to be upward-pointing in Example 7.21, as required in this proposition.

Therefore, we have:

$$K(p) = \frac{\det[h]}{\det[g]}(p) = \det\left[\frac{\partial^2 f}{\partial u_i \partial u_j}(p)\right] = \left(f_{11}f_{22} - f_{12}^2\right)(p)$$

$$H(p) = \frac{1}{2}g^{ij}(p)h_{ij}(p) = \frac{1}{2}\delta^{ij}\frac{\partial^2 f}{\partial u_i \partial u_j}(p) = \frac{1}{2}\left(f_{11} + f_{22}\right)(p)$$

From the second derivative test in multivariable calculus, given a critical point p, if $f_{11}f_{22} - f_{12}^2 > 0$ and $f_{11} + f_{22} > 0$ at p, then p is a local minimum of f. The other cases can be proved similarly using the second derivative test. $\square$

Given any regular surface Σ (not necessarily the graph of a function) and any point $p \in \Sigma$, one can apply a rigid-motion motion $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ so that $T_p\Sigma$ is transformed into a horizontal plane. Then, the new surface $\Phi(\Sigma)$ becomes locally a graph of a function f near the point $\Phi(p)$. Recall that the Gauss curvatures of p and $\Phi(p)$ are the same as given by Exercise 7.8. If $K(p) > 0$ (and hence $K(\Phi(p)) > 0$), then Proposition 7.25 asserts that

$\Phi(p)$ is a local maximum or minimum of the function f and so the surface $\Phi(\Sigma)$ is locally above or below the tangent plane at $\Phi(p)$. In other words, near p the surface Σ is locally on one side of the the tangent plane $T_p\Sigma$. On the other hand, if $K(p) < 0$ then no matter how close to p the surface Σ would intersect $T_p\Sigma$ at points other than p.

7.2.4. Surfaces of Revolution. Surfaces of revolution are surfaces obtained by revolving a plane curve about a central axis. They are important class of surfaces, examples of which include spheres, torus, and many others. In this subsection, we will study the fundamental forms and curvatures of these surfaces.

For simplicity, we assume that the z-axis is the central axis. A surface of revolution (about the z-axis) is defined as follows.

Definition 7.26 (Surfaces of Revolution)**.** Consider the curve $\gamma(t) = (x(t),\, 0,\, z(t))$, where $t \in (a, b)$, on the xz-plane such that $x(t) \geq 0$ for any $t \in (a, b)$. The surface of revolution generated by γ is obtained by revolving γ about the z-axis, and it can be parametrized by:

$$F(t, \theta) = (x(t)\cos\theta,\, x(t)\sin\theta,\, z(t)), \quad (t, \theta) \in (a, b) \times [0, 2\pi].$$

It is a straight-forward computation to verify that:

$$\frac{\partial F}{\partial t} = (x'(t)\cos\theta,\, x'(t)\sin\theta,\, z'(t)) \tag{7.5}$$

$$\frac{\partial F}{\partial \theta} = (-x(t)\sin\theta,\, x(t)\cos\theta,\, 0) \tag{7.6}$$

$$\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta} = (-x(t)\, z'(t)\cos\theta,\; -x(t)\, z'(t)\sin\theta,\; x(t)\, x'(t)) \tag{7.7}$$

Exercise 7.9. Verify (7.5)-(7.7) and show that:

$$\left|\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta}\right| = |x(t)\, \gamma'(t)|\,.$$

Under what condition(s) will F be a smooth local parametrization when (t, θ) is restricted to $(a, b) \times (0, 2\pi)$?

Under the condition on $\gamma(t) = (x(t),\, 0,\, z(t))$ that its surface of revolution is smooth, one can easily compute that the first fundamental form is given by:

$$[g] = \begin{bmatrix} (x')^2 + (z')^2 & 0 \\ 0 & x^2 \end{bmatrix} \qquad \text{(matrix notation)} \tag{7.8}$$

$$g = \left[(x')^2 + (z')^2\right] dt^2 + x^2 d\theta^2 \qquad \text{(tensor notation)}$$

and the second fundamental form with respect to the Gauss map $\nu := \frac{\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta}}{\left|\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta}\right|}$ is given by:

$$[h] = \frac{1}{\sqrt{(x')^2 + (z')^2}} \begin{bmatrix} x'\, z'' - x''\, z' & 0 \\ 0 & x\, z' \end{bmatrix} \qquad \text{(matrix notation)} \tag{7.9}$$

$$h = \frac{1}{\sqrt{(x')^2 + (z')^2}} \left[(x'\, z'' - x''\, z')\, dt^2 + x\, z' d\theta^2\right] \qquad \text{(tensor notation)}$$

Exercise 7.10. Verify that the first and second fundamental forms of a surface of revolution with parametrization

$$F(t,\theta) = (x(t)\cos\theta,\ x(t)\sin\theta,\ z(t)), \quad (t,\theta) \in (a,b)\times[0,2\pi]$$

are given as in (7.8) and (7.9).

As both $[g]$ and $[h]$ are diagonal matrices, it is evident that the principal curvatures, i.e. the eigenvalues of $[g]^{-1}[h]$, are:

$$\begin{aligned} k_1 &= \frac{x'z'' - x''z'}{\left[(x')^2 + (z')^2\right]^{3/2}} = \frac{x'z'' - x''z'}{|\gamma'|^3} \\ k_2 &= \frac{z'}{x\sqrt{(x')^2 + (z')^2}} = \frac{z'}{x\,|\gamma'|} \end{aligned}$$

Note that here we are not using the convention that $k_1 \leq k_2$ as in before, since there is no clear way to tell which eigenvalue is larger.

Therefore, the Gauss and mean curvatures are given by:

$$K = k_1 k_2 = \frac{(x'z'' - x''z')\,z'}{x\,|\gamma'|^4} \tag{7.10}$$

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\left(\frac{x'z'' - x''z'}{|\gamma'|^3} + \frac{z'}{x\,|\gamma'|}\right) \tag{7.11}$$

Example 7.27. Let's verify that the round sphere (of any radius) has indeed constant Gauss and mean curvatures. Parametrize the sphere by:

$$F(t,\theta) = (R\sin t\cos\theta, R\sin t\sin\theta, R\cos t),$$

i.e. taking $x(t) = R\sin t$ and $z(t) = R\cos t$. By (7.10), then the Gauss curvature is clearly given by:

$$K = \frac{\left[(R\cos t)(-R\cos t) - (-R\sin t)(-R\sin t)\right](-R\sin t)}{(R\sin t)\cdot R^4} = \frac{1}{R^2}.$$

By (7.11), the mean curvature is given by:

$$H = \frac{1}{2}\left(\frac{-R^2}{R^3} + \frac{-R\sin t}{(R\sin t)\cdot R}\right) = -\frac{1}{R}.$$

□

Exercise 7.11. Compute the Gauss and mean curvatures of a round torus using (7.10) and (7.11).

7.3. Theorema Egregium

The goal of this section is to give the proof of a celebrated theorem due to Gauss, known in Latin as *Theorema Egregium* (a surprising/remarkable theorem). The theorem asserts that although the Gauss curvature of a two-dimensional regular surface in $\mathbb{R}^3$ was defined using both the first and second fundamental forms, it indeed depends only on the first fundamental form g.

It is remarkable in a sense that to define a notion of curvature, we no longer require the surface to have an ambient Euclidean space. So long as one can *declare* an appropriate 2-tensor g to act in lieu as the "first fundamental form", then one can still make sense of curvatures. Einstein's theory of general relativity relies very much on Riemannian geometry because it regards the Universe as an intrinsic 4-dimensional manifold without the ambient space (there is nothing *outside* the Universe). Riemannian geometry is a good fit mathematical language to formulate the theory of general relativity in a rigorous way.

To prove the Theorema Egregium, we first need to introduce covariant derivatives, which depend only on the first fundamental form. Then, the theorem can be proved by showing $\det[h]$ depends only on the covariant derivatives (and hence only on the first fundamental form).

7.3.1. Covariant Derivatives. Let's first recall *directional derivatives* in multivariable calculus. Let Σ be a regular hypersurface, and $\gamma(t) : (a, b) \to \Sigma$ be a smooth curve on Σ. Given a vector field X on Σ, the directional derivative of X at p along γ is given by

$$D_{\gamma'}X(p) := \left.\frac{d}{dt}\right|_{t=t_0} X(\gamma(t))$$

where t_0 is a time such that $\gamma(t_0) = p$.

When $\gamma(t)$ is a u_i-coordinate curve and X is any vector field, then

$$D_{\gamma'}X = \frac{\partial X}{\partial u_i}.$$

In particular, if $X = \frac{\partial F}{\partial u_j}$, then we have:

$$D_{\gamma'}X = \frac{\partial}{\partial u_i}\frac{\partial F}{\partial u_j} = \frac{\partial^2 F}{\partial u_i \partial u_j}.$$

In general, suppose $X = X^i \frac{\partial F}{\partial u_i}$ and given any curve $\gamma(t)$ on Σ locally expressed as $(u_1(t), \cdots, u_n(t))$, then by the chain rule we have:

$$\begin{aligned} D_{\gamma'}X &= \frac{d}{dt}X(\gamma(t)) = \frac{\partial X}{\partial u_i}\frac{du_i}{dt} \\ &= \frac{\partial}{\partial u_i}\left(X^j \frac{\partial F}{\partial u_j}\right)\frac{du_i}{dt} \\ &= \left(\frac{\partial X^j}{\partial u_i}\frac{\partial F}{\partial u_j} + X^j \frac{\partial^2 F}{\partial u_i \partial u_j}\right)\frac{du_i}{dt} \end{aligned}$$

Note that under a fixed local coordinate system $(u_1, \cdots, u_n)$, the quantities $\frac{\partial X^j}{\partial u_i}\frac{\partial F}{\partial u_j} + X^j \frac{\partial^2 F}{\partial u_i \partial u_j}$ are uniquely determined by the vector field X whereas $\frac{du_i}{dt}$ are uniquely determined by the tangent vector γ' of the curve. Now given another vector field $Y = Y^i \frac{\partial F}{\partial u_i}$, then one can define

$$D_Y X(p) := D_{\gamma'}X(p)$$

where γ is any curve on Σ which solves the ODE $\gamma'(t) = Y(\gamma(t))$ and $\gamma(0) = p$. The Existence Theorem of ODEs guarantees there is such a curve γ that flows along Y. Locally, $D_Y X$ can be expressed as:

$$D_Y X = \left(\frac{\partial X^j}{\partial u_i} \frac{\partial F}{\partial u_j} + X^j \frac{\partial^2 F}{\partial u_i \partial u_j} \right) Y^i. \tag{7.12}$$

In short, *covariant derivatives* on hypersurfaces are projection of directional derivatives onto the tangent space. They play an important role in differential geometry as it can be shown to be depending only on the first fundamental form, in contrast to directional derivatives which sit in the ambient space $\mathbb{R}^{n+1}$. Therefore, we are able to generalize the notion of covariant derivatives to Riemannian manifolds.

Definition 7.28 (Covariant Derivatives). Let Σ be a regular surface with Gauss map ν, and $\gamma(t)$ be a smooth curve on Σ. Given two vector fields X and Y on Σ, we define the *covariant derivative of X at $p \in M$ along Y* to be

$$\nabla_Y X(p) := (D_Y X(p))^T = \big(D_Y X - \langle D_Y X, \nu \rangle \, \nu\big)(p).$$

Here $(D_Y X(p))^T$ represents the projection of $D_Y X(p)$ onto the tangent space $T_p\Sigma$.

Using the fact that $\langle D_Y X, \nu\rangle = -\langle X, D_Y \nu\rangle = h(X,Y) = h_{ij} X^i Y^j$ and (7.12), we can derive the local expression for $\nabla_Y X$:

$$\nabla_Y X = \underbrace{\sum_{i,j} \left(\frac{\partial X^j}{\partial u_i} \frac{\partial F}{\partial u_j} + X^j \frac{\partial^2 F}{\partial u_i \partial u_j} \right) Y^i}_{D_Y X} - \underbrace{\left(\sum_{i,j} h_{ij} X^i Y^j \right) \nu}_{\langle D_Y X, \nu\rangle \nu} \tag{7.13}$$

Suppose X, $\widetilde{X}$, Y and $\widetilde{Y}$ are vector fields and φ is a smooth scalar functions. One can verify that the following properties hold:

(a) $\nabla_{\varphi Y} X = \varphi \nabla_Y X$

(b) $\nabla_Y (\varphi X) = (\nabla_Y \varphi) X + \varphi \nabla_Y X$

(c) $\nabla_{Y+\widetilde{Y}} X = \nabla_Y X + \nabla_{\widetilde{Y}} X$

(d) $\nabla_Y (X + \widetilde{X}) = \nabla_Y X + \nabla_Y \widetilde{X}$

According to (7.12) and (7.13), given any vector fields $X = X^i \frac{\partial F}{\partial u_i}$ and $Y = Y^i \frac{\partial F}{\partial u_i}$, the second derivatives $\frac{\partial^2 F}{\partial u_i \partial u_j}$ determine both $D_Y X$ and $\nabla_Y X$. We are going to express $\frac{\partial^2 F}{\partial u_i \partial u_j}$ in terms of this tangent-normal basis $\left\{ \frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_n}, \nu \right\}$ of $\mathbb{R}^{n+1}$. From (7.13), we have:

$$\frac{\partial^2 F}{\partial u_i \partial u_j} = D_{\frac{\partial F}{\partial u_j}} \frac{\partial F}{\partial u_i} = \nabla_{\frac{\partial F}{\partial u_j}} \frac{\partial F}{\partial u_i} + h_{ij} \nu.$$

We denote the coefficients of the tangent vector $\nabla_{\frac{\partial F}{\partial u_j}} \frac{\partial F}{\partial u_i}$ by the following *Christoffel symbols*:

$$\nabla_{\frac{\partial F}{\partial u_j}} \left(\frac{\partial F}{\partial u_i} \right) = \Gamma^k_{ij} \frac{\partial F}{\partial u_k}. \tag{7.14}$$

The Christoffel symbols can be shown to be depending only on the first fundamental form g. We will use it to prove that Gauss curvature depends also only on g but not on h.

Proposition 7.29. *In a local coordinate system $(u_1, \cdots, u_n)$, the Christoffel symbols Γ_{ij}^k's can be locally expressed in terms of the first fundamental form as:*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \right). \tag{7.15}$$

Proof. First recall that $g_{ij} = \left\langle \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j} \right\rangle$. By differentiating both sides respect to u_l, we get:

$$\frac{\partial g_{ij}}{\partial u_l} = \left\langle \frac{\partial^2 F}{\partial u_l \partial u_i}, \frac{\partial F}{\partial u_j} \right\rangle + \left\langle \frac{\partial^2 F}{\partial u_l \partial u_j}, \frac{\partial F}{\partial u_i} \right\rangle.$$

Using (7.14), we get:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial u_l} &= \left\langle \underbrace{\Gamma_{li}^k \frac{\partial F}{\partial u_k} + h_{li}\nu}_{\frac{\partial^2 F}{\partial u_l \partial u_i}}, \frac{\partial F}{\partial u_j} \right\rangle + \left\langle \underbrace{\Gamma_{lj}^k \frac{\partial F}{\partial u_k} + h_{lj}\nu}_{\frac{\partial^2 F}{\partial u_l \partial u_j}}, \frac{\partial F}{\partial u_i} \right\rangle \\ &= \Gamma_{il}^k g_{kj} + \Gamma_{lj}^k g_{ki} \end{aligned} \tag{*}$$

By cyclic permutation of indices $\{i, j, l\}$, we also get:

$$\frac{\partial g_{il}}{\partial u_j} = \left(\Gamma_{ij}^k g_{kl} + \Gamma_{jl}^k g_{ki} \right) \tag{**}$$

$$\frac{\partial g_{jl}}{\partial u_i} = \left(\Gamma_{ji}^k g_{kl} + \Gamma_{il}^k g_{kj} \right) \tag{***}$$

Recall that $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $h_{ij} = h_{ji}$ for any i, j and k. By considering (**)+(***)-(*), we get:

$$\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l} = 2\Gamma_{ij}^k g_{kl} \tag{7.16}$$

Finally, by multiplying g^{lq} on both sides of (7.16) and summing up over all l, we get:

$$\begin{aligned} g^{lq} \left(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l} \right) &= 2g^{lq}\Gamma_{ij}^k g_{kl} \\ &= 2\Gamma_{ij}^k \delta_k^q \\ &= 2\Gamma_{ij}^q. \end{aligned}$$

Relabelling the index q by k, we complete the proof of (7.15). □

7.3.2. The Proof of Theorema Egregium. The key ingredient of Gauss's Theorema Egregium is the following Gauss-Codazzi's equations, which hold for any regular hypersurface in any dimension. When the dimension is two, the RHS of the Gauss's equation is similar to $\det[h]$ while the LHS depends only g. This gives Theorema Egregium as a direct consequence of the Gauss's equation.

Theorem 7.30 (Gauss-Codazzi's Equations). *On any regular hypersurface Σ in $\mathbb{R}^{n+1}$, the following equations hold:*

$$\frac{\partial \Gamma_{jk}^q}{\partial u_i} - \frac{\partial \Gamma_{ik}^q}{\partial u_j} + \Gamma_{jk}^l \Gamma_{il}^q - \Gamma_{ik}^l \Gamma_{jl}^q = g^{ql} \left(h_{jk} h_{li} - h_{ik} h_{lj} \right) \tag{Gauss}$$

$$\frac{\partial h_{jk}}{\partial u_i} - \frac{\partial h_{ik}}{\partial u_j} + \Gamma_{jk}^l h_{il} - \Gamma_{ik}^l h_{jl} = 0 \tag{Codazzi}$$

Proof. The key step of the proof is to start with the fact that:

$$\frac{\partial^3 F}{\partial u_i \partial u_j \partial u_k} = \frac{\partial^3 F}{\partial u_j \partial u_i \partial u_k}$$

for any i, j, k, and then rewrite both sides in terms of the tangent-normal basis of $\mathbb{R}^{n+1}$. Gauss's equation follows from equating the tangent coefficients, and Codazzi's equation is obtained by equating the normal coefficient.

By (7.14), we have:

$$\frac{\partial^2 F}{\partial u_j \partial u_k} = \Gamma^l_{jk} \frac{\partial F}{\partial u_l} + h_{jk}\nu.$$

Differentiating both sides with respect to u_i, we get:

$$\begin{aligned}
\frac{\partial^3 F}{\partial u_i \partial u_j \partial u_k} &= \frac{\partial}{\partial u_i}\left(\Gamma^l_{jk}\frac{\partial F}{\partial u_l} + h_{jk}\nu\right) \\
&= \frac{\partial \Gamma^l_{jk}}{\partial u_i}\frac{\partial F}{\partial u_l} + \Gamma^l_{jk}\frac{\partial^2 F}{\partial u_i \partial u_l} + \frac{\partial h_{jk}}{\partial u_i}\nu + h_{jk}\frac{\partial \nu}{\partial u_i} \\
&= \frac{\partial \Gamma^l_{jk}}{\partial u_i}\frac{\partial F}{\partial u_l} + \Gamma^l_{jk}\left(\Gamma^q_{il}\frac{\partial F}{\partial u_q} + h_{il}\nu\right) + \frac{\partial h_{jk}}{\partial u_i}\nu - h_{jk}h^q_i\frac{\partial F}{\partial u_q} \\
&= \underbrace{\frac{\partial \Gamma^q_{jk}}{\partial u_i}\frac{\partial F}{\partial u_q}}_{l \mapsto q} + \Gamma^l_{jk}\Gamma^q_{il}\frac{\partial F}{\partial u_q} + \left(\frac{\partial h_{jk}}{\partial u_i} + \Gamma^l_{jk}h_{il}\right)\nu - h_{jk}h^{ql}g_{il}\frac{\partial F}{\partial u_q} \\
&= \left(\frac{\partial \Gamma^q_{jk}}{\partial u_i} + \Gamma^l_{jk}\Gamma^q_{il} - g^{ql}h_{jk}h_{il}\right)\frac{\partial F}{\partial u_q} + \left(\frac{\partial h_{jk}}{\partial u_i} + \Gamma^l_{jk}h_{il}\right)\nu
\end{aligned}$$

By switching i and j, we get:

$$\frac{\partial^3 F}{\partial u_j \partial u_i \partial u_k} = \left(\frac{\partial \Gamma^q_{ik}}{\partial u_j} + \Gamma^l_{ik}\Gamma^q_{jl} - g^{ql}h_{ik}h_{lj}\right)\frac{\partial F}{\partial u_q} + \left(\frac{\partial h_{ik}}{\partial u_j} + \Gamma^l_{ik}h_{jl}\right)\nu$$

The Gauss-Codazzi's equations can be obtained by equating the coefficients of each tangent and normal component. □

We derived the Gauss-Codazzi's equations (Theorem 7.30). It is worthwhile the note that the LHS of the Gauss's equation involves only Christoffel's symbols and their derivatives:

$$\underbrace{\frac{\partial \Gamma^q_{jk}}{\partial u_i} - \frac{\partial \Gamma^q_{ik}}{\partial u_j} + \Gamma^l_{jk}\Gamma^q_{il} - \Gamma^l_{ik}\Gamma^q_{jl}}_{\text{depends only on } \Gamma^k_{ij}\text{'s}} = g^{ql}\left(h_{jk}h_{li} - h_{ik}h_{lj}\right)$$

From (7.15) we also know that the Christoffel symbols depend only on the first fundamental form g but not on h. For simplicity, we denote:

$$R^q_{ijk} := \frac{\partial \Gamma^q_{jk}}{\partial u_i} - \frac{\partial \Gamma^q_{ik}}{\partial u_j} + \Gamma^l_{jk}\Gamma^q_{il} - \Gamma^l_{ik}\Gamma^q_{jl}.$$

The lower and upper indices for R^q_{ijk} are chosen so as to preserve their positions in the RHS expression (q being upper, and i, j, k being lower). We will see later that R^q_{ijk}'s are the components of the Riemann curvature tensor.

We are now in a position to give a proof of Gauss's Theorema Egregium as a direct consequence of the Gauss's equation.

Theorem 7.31 (Theorema Egregium, Gauss)**.** *On any regular surface Σ^2 in $\mathbb{R}^3$, the Gauss curvature K depends only on its first fundamental form g. In other words, K is intrinsic for regular surfaces.*

Proof. Consider the Gauss's equation, which asserts that for any i, j, k and q:

$$R^q_{ijk} = g^{ql}\left(h_{jk}h_{li} - h_{ik}h_{lj}\right).$$

Multiplying both sides by g_{pq}, and summing up all q's, we get:

$$g_{pq}R^q_{ijk} = g_{pq}g^{ql}\left(h_{jk}h_{li} - h_{ik}h_{lj}\right) = h_{jk}h_{pi} - h_{ik}h_{pj}.$$

The above result is true for any i, j, k and p. In particular, when $(i, j, k, p) = (1, 2, 2, 1)$, we get:

$$g_{1q}R^q_{122} = h_{22}h_{11} - h_{12}^2 = \det[h].$$

This shows $\det[h]$ depends only on g since R^q_{122} does so.

Finally, recall that the Gauss curvature is given by:

$$K = \frac{\det[h]}{\det[g]}.$$

Therefore, we have completed the proof that K depends only g. □

Remark 7.32. It is important to note that Theorem 7.31 holds for surfaces (i.e. $\dim \Sigma = 2$).

The long R^q_{ijk}-term can be interpreted in a nicer way using covariant derivatives. For simplicity, we denote

$$\begin{aligned}
\partial_i &:= \frac{\partial F}{\partial u_i} \\
\nabla_i &:= \nabla_{\partial_i} = \nabla_{\frac{\partial F}{\partial u_i}}
\end{aligned}$$

where $(u_1, \cdots, u_n)$ is a local coordinate system of regular hypersurface Σ in $\mathbb{R}^{n+1}$. By direct computations, we can verify that:

$$\begin{aligned}
\nabla_j \partial_k &= \Gamma^l_{jk}\partial_l \\
\nabla_i\left(\nabla_j \partial_k\right) &= \nabla_i\left(\Gamma^l_{jk}\partial_l\right) \\
&= \frac{\partial \Gamma^l_{jk}}{\partial u_i}\partial_l + \Gamma^l_{jk}\nabla_i\partial_l \\
&= \frac{\partial \Gamma^q_{jk}}{\partial u_i}\partial_q + \Gamma^l_{jk}\Gamma^q_{il}\partial_q
\end{aligned}$$

Similarly, we also have:

$$\nabla_j\left(\nabla_i\partial_k\right) = \frac{\partial \Gamma^q_{ik}}{\partial u_j}\partial_q + \Gamma^l_{ik}\Gamma^q_{jl}\partial_q.$$

Hence, the term R^q_{ijk} is the commutator of ∇_i and ∇_j:

$$\begin{aligned}
[\nabla_i, \nabla_j]\partial_k &= \nabla_i\left(\nabla_j\partial_k\right) - \nabla_j\left(\nabla_i\partial_k\right) \\
&= \left(\frac{\partial \Gamma^q_{jk}}{\partial u_i} - \frac{\partial \Gamma^q_{ik}}{\partial u_j} + \Gamma^l_{jk}\Gamma^q_{il} - \Gamma^l_{ik}\Gamma^q_{jl}\right)\partial_q \\
&= R^q_{ijk}\partial_q.
\end{aligned}$$

In summary, we see that once g_{ij} is fixed, the covariant derivative ∇ and hence the curvature term R^q_{ijk} are uniquely determined. It motivates the idea that if we can declare the g_{ij}'s on an abstract manifold M, then we can define its curvatures using the prescribed g_{ij} even if M is not a submanifold of an Euclidean space. This motivates the development of intrinsic geometry, a branch of geometry that plays no reference to the ambient space but only the manifold itself. This branch is called *Riemannian Geometry*, which is what this course is about!

Part 3

Riemannian Geometry

Chapter 8

Riemannian Manifolds

8.1. Riemannian Metrics

On a regular hypersurface in a Euclidean space, the first fundamental form g encodes many of its geometric properties such as length, area, and in two-dimensional case, the Gauss curvature. Now we want to extend all these geometric notions to abstract manifold which needs not have an ambient Euclidean space.

Without the ambient space, the "first fundamental form" g (which will be renamed as the *Riemannian metric*) is now not being *induced* from the dot product of the ambient space, but instead is being *defined*. It is analogous to the development of topological spaces from metric spaces. Open sets in a topological space are no longer characterized using metric balls, but are instead being *declared* as a collection of subsets which is called the *topology* of the space.

Definition 8.1 (Riemannian Metrics)**.** On a smooth manifold M, a *Riemannian metric* g is a C^∞ $(2,0)$-tensor on M such that

symmetry: $g(X,Y) = g(Y,X)$ for any $X, Y \in T_pM$;

positivity: $g(X,X) \geq 0$ for any $X \in T_pM$, with equality holds if and only if $X = 0$.

The pair (M,g) is called a *Riemannian manifold*.

We denote the local components of g by:

$$g_{ij} = g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right),$$

so that $g = g_{ij}\, du^i \otimes du^j$.

Example 8.2. The Euclidean space $\mathbb{R}^n$ is a Riemannian manifold with a Riemannian metric

$$\delta := \sum_{i=1}^n dx^i \otimes dx^i.$$

It is called the *flat metric* on $\mathbb{R}^n$. □

Example 8.3 (Hyperbolic Spaces - Poincaré Disc Model)**.** The n-dimensional hyperbolic space $\mathbb{H}^n$ (under the Poincaré model) is topologically an open unit ball in $\mathbb{R}^n$:

$$\mathbb{H}^n := \{x \in \mathbb{R}^n : |x| < 1\}$$

equipped with the Poincaré metric

$$g = \sum_{i=1}^{n} \frac{4\,dx^i \otimes dx^i}{\left(1 - |x|^2\right)^2}.$$

□

Example 8.4. Any regular hypersurface surface Σ^n in $\mathbb{R}^{n+1}$ is a Riemannian manifold with Riemannian metric g given by the first fundamental form. The symmetry and strict positivity conditions are inherited from the flat metric on $\mathbb{R}^{n+1}$.

For instance, the round sphere $\mathbb{S}^2$ with radius R has a Riemannian metric given by:

$$g = R^2\,d\varphi^2 + R^2 \sin^2\varphi\,d\theta^2.$$

□

Example 8.5. Given any smooth immersion $\Phi : \Sigma \to M$ between two C^∞ smooths, and suppose M is a Riemannian manifold with metric g. Then, Σ has an induced Riemannian metric given by $\widetilde{g} := \Phi^* g$. The symmetry condition holds trivially. To verify the strict positivity condition, we consider $X \in T_p\Sigma$, then

$$\widetilde{g}(X, X) = g(\Phi_* X, \Phi_* X) \geq 0,$$

and by strict positivity of g, we have equality holds if and only if $\Phi_* X = 0$. Since Φ_* is injective (as Φ is an immersion), we must have $X = 0$.

In particular, any submanifold of a Riemannian manifold is itself a Riemannian manifold with induced metric defined using the pull-back of the inclusion map. □

Example 8.6 (Product Manifolds)**.** Suppose (M, g_M) and (N, g_N) are two Riemannian manifolds, then the product $M \times N$ is also a Riemannian manifold with metric given by:

$$g_M \oplus g_N := \pi_M^* g_M + \pi_N^* g_N$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are projection maps. A tangent vector $X \in T_p(M \times N)$ can be expressed as $(X_M, X_N) \in T_pM \oplus T_pN$. The product metric acts on tangent vectors by:

$$(g_M \oplus g_N)\big((X_M, X_N), (Y_M, Y_N)\big) = g_M(X_M, Y_M) + g_N(X_N, Y_N).$$

□

Example 8.7 (Conformal Metrics)**.** Given any Riemannian manifold (M, g) and a smooth function $f : M \to \mathbb{R}$, one can define another Riemannian metric $\widetilde{g}$ on M by conformal rescaling:

$$\widetilde{g} = e^f g.$$

□

Example 8.8 (Fubini-Study Metric on $\mathbb{CP}^n$)**.** On $\mathbb{CP}^n$, there is an important Riemannian metric called the *Fubini-Study metric*, which is more convenient to be expressed using complex coordinates. Parametrize $\mathbb{CP}^n$ using standard local coordinates

$$F^{(j)}\left(z_0^{(j)}, \cdots, z_{j-1}^{(j)}, z_{j+1}^{(j)}, \cdots, z_n^{(j)}\right) = \left[z_0^{(j)} : \cdots : z_{j-1}^{(j)} : 1 : z_{j+1}^{(j)} : \cdots : z_n^{(j)}\right].$$

For simplicity, we let $z_j^{(j)} = 1$. The Fubini-Study metric is defined to be:

$$g_{\text{FS}} := 2\,\mathrm{Re}\left(\frac{\partial^2 \log \sum_{k=1}^{n+1} \left|z_k^{(j)}\right|^2}{\partial z_p^{(j)} \partial \bar{z}_q^{(j)}}\, dz_p^{(j)} \otimes d\bar{z}_q^{(j)}\right).$$

We leave it as an exercise for readers to show that g_{FS} is independent of j, and is a Riemannian metric. □

Given a Riemannian metric, one can then define the length of curve and volume of a region of a manifold. Suppose $\gamma(t) : [a,b] \to M$ is a C^1 curve on a Riemannian manifold (M,g), then we define the length of γ by (with respect to g) by:

$$L_g(\gamma) := \int_a^b \sqrt{g\big(\gamma'(t), \gamma'(t)\big)}\, dt$$

where $\gamma'(t) := \gamma_* \frac{\partial}{\partial t}$. Note that $L_g(\gamma)$ depends on the metric g.

Given an open subset $\mathcal{O}$ of Riemannian manifold (M,g) such that $\mathcal{O}$ can be covered by one single parametrization chart $(u_1, \cdots, u_n)$, the volume of $\mathcal{O}$ (with respect to g) is defined to be:

$$\mathrm{Vol}_g(\mathcal{O}) = \left| \int_{\mathcal{O}} \sqrt{\det[g_{ij}]}\, du^1 \wedge \cdots \wedge du^n \right|$$

where $g_{ij} = g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$. The total volume of an orientable Riemannian manifold (M,g) is defined by:

$$\mathrm{Vol}_g(M) := \left| \sum_\alpha \int_{\mathcal{O}_\alpha} \rho_\alpha \sqrt{\det[g_{ij}^\alpha]}\, du_\alpha^1 \wedge \cdots \wedge du_\alpha^n \right|$$

where $\mathcal{A} := \{F_\alpha(u_\alpha^i) : \mathcal{U}_\alpha \to \mathcal{O}_\alpha\}$ is an oriented atlas of M and ρ_α is a partition of unity subordinate to $\mathcal{A}$.

8.1.1. Isometries. Recall that two smooth manifolds M and N are considered to be the same in *topological* sense if there exists a diffeomorphism between them. Assume further that they are Riemannian manifolds with metrics g_M and g_N. Heuristically, we consider M and N to be *geometrically* the same if their Riemannian metrics are the "same". Precisely, take a diffeomorphism $\Phi : M \to N$, which sets up a one-one correspondence between $p \in M$ and $\Phi(p) \in N$, and a tangent vector $X \in T_pM$ corresponds to $\Phi_* X \in T_{\Phi(p)}N$. We consider g_M and g_N are "the same" if for any $X, Y \in T_pM$, we have $g_M(X,Y) = g_N(\Phi_*X, \Phi_*Y)$ meaning that inner product between X and Y is the same as that between their correspondences Φ_*X and Φ_*Y. In this case, we then call Φ is an *isometry*, and (M, g_M) and (N, g_N) are said to be *isometric*. Using the pull-back map, we can rephrase these definitions in the following way:

Definition 8.9 (Isometries)**.** Two Riemannian manifolds (M, g_M) and (N, g_N) are said to be *isometric* if there exists a diffeomorphism $\Phi : M \to N$ such that $\Phi^* g_N = g_M$. Such a diffeomorphism Φ is called an *isometry* between (M, g_M) and (N, g_N).

Example 8.10. Consider the unit disk model $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ of the hyperbolic 2-space with the Poincaré's metric introduced in Example 8.3:

$$g = \frac{4(dx^2 + dy^2)}{1 - x^2 - y^2}.$$

It is well-known that the hyperbolic plane can be equivalently modeled by the upper-half plane $U = \{(u,v) : v > 0\}$ with metric

$$\widetilde{g} = \frac{du^2 + dv^2}{v^2}.$$

These two models can be shown to be isometric. The isometry $\Phi : D \to U$ can be described using complex coordinates in an elegant way:

$$\Phi(z) = i\frac{1-z}{1+z}$$

where $z = x + yi$. $\square$

Exercise 8.1. Complete the calculations in Example 8.10 to show that Φ is indeed an isometry.

Without the ambient space, we now make sense of *symmetries* in an intrinsic way. A C^∞ vector field V on (M, g) generates a 1-parameter subgroup of diffeomorphisms $\Phi_t : M \to M$ such that

$$\frac{\partial \Phi_t}{\partial t}(p) = V(\Phi_t(p)), \quad \Phi_0 = \mathrm{id}_M.$$

We then say g is *symmetric along* V if $\Phi_t^* g_{\Phi_t(p)} = g_p$ for any $t \in \mathbb{R}$ and $p \in M$.

Generally, such a diffeomorphism family Φ_t is hard to compute. Fortunately, one can use the Lie derivative to check if g is symmetric along a given vector field without actually computing Φ_t. The condition $\Phi_t^* g_{\Phi_t(p)} = g_p$ for any $t \in \mathbb{R}$ and $p \in M$ is equivalent to

$$\mathcal{L}_V g = 0.$$

8.1.2. Covering Spaces and Quotient Manifolds. A surjective smooth map $\pi : \widetilde{M} \to M$ between two smooth manifolds is called a *covering map* if for any $p \in M$, there exists an open neighborhood U containing p such that $\pi^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$ in $\widetilde{M}$ each of which is diffeomorphic to U via $\pi\big|_{V_\alpha}$. The manifold $\widetilde{M}$ is then said to be a *covering space* of M.

The quotient map $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ taking x to its equivalent class $[x]$ is an example of a covering map. Similar for the quotient map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ (where $\mathbb{RP}^n$ is regarded as $\mathbb{S}^n$ with all antipodal points identified).

Some smooth manifolds are constructed using quotients. The torus $\mathbb{R}^n/\mathbb{Z}^n$ and $\mathbb{RP}^n$ are good examples. Subject to certain conditions, one can define a Riemannian metric of the quotient manifold using that of the covering manifold, and vice versa. The crucial condition is that $\widetilde{M}$ has to be symmetric enough. Precisely, we have the following results:

Proposition 8.11. *Suppose $\pi : \widetilde{M} \to M$ is a covering map. Consider the following Deck transformation group defined by*

$$\mathrm{Deck}(\pi) := \{\Phi \in \mathrm{Diff}(\widetilde{M}) : \pi \circ \Phi = \pi\}.$$

Then, any Riemannian metric g of M induces a Riemannian metric $\pi^* g$ on $\widetilde{M}$ which is invariant under any $\Phi \in \mathrm{Deck}(\pi)$, i.e. $\Phi^*(\pi^* g) = \pi^* g$.

Conversely, suppose the Deck transformation group acts transitively on $\pi^{-1}(p)$ for any $p \in M$ (which is true when $\widetilde{M}$ is simply-connected). Then, given any Riemannian metric $\widetilde{g}$ of $\widetilde{M}$ such that any Deck transformation $\Phi : \widetilde{M} \to \widetilde{M}$ is an isometry of $(\widetilde{M}, \widetilde{g})$, i.e. $\Phi^* \widetilde{g} = \widetilde{g}$, then there exists a Riemannian metric g on M such that $\pi^* g = \widetilde{g}$.

Proof. Suppose g is a Riemannian metric of M. Since π is a covering map, in particular it is an immersion (and submersion too although it is not needed). This implies $\pi^* g$ is a Riemannian metric of $\widetilde{M}$ (see Example 8.5). For any Deck transformation $\Phi : \widetilde{M} \to \widetilde{M}$, we have $\pi \circ \Phi = \pi$ and so $\Phi^* \circ \pi^* = \pi^*$, this shows $\Phi^*(\pi^* g) = \pi^* g$, completing the proof of the first part of the proposition.

Conversely, given $\widetilde{g}$ a Riemannian metric on $\widetilde{M}$ such that $\Phi^* \widetilde{g} = \widetilde{g}$ for any $\Phi \in \mathrm{Deck}(\pi)$, we need to construct a Riemannian metric g on M such that $\Phi^* g = \widetilde{g}$. Given any $p \in M$, by the covering map condition, one can pick $q \in \pi^{-1}(p)$ and a neighborhood V of q such that $\pi\big|_V$ is a diffeomorphism onto its image. In particular, $(\pi_*)_q : T_q \widetilde{M} \to T_p M$ is invertible. We define $g_p : T_p M \times T_p M \to \mathbb{R}$ as follows:

$$g_p(X, Y) := \widetilde{g}_q\big((\pi_*)_q^{-1}(X), (\pi_*)_q^{-1}(Y)\big) \quad \text{for any } X, Y \in T_p M.$$

It is smooth as $\pi\big|_V : V \to \pi(V)$ is a diffeomorphism (so is its inverse). We need to justify that such a definition of g_p is independent of the choice of q in $\pi^{-1}(p)$. It thanks to the invariant condition of $\widetilde{g}$ under the Deck transformation group. Given another point $q' \in \pi^{-1}(p)$, by standard topology theory one can find a Deck transformation Φ such that $\Phi(q) = q'$. Then, we have

$$\begin{aligned}\widetilde{g}_{q'}\big((\pi_*)_{q'}^{-1}(X), (\pi_*)_{q'}^{-1}(Y)\big) &= \big((\Phi^{-1})^*\widetilde{g}_q\big)\big((\pi_*)_{q'}^{-1}(X), (\pi_*)_{q'}^{-1}(Y)\big)\\ &= \widetilde{g}_{\Phi(q')}\big(\Phi_*^{-1}(\pi_*)_{q'}^{-1}X, \Phi_*^{-1}(\pi_*)_{q'}^{-1}Y\big)\\ &= \widetilde{g}_q\big((\pi_*)_q^{-1}X, (\pi_*)_q^{-1}Y\big).\end{aligned}$$

The last steps follows from $\pi \circ \Phi = \pi$. Finally, we check that $\pi^* g = \widetilde{g}$:

$$\begin{aligned}(\pi^* g)(X, Y) &= g(\pi_* X, \pi_* Y)\\ &= \widetilde{g}\big(\pi_*^{-1}(\pi_* X), \pi_*^{-1}(\pi_* Y)\big)\\ &= \widetilde{g}(X, Y).\end{aligned}$$

This shows $\pi^* g = \widetilde{g}$. □

Example 8.12 (Flat Torus). A n-dimensional torus $\mathbb{T}^n$ can be regarded as the quotient manifold $\mathbb{R}^n/\mathbb{Z}^n$ whose elements are equivalent classes $[(x_1, \cdots, x_n)]$ under the relation

$$(x_1, \cdots, x_n) \sim (y_1, \cdots, y_n) \iff x_i - y_i \in \mathbb{Z} \text{ for any } i.$$

The quotient map $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is then a covering map. Any Deck transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ such that for any $(x_1, \cdots, x_n) \in \mathbb{R}^n$ we have $[\Phi(x_1, \cdots, x_1)] = [(x_1, \cdots, x_n)]$. In other words, there exists integers $m_1, \cdots, m_n$ such that

$$\Phi(x_1, \cdots, x_n) = (x_1, \cdots, x_n) + (m_1, \cdots, m_n).$$

These integers does not depend on x_i's by continuity. Hence, the Deck transformation group of the covering $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the set of translations by integer points.

It is clear that $\Phi^*\delta = \delta$ where δ is the Euclidean metric of $\mathbb{R}^n$. Hence by Proposition 8.11 (the converse part), there exists a metric g on $\mathbb{T}^n$ such that $\pi^* g = \delta$. □

Example 8.13 (Real Projective Space). Consider $\mathbb{RP}^n$ constructed by identifying antipodal points on $\mathbb{S}^n$. The quotient map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ is a covering map, and Deck transformations $\Phi : \mathbb{S}^n \to \mathbb{S}^n$ of this covering are either $\Phi = \text{id}$ or $\Phi = -\text{id}$. Both are isometries of the round metric g_{round} of $\mathbb{S}^n$, so it induces a Riemannian metric g such that $\pi^* g = g_{\text{round}}$. □

8.2. Levi-Civita Connections

Our next goal is to extend the notion of covariant derivatives of Euclidean hypersurfaces to its analogue on Riemannian manifolds. On Euclidean hypersurfaces, the covariant derivatives are inherited from the directional derivatives on the Euclidean ambient space (which no longer exists for Riemannian manifolds. However, since the first fundamental form g determines the covariant derivative (see Proposition 7.29) for Euclidean hypersurfaces, it motivates us to *define* "covariant derivatives" (renamed as *Levi-Civita connections* or *Riemannian connections*) using Riemannian metrics for an abstract manifold.

Let g be a Riemannian metric on a manifold M. Suppose under local coordinates $\{u_i\}$ on M the metric has local components $g_{ij} = g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$, then we *define* the Christoffel symbol as:

$$\Gamma_{ij}^k := \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \right)$$

where g^{kl} is the (k,l)-th entry of the matrix $[g]^{-1}$. From now on we will denote

$$\Gamma^\infty(TM) = \text{ the space of } C^\infty \text{ vector fields on } M.$$

We then *define* an operator

$$\nabla : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$$

by *declaring* that it acts on coordinate vectors by

$$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} := \Gamma_{ij}^k \frac{\partial}{\partial u_k}$$

and then is extended to general vector fields by the rules

$$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \tag{8.1}$$

$$\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y \tag{8.2}$$

$$\nabla_{\varphi X} (fY) = [X(f) \cdot Y + f \nabla_X Y] \, \varphi \tag{8.3}$$

Exercise 8.2. Let $X = X^i \frac{\partial}{\partial u_i}$ and $Y = Y^j \frac{\partial}{\partial u_j}$. Show that $\nabla_X Y$ has the following local expression:

$$\nabla_X Y = \left(X^i \frac{\partial Y^k}{\partial u_i} + X^i Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial u_k}. \tag{8.4}$$

Check also that (8.4) is independent of local coordinates, i.e. given another local coordinates $\{v_\alpha\}$ so that $X = \widetilde{X}^\alpha \frac{\partial}{\partial v_\alpha}$, $Y = \widetilde{Y}^\beta \frac{\partial}{\partial v_\alpha}$, and $\widetilde{g}_{\alpha\beta} = g(\frac{\partial}{\partial v_\alpha}, \frac{\partial}{\partial v_\beta})$, verify that:

$$\left(\widetilde{X}^\alpha \frac{\partial \widetilde{Y}^\gamma}{\partial v_\alpha} + \widetilde{X}^\alpha \widetilde{Y}^\beta \widetilde{\Gamma}_{\alpha\beta}^\gamma \right) \frac{\partial}{\partial v_\gamma} = \left(X^i \frac{\partial Y^k}{\partial u_i} + X^i Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial u_k}$$

where the Christoffel symbols $\widetilde{\Gamma}_{\alpha\beta}^\gamma$ in the new coordinate system are defined as

$$\widetilde{\Gamma}_{\alpha\beta}^\gamma = \frac{1}{2} \widetilde{g}^{\gamma\eta} \left(\frac{\partial \widetilde{g}_{\beta\eta}}{\partial \widetilde{v}_\alpha} + \frac{\partial \widetilde{g}_{\alpha\eta}}{\partial \widetilde{v}_\beta} - \frac{\partial \widetilde{g}_{\alpha\beta}}{\partial \widetilde{v}_\eta} \right).$$

The operator ∇ described above is called the *Levi-Civita connection* (or *Riemannian connection*) of (M, g). Note that the Christoffel symbols Γ_{ij}^k, and hence the Levi-Civita connection, depend on g.

The term *connection* has a broader meaning. Any operator $D : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ satisfying all of (8.1)-(8.3) is called a *connection* on M.

Exercise 8.3. Given a connection $D : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ (not necessarily the Levi-Civita connection), show that if $X_1(p) = X_2(p)$ at a point $p \in M$, then we have $D_{X_1}Y(p) = D_{X_2}Y(p)$.

Exercise 8.4. Consider a connection $D : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ on M with local coordinates (u_i). Let γ_{ij}^k be the coefficients when D acts on coordinate vectors, i.e.

$$D_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \gamma_{ij}^k \frac{\partial}{\partial u_k}.$$

Show that $D_X Y$ is uniquely determined by X^i's Y^i's, and γ_{ij}^k's where $X = X^i \frac{\partial}{\partial u_i}$ and $Y = Y^i \frac{\partial}{\partial u_i}$.

It is straight-forward to verify that Γ_{ij}^k and g_{ij} satisfy the following relations:

$$\Gamma_{ij}^k = \Gamma_{ji}^k \tag{8.5}$$

$$\frac{\partial g_{ij}}{\partial u_k} = \Gamma_{ik}^l g_{jl} + \Gamma_{jk}^l g_{il} \tag{8.6}$$

for any i, j, k. Using invariant notations (i.e. without using local coordinates), these relations can be written in equivalent form as:

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{8.7}$$

$$Z\left(g(X, Y)\right) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{8.8}$$

for any vector fields X, Y, Z. Here $[X, Y]$ denotes the Lie brackets between X and Y.

Let us verify that (8.6) and (8.8) are equivalent. Write $X = X^i \frac{\partial}{\partial u_i}$, $Y = Y^i \frac{\partial}{\partial u_i}$, and $Z = Z^i \frac{\partial}{\partial u_i}$, then

$$Z\left(g(X, Y)\right) = Z^k \frac{\partial}{\partial u_k}\left(g_{ij} X^i Y^j\right) = Z^k \left(\frac{\partial g_{ij}}{\partial u_k} X^i Y^j + g_{ij} Y^j \frac{\partial X^i}{\partial u_k} + g_{ij} X^i \frac{\partial Y^j}{\partial u_k}\right).$$

On the other hand, we have (for simplicity, denote $\partial_k := \frac{\partial}{\partial u_k}$ and $\nabla_k := \nabla_{\partial_k}$):

$$\begin{aligned}
&g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\
&= g\left(Z^k \nabla_k (X^i \partial_i), Y^j \partial_j\right) + g\left(X^i \partial_i, Z^k \nabla_k (Y^j \partial_j)\right) \\
&= g\left(Z^k \frac{\partial X^i}{\partial u_k} \frac{\partial}{\partial u_i} + Z^k X^i \Gamma_{ki}^l \frac{\partial}{\partial u_l}, Y^j \frac{\partial}{\partial u_j}\right) \\
&\quad + g\left(X^i \frac{\partial}{\partial u_i}, Z^k \frac{\partial Y^j}{\partial u_k} \frac{\partial}{\partial u_j} + Z^k Y^j \Gamma_{kj}^l \frac{\partial}{\partial u_l}\right) \\
&= \underbrace{Z^k \left(g_{lj} \Gamma_{ki}^l X^i Y^j + g_{il} \Gamma_{kj}^l X^i Y^j\right)}_{*} + g_{ij} Z^k \frac{\partial X^i}{\partial u_k} Y^j + g_{ij} Z^k \frac{\partial Y^j}{\partial u_k} X^i.
\end{aligned}$$

From the above calculations, it is clear that if (8.6) holds, then

$$* = Z^k \frac{\partial g_{ij}}{\partial u_k} X^i Y^j,$$

and so we have proved (8.8). To prove (8.8) implies (8.6), one may simply take $Z = \frac{\partial}{\partial u_k}$, $X = \frac{\partial}{\partial u_i}$, and $Y = \frac{\partial}{\partial u_j}$ and substitute them into (8.8).

Exercise 8.5. Show that (8.5) and (8.7) are equivalent.

While we define Levi-Civita connections using local coordinates, it is possible to give a more global definition, since we have the following uniqueness result:

Proposition 8.14. *Any connection $D : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ on a Riemannian manifold (M, g) satisfying both:*

- $D_X Y - D_Y X = [X, Y]$ *for any* $X, Y \in \Gamma^\infty(TM)$
- $Z\left(g(X, Y)\right) = g(D_Z X, Y) + g(X, D_Z Y)$ *for any* $X, Y, Z \in \Gamma^\infty(TM)$

must be equal to the Levi-Civita connection.

Proof. Let (u_i) be local coordinates of M and define γ_{ij}^k by

$$D_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \gamma_{ij}^k \frac{\partial}{\partial u_k}.$$

Then, the given two conditions of this proposition are equivalent to

$$\gamma_{ij}^k = \gamma_{ji}^k$$
$$\frac{\partial g_{ij}}{\partial u_k} = \gamma_{ik}^l g_{jl} + \gamma_{jk}^l g_{il}$$

for any i, j, k. The proof is identical to the one that shows (8.5)-(8.6) and (8.7)-(8.8) are equivalent (see page 215). By cyclic permutation of indices, we can then show:

$$\frac{\partial g_{jk}}{\partial u_i} + \frac{\partial g_{ik}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k} = 2\gamma_{ij}^l g_{kl}$$

for any i, j, k. Multiplying g^{-1} on both sides, we get:

$$\gamma_{ij}^l = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jk}}{\partial u_i} + \frac{\partial g_{ik}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k} \right),$$

which is exactly the Christoffel symbols Γ_{ij}^k for the Levi-Civita connection of g. Since the action of D is uniquely determined by its action on coordinate vectors, we must have $D = \nabla$.

□

Therefore, one can also *define* the Levi-Civita connection with respect to (M, g) as the unique operator that satisfies all of (8.1)-(8.3) and (8.7)-(8.8).

Exercise 8.6. Let (M, g) be a Riemannian manifold, and Σ be a submanifold of M. Denote $\iota : \Sigma \to M$ to be the inclusion map. Then, $\bar{g} := \iota^* g$ is a Riemannian metric on Σ. Show that the Levi-Civita connection of $(\Sigma, \bar{g})$, denoted by $\bar{\nabla}$, is given by:

$$\bar{\nabla}_X Y = (\nabla_X Y)^T := \text{ projection of } \nabla_X Y \text{ onto } T\Sigma$$

for any $X, Y \in \Gamma^\infty(T\Sigma)$.

8.2.1. Tensorial quantities. A linear operator T acting on vector fields and 1-forms $(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) \in \Gamma^\infty(TM)^r \times \Gamma^\infty(T^*M)^s$ is said to be *tensorial* if one can factor out a scalar function in each slot, i.e.

$$T(\varphi_1 X_1, \cdots, \varphi_r X_r, \psi_1 \omega_1, \cdots, \psi_s \omega_s) = \varphi_1 \cdots \varphi_r \psi_1 \cdots \psi_s T(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s)$$

for any scalar functions φ_i's and ψ_i's.

A Riemannian metric g is tensorial as it is required to be. For Euclidean hypersurfaces, the second fundamental form h is tensorial since $D_{fX}\nu = fD_X$. However, the operator

$S(X,Y) := D_X Y - D_Y X$, where D is any connection, is not tensorial. One can compute that

$$S(fX,Y) = D_{fX}Y - D_Y(fX) = fD_X Y - (Yf)X - fD_Y X = fS(X,Y) - (Yf)X.$$

However, one can modify it a little to make it to make it tensorial. Define

$$T(X,Y) = D_X Y - D_Y X - [X,Y].$$

Then, we have

$$T(fX,Y) = \underbrace{fD_X Y - fD_Y X - (Yf)X}_{D_{fX}Y - D_Y(fX)} - [fX,Y].$$

For any scalar function φ, we have

$$\begin{aligned}[fX,Y]\varphi &= fX(Y(\varphi)) - Y(fX(\varphi)) \\ &= fXY(\varphi) - Y(f)X(\varphi) - fYX(\varphi).\end{aligned}$$

Therefore, $[fX,Y] = f[X,Y] - (Yf)X$. By cancellations, we get

$$T(fX,Y) = f(D_X Y - D_Y X) - f[X,Y] = fT(X,Y).$$

One can also check $T(X,fY) = fT(X,Y)$ in a similar way.

It is important to note that the a connection, written as $D(X,Y) := D_X Y$, is NOT tensorial! While it is still true that $D(fX,Y) = fD(X,Y)$, it fails to be tensorial in the second slot:

$$D(X,fY) = D_X(fY) = (Xf)Y + fD_X Y \neq fD(X,Y).$$

However, it is interesting (also an important fact) that the *difference* between two connections is nonetheless tensorial! Suppose D and $\widetilde{D}$ are two connections on a manifold. Then, one can easily check that

$$\begin{aligned}(D - \widetilde{D})(X,fY) &= D_X(fY) - \widetilde{D}_X(fY) \\ &= (Xf)Y + fD_X Y - (Xf)Y - f\widetilde{D}_X Y \\ &= f(D - \widetilde{D})(X,Y).\end{aligned}$$

If an operator T is tensorial, then at any fixed point $p \in M$ the action of T on tangents and cotangents at p is uniquely determined by its action on a basis for T_pM and T_p^*M. Take a 2-tensor for example, if $\{e_i\}$ is a basis of T_pM for a fixed $p \in M$, then any vectors $X_p, Y_p \in T_pM$ can be expressed as $X_p = X^i e_i$ and $Y_p = Y^i e_i$, and so

$$T(X_p, Y_p) = X^i Y^j T(e_i, e_j)$$

which depends only on the point p. However, if we consider a connection D instead (which is not tensorial), we will see that

$$D(X_p, Y_p) = D_{X^i e_i}(Y^j e_j) = X^i (D_{e_i} Y^j) + X^i Y^j D_{e_i} e_j.$$

The quantity $D_{e_i} Y^j$ is a derivative of Y^j which depends on a *neighborhood* of p (not just p itself).

8.2.2. Levi-Civita connection on tensor bundles. From now on we will denote ∇ as the Levi-Civita connection of a given Riemannian metric g, and may sometimes write ∇^g to specify the metric. We will use D for an unspecified connection of a manifold.

The Levi-Civita connection ∇ (in fact any connection D) extends to an operator on tensor bundles $T^{r,s}(M) := T^*M^{\otimes r} \otimes TM^{\otimes s}$. First, the operator ∇_X is extended to act on 1-forms by the following rule.

Given any vector field X and 1-form α, the output $\nabla_X \alpha$ is a 1-form such that given any vector field Y the product rule holds formally:

$$X\left(\alpha(Y)\right) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y)$$

or equivalently,

$$(\nabla_X \alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y).$$

Locally, we have:

$$\begin{aligned}
(\nabla_i \, du^j)\left(\frac{\partial}{\partial u_k}\right) &= \frac{\partial}{\partial u_i}\left(du^j\left(\frac{\partial}{\partial u_k}\right)\right) - du^j\left(\nabla_i \frac{\partial}{\partial u_k}\right) \\
&= \frac{\partial}{\partial u_i}\delta^j_k - du^j\left(\Gamma^l_{ik}\frac{\partial}{\partial u_l}\right) \\
&= 0 - \Gamma^l_{ik}\delta^j_l = -\Gamma^j_{ik}.
\end{aligned}$$

In other words, we have

$$\nabla_i \, du^j = -\Gamma^j_{ik}\, du^k.$$

Then, given any vector fields $X, Y_1, \cdots, Y_s$ and 1-forms $\alpha_1, \cdots, \alpha_r$, we define the operator ∇_X by:

$$\nabla_X(Y_1 \otimes Y_2) := (\nabla_X Y_1) \otimes Y_2 + Y_1 \otimes (\nabla_X Y_2)$$

and more generally,

$$\begin{aligned}
\nabla_X\left((\otimes_{i=1}^r \omega_i) \otimes (\otimes_{j=1}^s Y_j)\right) &= \sum_{i=1}^r \omega_1 \otimes \cdots \otimes \left(\nabla_X \omega_i\right) \otimes \cdots \otimes \omega_r \otimes Y_1 \otimes \cdots \otimes Y_s \\
&\quad + \sum_{j=1}^s \omega_1 \otimes \cdots \otimes \omega_r \otimes Y_1 \otimes \cdots \otimes \left(\nabla_X Y_j\right) \otimes \cdots \otimes Y_s.
\end{aligned}$$

For instance, for a 2-tensor $T = T_{ij}\, du^i \otimes du^j$, we have

$$\begin{aligned}
\nabla_k T &= \nabla_k\left(T_{ij}\, du^i \otimes du^j\right) \\
&= \frac{\partial T_{ij}}{\partial u_k}\, du^i \otimes du^j + T_{ij}(\nabla_k\, du^i) \otimes du^j + T_{ij}\, du^i \otimes (\nabla_k\, du^j) \\
&= \frac{\partial T_{ij}}{\partial u_k}\, du^i \otimes du^j - T_{ij}\Gamma^i_{kl}\, du^l \otimes du^j - T_{ij}\Gamma^j_{kl}\, du^i \otimes du^l \\
&= \left(\frac{\partial T_{ij}}{\partial u_k} - T_{lj}\Gamma^l_{ki} - T_{il}\Gamma^l_{kj}\right) du^i \otimes du^j.
\end{aligned}$$

We denote $\nabla_k T_{ij}$ to be the local expression such that

$$\nabla_k\left(T_{ij}\, du^i \otimes du^j\right) =: (\nabla_k T_{ij})\, du^i \otimes du^j,$$

so from the above calculation, we have:

$$\nabla_k T_{ij} = \frac{\partial T_{ij}}{\partial u_k} - T_{lj}\Gamma^l_{ki} - T_{il}\Gamma^l_{kj}.$$

While $\nabla_k f := \frac{\partial f}{\partial u_k}$ for a scalar function f, note that $\nabla_k T_{ij}$ does *not* mean the partial derivative of the scalar T_{ij} with respect to u_k.

Likewise, given a vector field $Y = Y^i \frac{\partial}{\partial u_i}$, we denote $\nabla_j Y^i$ to be the local expression such that

$$\nabla_j\left(Y^i \frac{\partial}{\partial u_i}\right) =: (\nabla_j Y^i)\frac{\partial}{\partial u_i}.$$

By direct computatios, we have:

$$\begin{aligned}\nabla_j\left(Y^i\frac{\partial}{\partial u_i}\right) &= \frac{\partial Y^i}{\partial u_j}\frac{\partial}{\partial u_i}+Y^i\nabla_j\frac{\partial}{\partial u_i}\\ &= \frac{\partial Y^i}{\partial u_j}\frac{\partial}{\partial u_i}+Y^i\Gamma^k_{ji}\frac{\partial}{\partial u_k}\\ &= \left(\frac{\partial Y^i}{\partial u_j}+Y^k\Gamma^i_{jk}\right)\frac{\partial}{\partial u_i}.\end{aligned}$$

Therefore, we conclude that

$$\nabla_j Y^i = \frac{\partial Y^i}{\partial u_j}+Y^k\Gamma^i_{jk}.$$

Given a (r,s)-tensor T locally expressed as

$$T = T^{j_1\cdots j_s}_{i_1\cdots i_r}\,du^{i_1}\otimes\cdots\otimes du^{i_r}\otimes\frac{\partial}{\partial u_{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial u_{j_s}},$$

we define $\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r}$ to be the local expression such that

$$\begin{aligned}&\nabla_k\left(T^{j_1\cdots j_s}_{i_1\cdots i_r}\,du^{i_1}\otimes\cdots\otimes du^{i_r}\otimes\frac{\partial}{\partial u_{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial u_{j_s}}\right)\\ &=\left(\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r}\right)du^{i_1}\otimes\cdots\otimes du^{i_r}\otimes\frac{\partial}{\partial u_{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial u_{j_s}}.\end{aligned}$$

Exercise 8.7. Show that:

$$\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r} = \frac{\partial T^{j_1\cdots j_s}_{i_1\cdots i_r}}{\partial u_k}+\sum_{p=1}^{r}T^{j_1\cdots j_{p-1}lj_{p+1}\cdots j_s}_{i_1\cdots i_r}\Gamma^{j_p}_{kl}-\sum_{q=1}^{s}T^{j_1\cdots j_s}_{i_1\cdots i_{q-1}li_{q+1}\cdots i_r}\Gamma^l_{ki_q}.$$

Exercise 8.8. Show that (8.6) can be rephrased as:

$$\nabla_k g_{ij} = 0 \quad \text{for any } i,j,k.$$

Exercise 8.9. Show that the Codazzi equation (Theorem 7.30) is equivalent to

$$\nabla_i h_{jk} = \nabla_j h_{ik}$$

for any i,j,k.

Since ∇_X is tensorial in the X-slot, given a (r,s)-tensor T, one can then view X as an input and define a new $(r+1,s)$-tensor ∇T as

$$(\nabla T)(X,Y_1,\cdots,Y_r,\alpha_1,\cdots,\alpha_s) := (\nabla_X T)(Y_1,\cdots,Y_r,\alpha_1,\cdots,\alpha_s).$$

Locally, given a vector field $Y = Y^i\frac{\partial}{\partial u_i}$, we have:

$$\nabla_j Y = (\nabla_j Y^i)\frac{\partial}{\partial u_i} \implies \nabla Y := du^j\otimes(\nabla_j Y) = (\nabla_j Y^i)\,du^j\otimes\frac{\partial}{\partial u_i}.$$

Now that ∇Y is a $(1,1)$-tensor, one can then differentiate it again:

$$\begin{aligned}
&\nabla_l(\nabla Y)\\
&= \nabla_l\left((\nabla_j Y^i)\,du^j \otimes \frac{\partial}{\partial u_i}\right)\\
&= \frac{\partial(\nabla_j Y^i)}{\partial u_l}\,du^j \otimes \frac{\partial}{\partial u_i} - (\nabla_j Y^i)\Gamma^j_{lp}\,du^p \otimes \frac{\partial}{\partial u_i} + (\nabla_j Y^i)\Gamma^p_{li}\,du^j \otimes \frac{\partial}{\partial u_p}\\
&= \left(\frac{\partial(\nabla_j Y^i)}{\partial u_l} - (\nabla_p Y^i)\Gamma^p_{lj} + (\nabla_j Y^p)\Gamma^i_{lp}\right) du^j \otimes \frac{\partial}{\partial u_i}.
\end{aligned}$$

Similarly, we will define $\nabla_l\nabla_j Y^i$ to be the local component such that

$$\nabla_l\left((\nabla_j Y^i)\,du^j \otimes \frac{\partial}{\partial u_i}\right) =: (\nabla_l\nabla_j Y^i)\,du^j \otimes \frac{\partial}{\partial u_i}$$

so that

$$\begin{aligned}
\nabla_l\nabla_j Y^i &= \frac{\partial(\nabla_j Y^i)}{\partial u_l} - (\nabla_p Y^i)\Gamma^p_{lj} + (\nabla_j Y^p)\Gamma^i_{lp}\\
&= \frac{\partial}{\partial u_l}\left(\frac{\partial Y^i}{\partial u_j} + Y^k\Gamma^i_{jk}\right) - \left(\frac{\partial Y^i}{\partial u_p} + Y^k\Gamma^i_{pk}\right)\Gamma^p_{lj}\\
&\quad + \left(\frac{\partial Y^p}{\partial u_j} + Y^k\Gamma^p_{jk}\right)\Gamma^i_{lp}.
\end{aligned}$$

Exercise 8.10. Let $\alpha = \alpha_i\,du^i$ be a 1-form, compute the local expression of $\nabla_j\alpha_i$ and $\nabla_l\nabla_j\alpha_i$.

It is important to note that ∇_i and ∇^i are different! The former was discussed above, but we define

$$\nabla^i := g^{ij}\nabla_j.$$

For instance, when acting on scalar functions, we have

$$\nabla_i f = \frac{\partial f}{\partial u_i} \quad \text{whereas} \quad \nabla^i f = g^{ij}\nabla_j f = g^{ij}\frac{\partial f}{\partial u_j}.$$

For a vector field $Y = Y^i\frac{\partial}{\partial u_i}$, we have:

$$\nabla^j Y^i = g^{jk}\nabla_k Y^i = g^{jk}\left(\frac{\partial Y^i}{\partial u_k} + Y^l\Gamma^i_{kl}\right).$$

While we can define ∇Y as a $(1,1)$-tensor $(\nabla_i Y^j)\,du^i \otimes \frac{\partial}{\partial u_j}$, it also makes sense to regard it as a $(0,2)$-tensor:

$$(\nabla^i Y^j)\frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j} = (g^{ik}\nabla_k Y^j)\frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j}.$$

Very often, it is not difficult to judge whether ∇Y means a $(1,1)$-tensor or a $(0,2)$-tensor according to the context. Similar notations apply to other higher-rank tensors.

However, when acting on scalar functions f, it is a convention to regard ∇f as a *vector field* but not a 1-form, that is:

$$\nabla f := (\nabla^i f)\frac{\partial}{\partial u_i} = g^{ij}\frac{\partial f}{\partial u_j}\frac{\partial}{\partial u_i}$$

instead of $(\nabla_i f)\,du^i = \frac{\partial f}{\partial u_i}\,du^i$. We have this convention because we already have another symbol to denote $\frac{\partial f}{\partial u_i}\,du^i$, namely the exterior derivative df. The vector field ∇f is called the *gradient* of f with respect to g. When g is the Euclidean metric on $\mathbb{R}^n$, the gradient ∇f is the standard gradient in multivariable calculus.

Exercise 8.11. Given $f : M \to \mathbb{R}$ is a scalar function on a smooth manifold M. Check that for any vector field X on M, we have

$$g(\nabla f, X) = df(X).$$

Suppose $f^{-1}(c)$ is a submanifold of M. Show that for any $p \in f^{-1}(c)$, the vector $\nabla f(p)$ is a normal to the level set $f^{-1}(c)$, i.e.

$$g(\nabla f, T) = 0$$

for any $T \in T_p f^{-1}(c) \subset T_p M$.

Using the Levi-Civita connection, one can also define divergence and Laplacian operators. Given a vector field $X = X^i \frac{\partial}{\partial u_i}$ on a Riemannian manifold (M, g), we define its divergence with respect to g by

$$\mathrm{div}_g X := \nabla_i X^i.$$

The Laplacian operator Δ acting on functions is the divergence of the gradient with respect to g:

$$\Delta_g f := \mathrm{div}_g(\nabla f) = \nabla_i \nabla^i f.$$

Since $\nabla^i = g^{ij}\nabla_j$ and g is constant under ∇, we have

$$\Delta_g f = \nabla_i(g^{ij}\nabla_j f) = g^{ij}\nabla_i\nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial u_i \partial u_j} - \Gamma_{ij}^k \frac{\partial f}{\partial u_k}\right).$$

8.2.3. Contraction of tensors. Given two tensors of different types, one can "cook them up" to form new tensors. For instance, consider the first fundamental form g_{ij} (and its inverse g^{ij}), and the second fundamental form h_{ij} of a regular Euclidean hypersurface. They are both $(2,0)$-tensors. We can define "cook up" two new tensors by summing up indices:

$$g^{ij} h_{ij} \quad \text{or} \quad g^{ik} h_{kj}.$$

They are different tensors. The first one $g^{ij}h_{ij}$ sums over both i and j, so there is no free component and it is a scalar function (this function is the mean curvature). For the second one $g^{ik}h_{kj}$, we sum them up over k leaving i and j to be free. It gives a new tensor denoted by, say, A with one lower component and one upper component (i.e. a $(1,1)$-tensor), that if we input $\frac{\partial}{\partial u_j}$ into A, it will output

$$A\left(\frac{\partial}{\partial u_j}\right) = g^{ik} h_{kj} \frac{\partial}{\partial u_i}, \quad \text{or equivalently,} \quad A = g^{ik} h_{kj}\, du^j \otimes \frac{\partial}{\partial u_i}.$$

In this case, the operator A is simply the shape operator discussed in Section 7.2. We often denote the components of A by $h^i_j := g^{ik} h_{kj}$.

For more complicated tensors, we can "cook them up" in similar ways. For example, given $S = S^k_{ij}\, du^i \otimes du^j \otimes \frac{\partial}{\partial u_k}$ and $T = T^i_j\, du^j \otimes \frac{\partial}{\partial u_i}$, we can form many combinations:

component	type	full form
$W^l_{ij} := S^k_{ij} T^l_k$	$(2,1)$-tensor	$W = S^k_{ij} T^l_k\, du^i \otimes du^j \otimes \frac{\partial}{\partial u_l}$
$U^l_{ik} := S^l_{ij} T^j_k$	$(2,1)$-tensor	$U = S^l_{ij} T^j_k\, du^i \otimes du^k \otimes \frac{\partial}{\partial u_l}$
$\alpha_i := S^k_{ij} T^j_k$	$(1,0)$-tensor	$\alpha = S^k_{ij} T^j_k\, du^i$

It is interesting to note that if each summations is over exactly *one upper* index and *one lower* index, then the new tensor produced is independent of local coordinates. Take

S and T above as an example, if we write them in two different coordinate systems:

$$S = S_{ij}^k \, du^i \otimes du^j \otimes \frac{\partial}{\partial u_k} = S_{\alpha\beta}^\gamma \, dv^\alpha \otimes dv^\beta \otimes \frac{\partial}{\partial v_\gamma}$$
$$T = T_j^i \, du^j \otimes \frac{\partial}{\partial u_i} = T_\beta^\alpha \, dv^\beta \otimes \frac{\partial}{\partial v_\alpha},$$

then it can be easily checked that

$$S_{ij}^k T_k^l \, du^i \otimes du^j \otimes \frac{\partial}{\partial u_l} = S_{\alpha\beta}^\gamma T_\gamma^\eta \, dv^\alpha \otimes dv^\beta \otimes \frac{\partial}{\partial v_\eta}. \tag{8.9}$$

Exercise 8.12. Verify (8.9).

However, it is generally not true that $S_{ij}^k T_j^l$ gives a well-defined $(1, 2)$-tensor, since both j's are lower indices.

It is interesting and very useful to note that the product rule holds for contractions of tensors when taking derivatives by the Levi-Civita connection. For example, we have

$$\nabla_i (S_{jk} T_q^k) = (\nabla_i S_{jk}) T_q^k + S_{jk} (\nabla_i T_q^k),$$

which precisely means

$$\nabla_{\frac{\partial}{\partial u_i}} \left(S_{jk} T_q^k \, du^j \otimes du^q \right) = \left((\nabla_i S_{jk}) T_q^k + S_{jk} (\nabla_i T_q^k) \right) du^j \otimes du^q.$$

Let's verify this by direct computations. The LHS equals

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial u_i}} \left(S_{jk} T_q^k \, du^j \otimes du^q \right) \\
&= \frac{\partial}{\partial u_i} (S_{jk} T_q^k) \, du^j \otimes du^q + S_{jk} T_q^k (\nabla_i \, du^j) \otimes du^q + S_{jk} T_q^k \, du^j \otimes (\nabla_i \, du^q) \\
&= \left(\frac{\partial S_{jk}}{\partial u_i} T_q^k + S_{jk} \frac{\partial T_q^k}{\partial u_i} \right) du^j \otimes du^q - S_{jk} T_q^k \Gamma_{ip}^j \, du^p \otimes du^q - S_{jk} T_q^k \, du^j \otimes (\Gamma_{ip}^q \, du^p) \\
&= \left(\frac{\partial S_{jk}}{\partial u_i} T_q^k + S_{jk} \frac{\partial T_q^k}{\partial u_i} \right) du^j \otimes du^q - S_{pk} T_q^k \Gamma_{ij}^p \, du^j \otimes du^q - S_{jk} T_p^k \Gamma_{iq}^p \, du^j \otimes du^q.
\end{aligned}$$

The last step follows from relabelling indices of the last two terms. By taking out the common factor $du^j \otimes du^q$, we have proved

$$\nabla_i (S_{jk} T_q^k) = \frac{\partial S_{jk}}{\partial u_i} T_q^k + S_{jk} \frac{\partial T_q^k}{\partial u_i} - S_{pk} T_q^k \Gamma_{ij}^p - S_{jk} T_p^k \Gamma_{iq}^p.$$

For the RHS, we recall that:

$$\nabla_i S_{jk} = \frac{\partial S_{jk}}{\partial u_i} - \Gamma_{ij}^p S_{pk} - \Gamma_{ik}^p S_{jp},$$
$$\nabla_i T_q^k = \frac{\partial T_q^k}{\partial u_i} - \Gamma_{iq}^p T_p^k + \Gamma_{ip}^k T_q^p.$$

Combining these results, we can easily get that

$$(\nabla_i S_{jk}) T_q^k + S_{jk} (\nabla_i T_q^k) = \nabla_i (S_{jk} T_q^k)$$

by cancellations and relabelling of indices.

A similar product rule holds for all other legitimate contractions (i.e. one "up" paired with one "down"). The proof is similar to the above special case although it is quite tedious. We omit the proof here.

Exercise 8.13. Prove that $\nabla_i(S^{kl}T_{kj}) = (\nabla_i S^{kl})T_{kj} + S^{kl}\nabla_i T_{kj}$.

In particular, as we have $\nabla_i g_{jk} = 0$ for any i, j, k, one can apply the product rule to see that g_{ij} can be treated as a constant when we differentiate it by ∇. For example:

$$\nabla_i(g_{jk}T_l^k) = g_{jk}\nabla_i T_l^k.$$

Recall also that $g_{ik}g^{kj} = \delta_{ij}$. Applying the product rule one can get

$$\nabla_p(g_{ik}g^{kj}) = 0 \implies g_{ik}\nabla_p g^{kj} = 0.$$

Taking g^{-1} on both sides, we can get $\nabla_p g^{kj} = 0$ as well.

Chapter 9

Parallel Transport and Geodesics

Let (M, g) be a Riemannian manifold. From now on unless otherwise is said, we will denote ∇ to be the Levi-Civita connection with respect to g. Furthermore, as ∇ coincides with the covariant derivative in case when M is an Euclidean hypersurface and g is the first fundamental form, we will use also use the term *covariant derivative* for the Levi-Civita connection.

9.1. Parallel Transport

9.1.1. Parallel Transport Equation. On an Euclidean space, we can make good sense of *translations* of vectors as $T_p\mathbb{R}^n$ is naturally identified with $T_q\mathbb{R}^n$ for any other point q. However, on an abstract manifold M, the tangent spaces T_pM and T_qM may not be naturally related to each other, so it is non-trivial to make sense of translating a vector $V \in T_pM$ to a vector in T_qM.

If one translates a vector V at $p \in \mathbb{R}^n$ along a path $\gamma(t)$ connecting p and q, then it is ease to see that $\frac{d}{dt}V(\gamma(t)) = 0$ along the path. In other words, $D_{\gamma'(t)}V = 0$ where D is the directional derivative of $\mathbb{R}^n$. Now on a Riemannian manifold we have the notion of the covariant derivative ∇. Using ∇ in place of D in $\mathbb{R}^n$, one can still define the notion of translations as follows:

Definition 9.1 (Parallel Transport). Given a curve $\gamma : [0, T] \to M$ and a vector $V_0 \in T_{\gamma(0)}M$, we define the *parallel transport* of V_0 along $\gamma(t)$ to be the unique solution $V(t)$ to the ODE:

$$\nabla_{\gamma'(t)}V(t) = 0$$

for any $t \in [0, T]$ with the initial condition $V(0) = V_0$.

In terms of local coordinates $F(u_1, \cdots, u_n)$, the parallel transport equation can be expressed as follows. Let $\gamma(t) = F(\gamma^1(t), \cdots, \gamma^n(t))$ and $V(t) = V^i(t)\frac{\partial}{\partial u_i}$, then

$$\gamma'(t) = \frac{d\gamma^i}{dt}\frac{\partial}{\partial u_i}$$

$$\begin{aligned}\nabla_{\gamma'(t)}V(t) &= \nabla_{\frac{d\gamma^i}{dt}\frac{\partial}{\partial u_i}}\left(V^j(t)\frac{\partial}{\partial u_j}\right)\\ &= \frac{d\gamma^i}{dt}\left(\frac{\partial V^j}{\partial u_i}\frac{\partial}{\partial u_j} + V^j\Gamma^k_{ij}\frac{\partial}{\partial u_k}\right)\\ &= \frac{dV^j}{dt}\frac{\partial}{\partial u_j} + V^j\Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{\partial}{\partial u_k}.\end{aligned}$$

Therefore, the parallel transport equation $\nabla_{\gamma'(t)}V(t) = 0$ is equivalent to

$$\frac{dV^k}{dt} + V^j\Gamma^k_{ij}\frac{d\gamma^i}{dt} = 0 \quad \text{for any } k.$$

Notably, the equation depends on $\gamma'(t)$ only but not on $\gamma(t)$ itself, so it also makes sense of parallel transporting V_0 along a vector field X. It simply means parallel transport of V_0 along the integral curve $\gamma(t)$ such that $\gamma'(t) = X(\gamma(t))$.

Even though the parallel transport equation is a first-order ODE whose existence and uniqueness of solutions are guaranteed, it is often impossible to solve it explicitly (and often not necessary to). Nonetheless, there are many remarkable uses of parallel transports. When we have an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM at a fixed point $p \in M$, one can naturally extend it along a curve to become a moving orthonormal basis $\{e_i(t)\}_{i=1}^n$ along that curve. The reason is that parallel transport preserves the angle between vectors.

Given a curve $\gamma(t)$ on a Riemannian manifold (M, g), and let $V(t)$ be the parallel transport of $V_0 \in T_pM$ along γ, and $W(t)$ be that of $W_0 \in T_pM$. Then one can check easily that

$$\frac{d}{dt}g\big(V(t), W(t)\big) = g\big(\nabla_{\gamma'}V, W\big) + g\big(V, \nabla_{\gamma'}W\big) = 0$$

by the parallel transport equations $\nabla_{\gamma'}V = \nabla_{\gamma'}W = 0$. Therefore, we have

$$g\big(V(t), W(t)\big) = g\big(V_0, W_0\big)$$

for any t. In particular, if V_0 and W_0 are orthogonal, then so are $V(t)$ and $W(t)$ for any t. If we take $W_0 = V_0$, by uniqueness theorem of ODE we have $W(t) \equiv V(t)$. The above result shows the length of $V(t)$, given by $\sqrt{g(V(t), V(t))}$, is also a constant. Hence, the parallel transport of an orthonormal basis remains to be orthonormal along the curve.

9.1.2. Holonomy and de Rham Splitting Theorem. Another remarkable consequence of the above observation is that parallel transport can be used to define an $O(n)$-action on T_pM. Consider a *closed* piecewise smooth curve $\gamma(t)$ where $\gamma(0) = p \in M$. The curve needs not to be smooth at the closing point p. Given a vector $V \in T_pM$, we parallel transport it subsequentially along each smooth segment of γ. Precisely, suppose γ is defined on $[t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{k-1}, t_k]$ with $\gamma(t_0) = \gamma(t_k) = p$ and that γ is smooth on each (t_i, t_{i_1}) and is continuous on $[t_0, t_k]$. We solve the parallel transport equation to get $V(t)$ which is continuous on $[t_0, t_k]$ and satisfies $\nabla_{\gamma'}V = 0$ on each (t_i, t_{i-1}). We denote the final vector $V(t_k)$ by $P_\gamma(V) \in T_pM$. It then defines a map

$$P_\gamma : T_pM \to T_pM.$$

By linearity of $\nabla_{\gamma'}$, it is easy to show that P_γ is a linear map. Moreover, as $P_\gamma(V)$ and V have the same length, we have in fact $P_\gamma \in O(T_pM)$, the orthogonal group acting on T_pM.

The set of all P_γ's, where γ is any closed piecewise smooth curve based at p, is in fact a group with multiplication given by compositions, and inverse given by parallel transporting vectors along the curve backward. We call this:

Definition 9.2 (Holonomy Group)**.** The *holonomy group* of (M, g) based at $p \in M$ is given by:

$$\mathrm{Hol}_p(M, g) := \{P_\gamma : \gamma \text{ is a closed piecewise smooth curve on } M \text{ based at } p\}.$$

Exercise 9.1. Let (M, g) be a connected complete Riemannian manifold, and p and q be two distinct points on M. Show that $\mathrm{Hol}_p(M, g)$ and $\mathrm{Hol}_q(M, g)$ are related by conjugations (hence are isomorphic).

When (M, g) is the flat Euclidean space $\mathbb{R}^n$, parallel transport is simply translations. Any vector will end up being the same vector after transporting back to its based point. Therefore, $\mathrm{Hol}_p(\mathbb{R}^n, \delta)$ is the trivial group for any p.

For the round sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, we pick two points P, Q on the equator, and mark N to be the north pole. Construct a piecewise great circle path $P \to N \to Q \to P$, then one can see that what parallel transport along this path does is a rotation on vectors in $T_P\mathbb{S}^2$ by an angle depending on the distance between P and Q. For instance, when P and Q are antipodal, the parallel transport map is rotation by π. When $P = Q$, then the parallel transport map is simply the identity map. By varying the position of Q, one can obtain all possible angles from 0 to 2π. To explain this rigorously, one way is to solve the parallel transport equation in Definition 9.1. There is a more elegant to explain this after we learn about geodesics.

It is remarkable that the holonomy group reveals a lot about the topological structure about a Riemannian manifold, and is a extremely useful tool for classification problems of manifolds. There is a famous theorem due to Ambrose-Singer that the Lie algebra of $\mathrm{Hol}_p(M, g)$ is related to how curved (M, g) is around p.

Another beautiful theorem which demonstrates the usefulness of parallel transport is the following one due to de Rham. It is widely used in Ricci flow to classify the topology of certain class of manifolds by decomposing them into lower dimensional ones.

Theorem 9.3 (de Rham Splitting Theorem - local version)**.** *Suppose the tangent bundle TM of a Riemannian manifold (M, g) can be decomposed orthogonally into $TM = E_1 \oplus E_2$ such that each of E_i's is invariant under parallel transport, i.e. whenever $V \in E_i$, any parallel transport of V stays in E_i. Then, M is locally a product manifold $(N_1, h) \times (N_2, k)$ such that $TN_i = E_i$, and g is a locally product metric $\pi_1^* h + \pi_2^* k$.*

Proof. The proof begins by constructing a local coordinate system $\{x_i, y_\alpha\}$ such that $\operatorname{span}\left\{\frac{\partial}{\partial x_i}\right\} = E_1$ and $\operatorname{span}\left\{\frac{\partial}{\partial y_\alpha}\right\} = E_2$. This is done by Frobenius' Theorem which asserts that such that such local coordinate system would exist if each E_i is closed under the Lie brackets, i.e. whenever $X, Y \in E_i$, we have $[X, Y] \in E_i$.

Let's prove E_1 is closed under the Lie brackets (and the proof for E_2 is exactly the same). The key idea is to use (8.7) that:

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

First pick an orthonormal basis $\{e_i, e_\alpha\}$ at a fixed point $p \in M$, such that $\operatorname{span}\{e_i\} = E_1$ and $\operatorname{span}\{e_\alpha\} = E_2$. Extend them locally around p using parallel transport, then one would have $\nabla e_i = \nabla e_\alpha = 0$ for any i and α. Now suppose $X, Y \in E_1$, we want to show $\nabla_X Y \in E_1$. Since $g(Y, e_\alpha) = 0$ and $\nabla_X e_\alpha = 0$, we have by (8.8):

$$0 = X\big(g(Y, e_\alpha)\big) = g(\nabla_X Y, e_\alpha).$$

This shows $\nabla_X Y \perp e_\alpha$ for any α, so $\nabla_X Y \in E_1$. The same argument shows $\nabla_Y X \in E_1$, and so does $[X, Y]$.

Frobenius' Theorem asserts that there exists a local coordinate system $\{x_i, y_\alpha\}$ with span$\{\frac{\partial}{\partial x_i}\} = E_1$ and span$\{\frac{\partial}{\partial y_\alpha}\} = E_2$. Next, we argue that the metric g is locally expressed as:

$$g = g_{ij}\, dx^i \otimes dx^j + g_{\alpha\beta}\, dy^\alpha \otimes dy^\beta,$$

and that g_{ij} depends only on $\{x_i\}$, and $g_{\alpha\beta}$ depends only on $\{y_\alpha\}$. To show this, we argue that Christoffel symbols of type Γ_{ij}^α, $\Gamma_{i\alpha}^j$, $\Gamma_{i\alpha}^\beta$, and $\Gamma_{\alpha\beta}^i$ are all zero.

Consider $V_0 := \frac{\partial}{\partial x_i} \in T_pM$, and let $V(t)$ be its parallel transport along $\frac{\partial}{\partial x_j}$. Since $V(t) \in E_1$, it can be locally expressed as $V(t) = V^k(t)\frac{\partial}{\partial x_k}$. We then have

$$0 = \nabla_j V(t) = \frac{\partial V^k}{\partial x_j}\frac{\partial}{\partial x_k} + V^k\left(\Gamma_{jk}^l \frac{\partial}{\partial x_l} + \Gamma_{jk}^\alpha \frac{\partial}{\partial y_\alpha}\right).$$

By equating coefficients, we get $V^k\Gamma_{jk}^\alpha = 0$. In particular, it implies $\Gamma_{ik}^\alpha = 0$ at p since $V(0) = \frac{\partial}{\partial x_i}$ at p. Apply the same argument at other points covered by the local coordinates $\{x_i, y_\alpha\}$, we have proved Christoffel symbols of type Γ_{ik}^α all vanish. Using a similar argument by parallel transporting V_0 along $\frac{\partial}{\partial y_\alpha}$ instead, one can also show $\Gamma_{\alpha i}^\beta = 0$. The other combinations $\Gamma_{i\alpha}^j$ and $\Gamma^i_{\alpha\beta}$ follow from symmetry argument.

Finally, we conclude that

$$\frac{\partial}{\partial y_\alpha} g_{ij} = \Gamma_{\alpha i}^k g_{kj} + \Gamma_{\alpha j}^k g_{ki} = 0,$$

so g_{ij} depends only on $\{x_i\}$. Similarly, $g_{\alpha\beta}$ depends only on $\{y_\alpha\}$. It completes our proof. □

Remark 9.4. There is a global version of de Rham splitting theorem. If one further assumes that M is simply-connected, then the splitting would be in fact global. Generally if (M, g) satisfies the hypothesis of the de Rham splitting theorem, one can consider its universal cover $(\widetilde{M}, \pi^*g)$ which *is* simply-connected. Applying the theorem one can assert that $(\widetilde{M}, \pi^*g)$ splits isometrically as a product manifold. □

9.1.3. Parallel Vectors and Tensors. A vector field X satisfying $\nabla X = 0$ is called a *parallel vector field*. It has the property that for any curve $\gamma(t)$ on M with $\gamma(0) = p$, if we parallel transport $X(p) \in T_pM$ along γ one would get the vectors $X(\gamma(t))$ for any t. If such a vector field exists and is non-vanishing, then by de Rham splitting theorem applied to $TM = \text{span}\{X\} \oplus \text{span}\{X\}^\perp$, one can assert that locally the manifold M splits off as a product of a 1-dimensional manifold and an $(\dim M - 1)$-manifold.

As one can take covariant derivatives on tensors of any type, we also have a notion of parallel tensors. A tensor T is said to be *parallel* if $\nabla T = 0$. Notably, the Riemannian metric g itself is a parallel $(2, 0)$-tensor, as $\nabla_i g_{jk} = 0$.

Now consider a parallel $(1, 1)$-tensor T, which operates on vector fields and output vector fields. Given $\lambda \in \mathbb{R}$, the λ-eigenspace E_λ of T turns out is invariant under parallel transport. To see this, consider $V \in E_\lambda$ at p such that $T(V) = \lambda V$, or in terms of local coordinates:

$$T_i^j V^i = \lambda V^j.$$

Now parallel transport V along an arbitrarily given curve γ starting from P, then we have:

$$\nabla_{\gamma'}\big(T_i^j V^i - \lambda V^j\big) = (\nabla_{\gamma'} T_i^j)V^i + T_i^j \nabla_{\gamma'} V^i - \lambda \nabla_{\gamma'} V^j = 0$$

by the fact that T is parallel. Therefore, $T(V) - \lambda V$ satisfies the parallel transport equation along γ. As it equals 0 at the starting point p, it remains so along the curve. This shows $V(t)$ is also an eigenvector of T with eigenvalue λ. If we assume further that T is self-adjoint respect to g, i.e. $g(T(X), Y) = g(X, T(Y))$ for any vector fields X and

Y, then the eigenspaces of T with distinct eigenvalues would be orthogonal to each other (by freshmen linear algebra). The de Rham splitting theorem can be applied to show the manifold splits according to the eigenspaces of T.

Exercise 9.2. Consider a Riemannian manifold (M, g) and a parallel $(1,1)$-tensor J such that $J^2 = -\mathrm{id}$. Suppose further that

$$g(JX, JY) = g(X, Y)$$

for any vector fields X and Y. Consider the two-form defined by

$$\omega(X, Y) := g(JX, Y).$$

(1) Verify that ω is indeed a two-form, i.e. $\omega(X, Y) = -\omega(Y, X)$.

(2) Show that $\nabla\omega = 0$ and $d\omega = 0$.

9.2. Geodesic Equations

One classic problem in differential geometry is to find the shortest path on a surface connecting two distinct points. Such a path is commonly called a *geodesic*, or more accurately *minimizing geodesic*. To find such a path is a problem in calculus of variations.

In this section, we will derive an equation for us to find geodesics connecting two given points. The technique for deriving such an equation is very common in calculus of variations. We consider a family of curves $\gamma_s(t)$ connecting the same given points with $\gamma_0(t)$ being the candidate curve for the geodesic. Then, we compute the first derivative $\frac{d}{ds}L(\gamma_s)$ of the length $L(\gamma_s)$, and see under what condition on γ_0 would guarantee that the first derivative at $s = 0$ equals to 0.

Proposition 9.5. *Let $\gamma_s(t) : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a 1-parameter family of curves with $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for any $s \in (-\varepsilon, \varepsilon)$. Here s is the parameter of the family, and t is the parameter of each curve γ_s. Then, the first variation of the arc-length is given by:*

$$\left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) = -\int_a^b g\left(\left.\frac{\partial\gamma}{\partial s}\right|_{s=0}, \nabla_{\gamma_0'(t)}\left(\frac{\gamma_0'(t)}{|\gamma_0'(t)|}\right)\right) dt. \tag{9.1}$$

Therefore, if γ_0 minimizes the length among all variations γ_s, it is necessarily that

$$\nabla_{\gamma_0'(t)}\left(\frac{\gamma_0'(t)}{|\gamma_0'(t)|}\right) = 0 \quad \text{for any } t \in [a, b]. \tag{9.2}$$

Proof. For simplicity, we denote

$$S := \frac{\partial\gamma}{\partial s}, \quad T := \frac{\partial\gamma}{\partial t}.$$

$$\begin{aligned}
\frac{d}{ds}L(\gamma_s) &= \frac{d}{ds}\int_a^b \sqrt{g(T,T)}\,dt \\
&= \int_a^b \frac{1}{2\,|T|}\frac{d}{ds}g(T,T)\,dt \\
&= \int_a^b \frac{1}{2\,|T|}\cdot 2g\left(\nabla_S T, T\right)\,dt \\
&= \int_a^b \frac{1}{|T|}g(\nabla_T S, T)\,dt.
\end{aligned}$$

Here we have used (8.7) so that $\nabla_S T = \nabla_T S + [S,T]$ and

$$[S,T] = \left[\gamma_*\left(\frac{\partial}{\partial s}\right), \gamma_*\left(\frac{\partial}{\partial t}\right)\right] = \gamma_*\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = \gamma_*(0) = 0.$$

Now evaluate the above derivative at $s = 0$. We get

$$\left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) = \int_a^b g\left(\nabla_T S, \frac{T}{|T|}\right)\,dt\bigg|_{s=0} = -\int_a^b g\left(S, \nabla_T\left(\frac{T}{|T|}\right)\right)\,dt\bigg|_{s=0}$$

as desired. We have used integration by parts in the last step. The boundary term vanishes because $\gamma_s(a)$ and $\gamma_s(b)$ are both independent of s, and hence $\frac{\partial\gamma}{\partial s} = 0$ at both $t = a$ and $t = b$.

□

Exercise 9.3. Compute the first variation of the *energy functional* $E(\gamma_s)$ of a 1-parameter family of curves $\gamma_s(t) : (-\varepsilon, \varepsilon) \times [a, b] \to M$ with $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for any s. The energy functional is given by

$$E(\gamma_s) := \frac{1}{2}\int_a^b g\left(\frac{\partial \gamma_s}{\partial t}, \frac{\partial \gamma_s}{\partial t}\right) dt.$$

Note that it is different from arc-lengths: there is not square root in the integrand.

Furthermore, show that if γ_0 minimizes $E(\gamma_s)$, then one has $\nabla_{\gamma_0'(t)}\gamma_0'(t) = 0$ for any $t \in [a, b]$.

Since every (regular) curve can be parametrized by constant speed, we can assume without loss of generality that $|\gamma_0'(t)| \equiv C$, so that (9.2) can be rewritten as

$$\nabla_{\gamma_0'}\gamma_0' = 0. \tag{9.3}$$

We call (9.3) the *geodesic equation* and any constant speed curve $\gamma_0(t)$ that satisfies (9.3) is called a *geodesic*. For simplicity, let's assume from now on that every geodesic has a constant speed. We can express (9.3) using local coordinates. Suppose under a local coordinate system $F(u_1, \cdots, u_n)$, the coordinate representation of γ_0 is given by:

$$F^{-1} \circ \gamma_0(t) = (\gamma^1(t), \cdots, \gamma^n(t)).$$

Then $\gamma_0'(t)$ is given by

$$\gamma_0' := \frac{d\gamma^i}{dt}\frac{\partial}{\partial u_i},$$

so we have

$$\begin{aligned}
\nabla_{\gamma_0'}\gamma_0' &= \nabla_{\gamma_0'}\left(\frac{d\gamma^j}{dt}\frac{\partial}{\partial u_j}\right) \\
&= \frac{d^2\gamma^j}{dt^2}\frac{\partial}{\partial u_j} + \frac{d\gamma^j}{dt}\nabla_{\frac{d\gamma^i}{dt}\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j} \\
&= \frac{d^2\gamma^j}{dt^2}\frac{\partial}{\partial u_j} + \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k\frac{\partial}{\partial u_k} \\
&= \left(\frac{d^2\gamma^k}{dt^2} + \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k\right)\frac{\partial}{\partial u_k}
\end{aligned}$$

Hence, the geodesic equation (9.3) is locally a second-order ODE (assuming γ_0 is arc-length parametrized):

$$\frac{d^2\gamma^k}{dt^2} + \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k = 0 \quad \text{for any } k. \tag{9.4}$$

Example 9.6. Consider the round sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ which, under spherical coordinates, has its Riemannian metric given by

$$g = \sin^2\theta\, d\varphi^2 + d\theta^2.$$

By direct computations using (7.15), we get:

$$\begin{aligned}
\Gamma_{\varphi\varphi}^{\theta} &= -\sin\theta\cos\theta \\
\Gamma_{\varphi\theta}^{\varphi} = \Gamma_{\varphi\theta}^{\varphi} &= \cot\theta
\end{aligned}$$

and all other Christoffel symbols are zero. For instance,

$$\Gamma_{\varphi\varphi}^{\theta} = \frac{1}{2}g^{\theta\theta}\left(\frac{\partial}{\partial\varphi}g_{\varphi\theta} + \frac{\partial}{\partial\varphi}g_{\varphi\theta} - \frac{\partial}{\partial\theta}g_{\varphi\varphi}\right) + \frac{1}{2}g^{\theta\varphi}\left(\frac{\partial}{\partial\varphi}g_{\varphi\varphi} + \frac{\partial}{\partial\varphi}g_{\varphi\varphi} - \frac{\partial}{\partial\varphi}g_{\varphi\varphi}\right).$$

Note that the second term vanishes since $[g]$ is diagonal under this local coordinate system.

Write $\gamma(t)$ locally as $(\gamma^\varphi(t), \gamma^\theta(t))$, then the geodesic equations are given by

$$\begin{aligned}\frac{d^2\gamma^\varphi}{dt^2} + \frac{d\gamma^\varphi}{dt}\frac{d\gamma^\theta}{dt}\Gamma^\varphi_{\varphi\theta} + \frac{d\gamma^\theta}{dt}\frac{d\gamma^\varphi}{dt}\Gamma^\varphi_{\theta\varphi} &= 0\\ \frac{d^2\gamma^\theta}{dt^2} + \frac{d\gamma^\varphi}{dt}\frac{d\gamma^\varphi}{dt}\Gamma^\theta_{\varphi\varphi} &= 0,\end{aligned}$$

that is

$$\begin{aligned}\frac{d^2\gamma^\varphi}{dt^2} + 2\frac{d\gamma^\varphi}{dt}\frac{d\gamma^\theta}{dt}\cot\theta &= 0\\ \frac{d^2\gamma^\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\gamma^\varphi}{dt}\right)^2 &= 0.\end{aligned}$$

Clearly, the path with $(\gamma^\theta(t), \gamma^\varphi(t)) = (t, c)$, where c is a constant, is a solution to the system. This path is a great circle. □

Exercise 9.4. Show that geodesic is an isometric property, i.e. if $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ is an isometry, then γ is a geodesic of (M, g) if and only if $\Phi(\gamma)$ is a geodesic of $(\widetilde{M}, \widetilde{g})$.

Exercise 9.5. Consider the surface of revolution $F(u, \theta) = (x(u)\cos\theta, x(u)\sin\theta, z(u))$ given by a profile curve $(x(u), 0, z(u))$ on the xz-plane. Show that the profile curve itself is a geodesic of the surface.

Exercise 9.6. Consider the hyperbolic space with the upper-half plane model, i.e.

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}, \quad g = \frac{dx^2 + dy^2}{y^2}.$$

Show that straight lines normal to the x-axis, as well as semi-circles intersecting the x-axis orthogonally, are geodesics of the hyperbolic space.

Since (9.4) is a second-order ODE system, it has local existence and uniqueness when given both initial position and velocity. That is, given $p \in M$ and $V \in T_pM$, there exists a unique (constant speed) geodesic $\gamma_V : (-\varepsilon, \varepsilon) \to M$ such that $\gamma_V(0) = p$ and $\gamma_V'(0) = V$.

Let $c > 0$, the geodesic γ_{cV} (starting from p) is one that the speed is $c|V|$, and so it travels c times faster than γ_V does. One should then expect that $\gamma_{cV}(t) = \gamma_V(ct)$ provided ct is in the domain of γ_V. It can be easily shown to be true using uniqueness theorem ODE, since $\gamma_V(ct)\big|_{t=0} = \gamma_V(0) = p$ and

$$\left.\frac{d}{dt}\gamma_V(ct)\right|_{t=0} = c\gamma_V'(ct)\big|_{t=0} = c\gamma_V'(0) = cV.$$

Therefore, $\gamma_V(ct)$ is a curve initiating from p with initial velocity cV. It can be shown using the chain rule that $\gamma_V(ct)$ also satisfies (9.4). Hence, by uniqueness theorem of ODE, we must have $\gamma_V(ct) = \gamma_{cV}(t)$.

Note that a geodesic $\gamma(t)$ may not be defined globally on $(-\infty, \infty)$. Easy counter-examples are straight-lines $ax + by = 0$ on $\mathbb{R}^2\backslash\{0\}$. If *every* geodesic γ passing through p on a Riemannian manifold (M, g) can be extended so that its domain becomes $(-\infty, \infty)$, then we say (M, g) is *geodesically complete at* p. If (M, g) is geodesically complete is every point $p \in M$, then we say (M, g) is *geodesically complete*.

On a connected Riemannian manifold (M, g) there is a natural *metric space* structure, with the *metric*[1] $d : M \times M \to \mathbb{R}$ defined as:

$$\begin{aligned} & d(p, q) \\ & := \inf\{L(\gamma) | \gamma : [0, \tau] \to M \text{ is a piecewise-}C^\infty \text{ curve such that } \gamma(0) = p, \gamma(\tau) = q\}. \end{aligned}$$

Exercise 9.7. Check that d is a metric on M (in the sense of a metric space).

On a metric space, we can talk about *Cauchy completeness*, meaning that every Cauchy sequence in M with respect to d converges to a limit in M. Interestingly, the two notions of completeness are equivalent! It thanks to:

Theorem 9.7 (Hopf-Rinow). *Let (M, g) be a connected Riemannian manifold, and let $d : M \times M \to \mathbb{R}$ be the distance function induced by g. Then the following are equivalent:*

(1) There exists $p \in M$ such that (M, g) is geodesically complete at p

(2) (M, g) is geodesically complete

(3) (M, d) is Cauchy complete

We omit the proof in this note as it "tastes" differently from other parts of the course. Interested readers may consult any standard reference of Riemannian geometry to learn about the proof.

[1] Here *metric* means the distance function of a metric space, not a Riemannian *metric* on a manifold!

9.3. Exponential Map

9.3.1. Definition of the Exponential Map. Thanks to the Hopf-Rinow's Theorem, we will simply call *geodesically complete* Riemannian manifold (M, g) to be a *complete* Riemannian manifold. Such a metric g is called a *complete* Riemannian metric. On a complete Riemannian manifold (M, g), given any $p \in M$ and $V \in T_pM$, the unique geodesic $\gamma_V(t)$ with $\gamma_V(0) = p$ and $\gamma_V'(0) = V$ is defined for any $t \in (-\infty, \infty)$, and in particular, $\gamma_V(1)$ is well-defined. This is a point on M which is $|V|$ unit away from p along the geodesic in the direction of V. The map $V \mapsto \gamma_V(1)$ is an important map called:

Definition 9.8 (Exponential Map)**.** Let (M, g) be a complete Riemannian manifold. Fix a point $p \in M$ and given any tangent vector $V \in T_pM$, we consider the unique geodesic γ_V with initial conditions $\gamma_V(0) = p$ and $\gamma_V'(0) = V$. We define $\exp_p : T_pM \to M$ by

$$\exp_p(V) := \gamma_V(1).$$

The map $\exp_p$ is called the *exponential map at p*.

Remark 9.9. Standard ODE theory shows $\exp_p$ is a smooth map. For detail, see e.g. p.74 of John M. Lee's book. □

Lemma 9.10. *Let $p \in M$ which is a complete Riemannian manifold and $V \in T_pM$ be a fixed tangent. The push-forward of $\exp_p$ at $0 \in T_pM$, $(\exp_p)_{*0} : T_0(T_pM) \to T_pM$, is given by $(\exp_p)_{*0}(V) = V$ for any $V \in T_0(T_pM)$. Here we identify $T_0(T_pM)$ with T_pM.*

Proof. The key observation is the rescaling property $\gamma_{cV}(t) = \gamma_V(ct)$, which implies $\exp_p(cV) = \gamma_{cV}(1) = \gamma_V(c)$, and hence

$$\left.\frac{d}{dt}\right|_{t=0} (\exp_p)(0 + tV) = \left.\frac{d}{dt}\right|_{t=0} \gamma_V(t) = \gamma_V'(t) = V.$$

In other words, $(\exp_p)_{*0}(V) = V$. □

Clearly, $(\exp_p)_{*0}$ is invertible, and by inverse function theorem we can deduce that $\exp_p$ is locally a diffeomorphism near $0 \in T_pM$:

Corollary 9.11. *There exists an open ball $B(0, \varepsilon)$ on T_pM such that $\exp_p\big|_{B(0,\varepsilon)}$ is a diffeomorphism onto its image.*

To sum up, $\exp_p(V)$ is well-defined for all $V \in T_pM$ provided that (M, g) is complete. However, it may fail to be a diffeomorphism if the length of V is too large. The maximum possible length is called:

Definition 9.12 (Injectivity Radius)**.** Let (M, g) be a complete Riemannian manifold, and let $p \in M$. The *injectivity radius* of (M, g) at p is defined to be:

$$\text{inj}(p) := \sup\{r > 0 : \exp_p\big|_{B(0,r)} \text{ is a diffeomorphism onto its image}\}.$$

The *injectivity radius* of (M, g) is the minimum possible injectivity radius over all points on M, i.e.

$$\text{inj}(M, g) := \inf_{p \in M} \text{inj}(p).$$

Example 9.13. Injectivity radii may not be easy to be computed, but we have the following intuitive examples:

- For the round sphere $\mathbb{S}^2$ with radius R, the injectivity radius at every point is given by πR.
- For the flat Euclidean space, the injectivity radius at every point is $+\infty$.
- For a round torus obtained by rotating a circle with radius r, the injectivity radius at every point is πr.

□

9.3.2. Geodesic Normal Coordinates.

A consequence of Corollary 9.11 is that $\exp_p$ gives a local parametrization of M covering p. With a suitable modification, one can construct an important local coordinate system $\{u_i\}$, called the *geodesic normal coordinates*:

Proposition 9.14 (Existence of Geodesic Normal Coordinates)**.** *Let (M, g) be a Riemannian manifold. Then at every $p \in M$, there exists a local parametrization $G(u_1, \cdots, u_n)$ covering p, such that all of the following hold:*

(1) $g_{ij} := g\left(\dfrac{\partial}{\partial u_i}, \dfrac{\partial}{\partial u_j}\right) = \delta_{ij}$ *for any i, j at the point p;*

(2) $\Gamma_{ij}^k = 0$ *for any i, j, k at p; and*

(3) $\dfrac{\partial g_{ij}}{\partial u_k} = 0$ *for any i, j, k at p*

This local coordinate system is called the ***geodesic normal coordinates*** *at p.*

Proof. The key idea is to slightly modify the map $\exp_p$. By Gram-Schmidt's orthogonalization, one can take a basis $\{\hat{e}_i\}_{i=1}^n$ for T_pM which are orthonormal with respect to g, i.e. $g(\hat{e}_i, \hat{e}_j) = \delta_{ij}$. Then we define an isomorphism $E : \mathbb{R}^n \to T_pM$ by:

$$E(u_1, \cdots, u_n) := u_i \hat{e}_i.$$

Next we define the parametrization $G : E^{-1}(B(0, \varepsilon)) \to M$ by $G := \exp_p \circ E$. Here $\varepsilon > 0$ is sufficiently small so that $\exp_p \big|_{B(0,\varepsilon)}$ is a diffeomorphism onto its image.

Next we claim such G satisfies all three required conditions. To prove (1), we compute that:

$$\underbrace{\frac{\partial}{\partial u_i}(p)}_{\in T_pM} := \left.\frac{\partial G}{\partial u_i}\right|_0 = G_*\Bigg(\underbrace{\frac{\partial}{\partial u_i}(0)}_{\in T_0\mathbb{R}^n}\Bigg) = (\exp_p)_* \circ E_*\left(\frac{\partial}{\partial u_i}(0)\right).$$

From the definition of E, we have $E_*(\frac{\partial}{\partial u_i}) = \hat{e}_i$, and also at p, we have:

$$(\exp_p)_{*0}(\hat{e}_i) = \hat{e}_i$$

according to Lemma 9.10. Now we have proved $\frac{\partial}{\partial u_i}(p) = \hat{e}_i$. Hence, (1) follows directly from the fact that $\{\hat{e}_i\}$ is an orthonormal basis with respect to g:

$$g\left(\frac{\partial}{\partial u_i}(p), \frac{\partial}{\partial u_j}(p)\right) = g(\hat{e}_i, \hat{e}_j) = \delta_{ij}.$$

Next we claim that (2) is an immediate consequence of the geodesic equation (9.4). Consider the curve $\gamma(t) = \exp_p(t(\hat{e}_i + \hat{e}_j))$, which is a geodesic passing through p. Then, the local expression of $\gamma(t)$ is given by:

$$G^{-1} \circ \gamma(t) = (\gamma^1(t), \cdots, \gamma^n(t))$$

where

$$\gamma^k(t) = \begin{cases} t & \text{if } k = i \text{ or } j \\ 0 & \text{otherwise} \end{cases}.$$

By (9.4), we have:

$$\underbrace{\frac{d^2\gamma^k}{dt^2}}_{=0} + \Gamma_{ij}^k \underbrace{\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}}_{=1} = 0.$$

This shows $\Gamma_{ij}^k = 0$ along $\gamma(t)$ (see footnote[2]), and in particular $\Gamma_{ij}^k = 0$ at $p = \gamma(0)$.

Finally, (3) is an immediate consequence of (8.6).

□

The geodesic normal coordinates are one of the "gifts" to Riemannian geometry. In the later part of the course when we discuss geometric flows, we will see that it simplifies many tedious tensor computations. It is important to note that different points give rise to different geodesic normal coordinate systems! Note also that a geodesic normal coordinate system satisfies the three properties in Proposition 9.14 at **one point** p **only**. It is not always possible to pick a local coordinate system such that $g_{ij} = \delta_{ij}$ on the whole chart, unless the Riemannian manifold is locally flat.

Here is one demonstration of how useful geodesic normal coordinates are. Suppose M is a Riemannian manifold with a smooth family of Riemannian metrics $g(t)$, $t \in [0, T)$, such that $g(t)$ evolves in the direction of a family of symmetric 2-tensor $v(t)$, i.e.

$$\frac{\partial}{\partial t} g(t) = v(t).$$

In terms of local components, we may write $\frac{\partial}{\partial t} g_{ij}(t) = v_{ij}(t)$ where

$$g(t) = g_{ij}(t)\, du^i \otimes du^j, \ v(t) = v_{ij}(t)\, du^i \otimes du^j.$$

We want to derive the rate of change of the Christoffel symbols $\Gamma_{ij}^k(t)$'s, which change over time.

First we fix a time and a point (t_0, p). Note that $\frac{\partial}{\partial t}\Gamma_{ij}^k$ is tensorial since the difference $\Gamma_{ij}^k(t) - \Gamma_{ij}^k(t_0)$ is tensorial (even though $\Gamma_{ij}^k(t)$ itself isn't). Therefore, if one can express $\frac{\partial}{\partial t}\Gamma_{ij}^k$ at (t_0, p) as another tensorial quantity using *one particular* local coordinate system, then this expression holds true under *all other* local coordinate systems.

Let's choose the geodesic normal coordinates $\{u_i\}$ at p with respect to the metric $g(t_0)$. Then, we have

$$g_{ij}(t_0, p) = \delta_{ij}, \ \Gamma_{ij}^k(t_0, p), \ \frac{\partial g_{ij}}{\partial u_k}(t_0, p) = 0.$$

Recall that from (7.15) we have

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \right).$$

Taking time derivatives, we have:

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial}{\partial t} g^{kl}\right)\left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \right) + \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial u_i}\frac{\partial g_{jl}}{\partial t} + \frac{\partial}{\partial u_j}\frac{\partial g_{il}}{\partial t} - \frac{\partial}{\partial u_l}\frac{\partial g_{ij}}{\partial t} \right).$$

[2]Note that Γ_{ij}^k here is in fact $\Gamma_{ij}^k(\gamma(t))$, so one can only claim $\Gamma_{ij}^k = 0$ along $\gamma(t)$ but not at other points. Also, $\gamma(t)$ itself depends on the choice of i and j. This argument does *not* imply there is a geodesic on which all Christoffel's symbols vanish.

However, when evaluating at (t_0, p), the space derivatives $\frac{\partial g_{jl}}{\partial u_i}$'s all become zero, so we can ignore the first term above and obtain

$$\frac{\partial \Gamma_{ij}^k}{\partial t}(t_0, p) = \frac{1}{2} g^{kl} \left(\frac{\partial v_{jl}}{\partial u_i} + \frac{\partial v_{il}}{\partial u_j} - \frac{\partial v_{ij}}{\partial u_l} \right) \bigg|_{(t_0,p)}.$$

Note that $\frac{\partial v_{jl}}{\partial u_i}$'s are not tensorial, but at (t_0, p) all Christoffel symbols are zero, so by Exercise 8.7 we have

$$\frac{\partial v_{jl}}{\partial u_i} = \nabla_i v_{jl} \quad \text{at } (t_0, p),$$

and so

$$\frac{\partial \Gamma_{ij}^k}{\partial t} = \frac{1}{2} g^{kl} \left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij} \right) \quad \text{at } (t_0, p).$$

Note that now both sides are tensorial, so given that the above equation holds for one particular coordinate system, it holds for all other coordinate systems. Say $\{y_\alpha\}$, we will still have

$$\frac{\partial \Gamma_{\alpha\beta}^\gamma}{\partial t} = \frac{1}{2} g^{\gamma\eta} \big(\nabla_\alpha v_{\beta\eta} + \nabla_\beta v_{\alpha\eta} - \nabla_\eta v_{\alpha\beta} \big) \quad \text{at } (t_0, p).$$

Furthermore, even though this derivative expression holds at (t_0, p) only, we can repeat the same argument using geodesic normal coordinates with respect $g(t)$ at other time and at other points, so that we can conclude under *any* local coordinates on M, we have:

$$\frac{\partial \Gamma_{ij}^k}{\partial t} = \frac{1}{2} g^{kl} \left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij} \right).$$

Exercise 9.8. Given $g(t)$ is a smooth family of Riemannian metrics on M satisfying

$$\frac{\partial g_{ij}}{\partial t} = v_{ij}$$

where $v(t)$ is a smooth family of symmetric 2-tensors on M. Recall that the Laplacian of a scalar function f with respect to g is defined to be

$$\Delta_g f = g^{ij} \nabla_i \nabla_j f,$$

so it depends on t if $g(t)$ is time-dependent. Compute the evolution formula for:

$$\frac{\partial}{\partial t} \Delta_{g(t)} f$$

where f is a fixed (time-indepedent) scalar function.

Hint: first show that

$$\frac{\partial}{\partial t} g^{ij} = -g^{ip} g^{jq} \frac{\partial}{\partial t} g_{pq}.$$

Chapter 10

Curvatures of Riemannian Manifolds

10.1. Riemann Curvature Tensor

10.1.1. Motivations and Definitions. Recall that Gauss's Theorema Egregium is an immediate consequence of the Gauss's equation, which can be written using covariant derivatives as:

$$\nabla_i \nabla_j \frac{\partial}{\partial u_k} - \nabla_j \nabla_i \frac{\partial}{\partial u_k} = g^{ql}\left(h_{jk}h_{li} - h_{ik}h_{lj}\right)\frac{\partial}{\partial u_q}.$$

Taking the inner product with $\frac{\partial}{\partial u_p}$ on both sides, we get

$$\begin{aligned} g\left(\nabla_i \nabla_j \frac{\partial}{\partial u_k} - \nabla_j \nabla_i \frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_p}\right) &= g^{ql}\left(h_{jk}h_{li} - h_{ik}h_{lj}\right)g_{pq} \\ &= \delta_p^l\left(h_{jk}h_{li} - h_{ik}h_{lj}\right) \\ &= h_{jk}h_{pi} - h_{ik}h_{pj}. \end{aligned}$$

In dimension 2, we get

$$\det[h] = h_{11}h_{22} - h_{12}^2 = g(\nabla_1\nabla_2\partial_2 - \nabla_2\nabla_1\partial_2, \partial_1).$$

It motivates us to define a $(3,1)$-tensor T and a $(4,0)$-tensor S so that

$$T(\partial_1, \partial_2, \partial_2) = \nabla_1\nabla_2\partial_2 - \nabla_2\nabla_1\partial_2 \quad \text{and} \quad S(\partial_1, \partial_2, \partial_2, \partial_1) = g(\nabla_1\nabla_2\partial_2 - \nabla_2\nabla_1\partial_2, \partial_1),$$

then the Gauss curvature in the 2-dimension case can be expressed in tensor notations as

$$K = \frac{T_{122}^q g_{1q}}{g_{11}g_{22} - g_{12}^2} = \frac{S_{1221}}{g_{11}g_{22} - g_{12}^2}.$$

Naturally, one may attempt:

$$T(X, Y, Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)$$

for the $(3,1)$-tensor. However, even though it indeed gives $T_{122} = \nabla_1\nabla_2\partial_2 - \nabla_2\nabla_1\partial_2$, one can easily verify that such T is not tensorial. We modify such an T a bit and define the following very important tensors in Riemannian geometry:

Definition 10.1 (Riemann Curvature Tensors)**.** Let (M, g) be a Riemannian manifold, then its *Riemann curvature* $(3,1)$*-tensor* is defined as

$$\mathrm{Rm}^{(3,1)}(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

and the *Riemann curvature* $(4,0)$*-tensor* is defined as:

$$\mathrm{Rm}^{(4,0)}(X, Y, Z, W) := g\big(\mathrm{Rm}^{(3,0)}(X, Y)Z, W\big).$$

If the tensor type is clear from the context, we may simply call it the *Riemann curvature tensor* and simply denote it by Rm. Alternative notations include R, $\mathcal{R}$, etc.

When $X = \frac{\partial}{\partial u_i}$ and $Y = \frac{\partial}{\partial u_j}$, we have $[X, Y] = 0$, so that it still gives

$$\mathrm{Rm}^{(3,1)}(\partial_1, \partial_2)\partial_2 = \nabla_1\nabla_2\partial_2 - \nabla_2\nabla_1\partial_2,$$

and at the same time $\mathrm{Rm}^{(3,1)}$ is tensorial.

Exercise 10.1. Verify that $\mathrm{Rm}^{(3,1)}$ is tensorial, i.e.

$$\mathrm{Rm}^{(3,1)}(fX, Y)Z = \mathrm{Rm}^{(3,1)}(X, fY)Z = \mathrm{Rm}^{(3,1)}(X, Y)fZ = f\mathrm{Rm}^{(3,1)}(X, Y)Z.$$

It is clear from the definition that $\mathrm{Rm}^{(3,1)}(X, Y)Z = -\mathrm{Rm}^{(3,1)}(Y, X)Z$. That explains why we intentionally write its input vectors by $(X, Y)Z$ instead of (X, Y, Z), so as to emphasize that X and Y are alternating.

We express the local components of $\mathrm{Rm}^{(3,1)}$ by R^l_{ijk}, so that

$$\mathrm{Rm}^{(3,1)} = R^l_{ijk}\, du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u^l}$$

where

$$\begin{aligned} R^l_{ijk}\frac{\partial}{\partial u_l} &= \mathrm{Rm}^{(3,1)}\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)\frac{\partial}{\partial u_k} \\ &= \nabla_i\big(\nabla_j\partial_k\big) - \nabla_j\big(\nabla_i\partial_k\big) - \nabla_{[\partial_i,\partial_j]}\partial_k \\ &= \left(\frac{\partial\Gamma^l_{jk}}{\partial u_i} - \frac{\partial\Gamma^l_{ik}}{\partial u_j} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}\right)\frac{\partial}{\partial u_l}. \end{aligned}$$

Therefore, the local components of $\mathrm{Rm}^{(3,1)}$ are given by

$$R^l_{ijk} = \frac{\partial\Gamma^l_{jk}}{\partial u_i} - \frac{\partial\Gamma^l_{ik}}{\partial u_j} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}.$$

For the $(4,0)$-tensor, the local components are given by:

$$R_{ijkl} = \mathrm{Rm}^{(4,0)}(\partial_i, \partial_j, \partial_k, \partial_l) = g\big(\mathrm{Rm}^{(3,1)}(\partial_i, \partial_j)\partial_k, \partial_l\big) = g\big(R^p_{ijk}\partial_p, \partial_l\big) = g_{pl}R^p_{ijk}.$$

10.1.2. Geometric Meaning of the Riemann Curvature $(3,1)$-Tensor. While R^l_{ijk} and R_{ijkl} are related to the Gauss curvature in the two dimension case, it is not obvious what are their geometric meanings in higher dimensions at the first glance. In fact, the Riemann curvature $(3,1)$-tensor has a non-trivial relation with parallel transports. Given tangent vectors X_1, X_2, Y at p, the vector $\mathrm{Rm}^{(3,1)}(X_1, X_2)Y$ at p in fact measures the defect between the parallel transport of Y along two different paths.

For simplicity, consider local coordinates $\{u_i\}$ such that p has coordinates $(0, \cdots, 0)$, and let's only consider the form $\mathrm{Rm}(\partial_1, \partial_2)Y$.

Take an arbitrary $Y(p) \in T_pM$. We are going to transport $Y(p)$ in two different paths. One path is first along the u_1-direction, then along the u_2-direction; another is the opposite: along u_2-direction first, followed by the u_1-direction.

Let's consider the former path: first u_1, then u_2. Let $Y = Y^k \frac{\partial}{\partial u_k}$ be the transported vector, then along the first u_1-segment $(u_1, u_2) = (u_1, 0)$, the components Y^k satisfy the equation:

$$\nabla_{\partial_1} Y = 0 \implies \frac{\partial Y^k}{\partial u_1} + Y^j \Gamma^k_{1j} = 0 \text{ for any } k.$$

Provided that we are sufficiently close to p, we have the Taylor expansion:

$$Y^k(u_1, 0) = Y^k(0,0) + \left.\frac{\partial Y^k}{\partial u_1}\right|_{(0,0)} (u_1 - 0) + \frac{1}{2} \left.\frac{\partial^2 Y^k}{\partial u_1^2}\right|_{(0,0)} (u_1 - 0)^2 + O(u_1^3).$$

Using the parallel transport equation, both the first and second derivatives of Y^k can be expressed in terms of Christoffel's symbols.

$$\begin{aligned}
\frac{\partial Y^k}{\partial u_1} &= -Y^j \Gamma^k_{1j} \\
\frac{\partial^2 Y^k}{\partial u_1^2} &= -\frac{\partial Y^j}{\partial u_1} \Gamma^k_{1j} - Y^j \frac{\partial \Gamma^k_{1j}}{\partial u_1} \\
&= Y^l \Gamma^j_{1l} \Gamma^k_{1j} - Y^j \frac{\partial \Gamma^k_{1j}}{\partial u_1}
\end{aligned}$$

These give the local expression of Y^k along the first segment $(u_1, 0)$:

$$Y^k(u_1, 0) = Y^k(0,0) - Y^j \Gamma^k_{1j}\Big|_{(0,0)} u_1 + \frac{1}{2}\left(Y^l \Gamma^j_{1l} \Gamma^k_{1j} - Y^j \frac{\partial \Gamma^k_{1j}}{\partial u_1}\right)\Bigg|_{(0,0)} u_1^2 + O(u_1^3).$$

Next, we consider the second segment: transporting $Y(u_1, 0)$ along the u_2-curve. By a similar Taylor expansion, we get:

$$Y^k(u_1, u_2) = Y^k(u_1, 0) + \left.\frac{\partial Y^k}{\partial u_2}\right|_{(u_1,0)} u_2 + \frac{1}{2} \left.\frac{\partial^2 Y^k}{\partial u_2^2}\right|_{(u_1,0)} u_2^2 + O(u_2^3).$$

As before, we next rewrite the derivative terms using Christoffel's symbols. The parallel transport equation along the u_2-curve gives:

$$\begin{aligned}
\frac{\partial Y^k}{\partial u_2} &= -Y^j \Gamma^k_{2j} \\
\frac{\partial^2 Y^k}{\partial u_2^2} &= Y^l \Gamma^j_{2l} \Gamma^k_{2j} - Y^j \frac{\partial \Gamma^k_{2j}}{\partial u_2}.
\end{aligned}$$

Plugging these in, we get:

$$Y^k(u_1, u_2) = Y^k(u_1, 0) - Y^j \Gamma^k_{2j}\Big|_{(u_1,0)} u_2 + \frac{1}{2}\left(Y^l \Gamma^j_{2l} \Gamma^k_{2j} - Y^j \frac{\partial \Gamma^k_{2j}}{\partial u_2}\right)\Bigg|_{(u_1,0)} u_2^2 + O(u_2^3).$$

Each term $Y^k(u_1, 0)$ above can then be expressed in terms of $Y^k(0,0)$ and $\Gamma^k_{ij}(0,0)$ using our previous calculations. It seems a bit tedious, but if we just keep terms up to second-order, then we can easily see that:

$$\begin{aligned}
Y^k(u_1,u_2) &= Y^k(0,0) - Y^j\Gamma^k_{1j}\Big|_{(0,0)} u_1 + \frac{1}{2}\left(Y^l\Gamma^j_{1l}\Gamma^k_{1j} - Y^j\frac{\partial\Gamma^k_{1j}}{\partial u_1}\right)\Big|_{(0,0)} u_1^2 \\
&\quad - \left(Y^j(0,0) - Y^l(0,0)\Gamma^j_{1l}(0,0)u_1\right)\Gamma^k_{2j}(u_1,0)\,u_2 \\
&\quad + \frac{1}{2}\left(Y^l\Gamma^j_{2l}\Gamma^k_{2j} - Y^j\frac{\partial\Gamma^k_{2j}}{\partial u_2}\right)\Big|_{(u_1,0)} u_2^2 + O\big(\left|(u_1,u_2)\right|^3\big) \\
&= Y^k(0,0) - Y^j\Gamma^k_{1j}\Big|_{(0,0)} u_1 + \frac{1}{2}\left(Y^l\Gamma^j_{1l}\Gamma^k_{1j} - Y^j\frac{\partial\Gamma^k_{1j}}{\partial u_1}\right)\Big|_{(0,0)} u_1^2 \\
&\quad - \left(Y^j(0,0) - Y^l(0,0)\Gamma^j_{1l}(0,0)u_1\right)\Gamma^k_{2j}(0,0)\,u_2 - Y^j(0,0)\frac{\partial\Gamma^k_{2j}}{\partial u_1}\Big|_{(0,0)} u_1u_2 \\
&\quad + \frac{1}{2}\left(Y^l\Gamma^j_{2l}\Gamma^k_{2j} - Y^j\frac{\partial\Gamma^k_{2j}}{\partial u_2}\right)\Big|_{(0,0)} u_2^2 + O\big(\left|(u_1,u_2)\right|^3\big).
\end{aligned}$$

Next, we consider the parallel transport in another path, first along u_2, then along u_1. Let $\widetilde{Y} = \widetilde{Y}^k\frac{\partial}{\partial u_k}$ be the local expression of the transported vector, then the local expression of $\widetilde{Y}^k$ at (u_1,u_2) can be obtained by switching 1 and 2 of that of $Y^k(u_1,u_2)$:

$$\begin{aligned}
\widetilde{Y}^k(u_1,u_2) &= \widetilde{Y}^k(0,0) - \widetilde{Y}^j\Gamma^k_{2j}\Big|_{(0,0)} u_2 + \frac{1}{2}\left(\widetilde{Y}^l\Gamma^j_{2l}\Gamma^k_{2j} - \widetilde{Y}^j\frac{\partial\Gamma^k_{2j}}{\partial u_2}\right)\Big|_{(0,0)} u_2^2 \\
&\quad - \left(\widetilde{Y}^j(0,0) - \widetilde{Y}^l(0,0)\Gamma^j_{2l}(0,0)u_2\right)\Gamma^k_{1j}(0,0)\,u_1 - \widetilde{Y}^j(0,0)\frac{\partial\Gamma^k_{1j}}{\partial u_2}\Big|_{(0,0)} u_1u_2 \\
&\quad + \frac{1}{2}\left(\widetilde{Y}^l\Gamma^j_{1l}\Gamma^k_{1j} - \widetilde{Y}^j\frac{\partial\Gamma^k_{1j}}{\partial u_1}\right)\Big|_{(0,0)} u_1^2 + O\big(\left|(u_1,u_2)\right|^3\big).
\end{aligned}$$

Recall that $Y(0,0) = \widetilde{Y}(0,0)$. By comparing $Y(u_1,u_2)$ and $\widetilde{Y}(u_1,u_2)$, we get:

$$\begin{aligned}
&Y^k(u_1,u_2) - \widetilde{Y}^k(u_1,u_2) \\
&= Y^l\left(\Gamma^j_{1l}\Gamma^k_{2j} - \Gamma^j_{2l}\Gamma^k_{1j} + \frac{\partial\Gamma^k_{1l}}{\partial u_2} - \frac{\partial\Gamma^k_{2l}}{\partial u_1}\right)\Big|_{(0,0)} u_1u_2 + O\big(\left|(u_1,u_2)\right|^3\big) \\
&= R^k_{21l}Y^l\Big|_{(0,0)} u_1u_2 + O\big(\left|(u_1,u_2)\right|^3\big).
\end{aligned}$$

In other words, we have:

$$Y(u_1,u_2) - \widetilde{Y}(u_1,u_2) = \mathrm{Rm}(\partial_2,\partial_1)Y\Big|_{(0,0)} u_1u_2 + O\big(\left|(u_1,u_2)\right|^3\big).$$

Therefore, the vector $\mathrm{Rm}(\partial_2,\partial_1)Y$ at p measures the difference between the parallel transports of those two different paths.

10.1.3. Symmetric properties and Bianchi identities. We will explore more about the geometric meaning of the $\mathrm{Rm}^{(4,0)}$-tensor in the next section. Meanwhile, let's discuss some nice algebraic properties of this tensor. The Riemann curvature $(4,0)$-tensor satisfies some nice symmetric properties. The first two indices ij, and the last two indices kl of the components R_{ijkl}, and is symmetric if one swap the whole ij with the whole kl. Precisely, we have:

Proposition 10.2 (Symmetric Properties of $\mathrm{Rm}^{(4,0)}$). *The local components of* $\mathrm{Rm}^{(4,0)}$ *satisfy the following properties:*

- $R_{ijkl} = -R_{jikl} = -R_{ijlk}$, *and*
- $R_{ijkl} = R_{klij}$.

Proof. The fact that $R_{ijkl} = -R_{jikl}$ follows immediately from the definition of $\mathrm{Rm}^{(3,1)}$. It only remains to show $R_{ijkl} = R_{klij}$, then $R_{ijkl} = -R_{ijlk}$ would follow immediately. We prove it by picking geodesic normal coordinates at a fixed point p, so that

$$g_{ij} = \delta_{ij},\ \partial_k g_{ij} = 0, \text{ and } \Gamma^k_{ij} = 0 \text{ at } p.$$

Then at p, we have:

$$\begin{aligned}
&R_{ijkl} - R_{klij}\\
&= g_{pl}R^p_{ijk} - g_{pj}R^p_{kli}\\
&= g_{pl}\left(\frac{\partial\Gamma^p_{jk}}{\partial u_i} - \frac{\partial\Gamma^p_{ik}}{\partial u_j} + \Gamma^q_{jk}\Gamma^p_{iq} - \Gamma^q_{ik}\Gamma^p_{jq}\right)\\
&\quad - g_{pj}\left(\frac{\partial\Gamma^p_{li}}{\partial u_k} - \frac{\partial\Gamma^p_{ki}}{\partial u_l} + \Gamma^q_{li}\Gamma^p_{kq} - \Gamma^q_{ki}\Gamma^p_{lq}\right)\\
&= \frac{\partial\Gamma^l_{jk}}{\partial u_i} - \frac{\partial\Gamma^l_{ik}}{\partial u_j} - \frac{\partial\Gamma^j_{li}}{\partial u_k} + \frac{\partial\Gamma^l_{ki}}{\partial u_l}.
\end{aligned}$$

Note that in geodesic normal coordinates we only warrant $\Gamma^k_{ij} = 0$ at a point, which does not imply its derivatives vanish at that point! Next we recall that

$$\Gamma^l_{jk} = \frac{1}{2}g^{lp}\left(\frac{\partial g_{kp}}{\partial u_j} + \frac{\partial g_{jp}}{\partial u_k} - \frac{\partial g_{jk}}{\partial u_l}\right),$$

and so at p we have:

$$\frac{\partial\Gamma^l_{jk}}{\partial u_i} = \frac{1}{2}g^{lp}\left(\frac{\partial^2 g_{kp}}{\partial u_i\partial u_j} + \frac{\partial^2 g_{jp}}{\partial u_i\partial u_k} - \frac{\partial^2 g_{jk}}{\partial u_i\partial u_l}\right) = \frac{1}{2}\left(\frac{\partial^2 g_{kl}}{\partial u_i\partial u_j} + \frac{\partial^2 g_{jl}}{\partial u_i\partial u_k} - \frac{\partial^2 g_{jk}}{\partial u_i\partial u_p}\right).$$

By permutating the indices, one can find out similar expressions for

$$\frac{\partial\Gamma^l_{ik}}{\partial u_j},\ \frac{\partial\Gamma^j_{li}}{\partial u_k},\ \text{and}\ \frac{\partial\Gamma^l_{ki}}{\partial u_l}.$$

Then by cancellations one can verify that $R_{ijkl} - R_{klij} = 0$ at p. Since $R_{ijkl} - R_{klij}$ is tensorial, it holds true under any local coordinate system and at every point. □

Another nice property of $\mathrm{Rm}^{(4,0)}$ is the following pair of Bianchi identities, which assert that the indices of the tensor exhibit some cyclic relations.

Proposition 10.3 (Bianchi Identities). *The local components of* $\mathrm{Rm}^{(4,0)}$ *satisfy the following: for any i, j, k, l, p, we have*

$$R_{(ijk)l} := R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$
$$\nabla_{(i}R_{jk)lp} := \nabla_i R_{jklp} + \nabla_j R_{kilp} + \nabla_k R_{ijlp} = 0.$$

Sketch of proof. Both can be proved using geodesic normal coordinates. Consider the geodesic normal coordinates $\{u_i\}$ at a point p, then as in the proof of Proposition 10.2, we have

$$R_{ijkl} = \frac{\partial\Gamma^l_{jk}}{\partial u_i} - \frac{\partial\Gamma^l_{ik}}{\partial u_j} \text{ at } p.$$

By relabelling indices, we can write down the local expressions of R_{jkil} and R_{kijl}. By summing them up, one can prove the first Bianchi identity by cancellations.

The second Bianchi identity can be proved in similar way. However, we have to be careful not to apply $\Gamma_{ij}^k(p) = 0$ too early! Even though one has $\Gamma_{ij}^k = 0$ at a point p, it does not warrant its derivatives equal 0 at p. The same for $\partial_k g_{ij}$, which are all zero at p, but it does not imply $\partial_l \partial_k g_{ij} = 0$ at p.

We sketch the proof and leave the detail for readers to fill in. Under geodesic normal coordinates at p, we have

$$\nabla_i R_{jklp} = \partial_i R_{jklp}, \quad \text{at } p.$$

Now consider

$$\frac{\partial}{\partial u_i} R_{jklp} = \frac{\partial}{\partial u_i} \left\{ g_{qp} \left(\frac{\partial \Gamma_{kl}^q}{\partial u_j} - \frac{\partial \Gamma_{jl}^q}{\partial u_k} + \Gamma_{kl}^m \Gamma_{jm}^q - \Gamma_{jl}^m \Gamma_{km}^q \right) \right\}.$$

Use (7.15) again to write Γ_{ij}^k's in terms of derivatives of g_{ij}'s. Be caution that to evaluate at p only after differentiation by $\frac{\partial}{\partial u_i}$. Get similar expressions for $\nabla_j R_{kilp}$ and $\nabla_k R_{ijlp}$ at p. One then should see all terms got cancelled when summing them up. □

Exercise 10.2. Prove the second Bianchi identity using geodesic normal coordinates.

All of the above-mentioned symmetric properties and Bianchi identities can be written using invariant notations instead of local coordinates as follows:

- $\mathrm{Rm}(X, Y, Z, W) = -\mathrm{Rm}(Y, X, Z, W) = -\mathrm{Rm}(X, Y, W, Z)$
- $\mathrm{Rm}(X, Y, Z, W) = \mathrm{Rm}(Z, W, X, Y)$
- $\mathrm{Rm}(X, Y, Z, W) + \mathrm{Rm}(Y, Z, X, W) + \mathrm{Rm}(Z, X, Y, W) = 0$
- $(\nabla_X \mathrm{Rm})(Y, Z, W, U) + (\nabla_Y \mathrm{Rm})(Z, X, W, U) + (\nabla_Z \mathrm{Rm})(X, Y, W, U) = 0$

for any vector fields X, Y, Z, W, U, V.

10.1.4. Isometric invariance. The Riemann curvature tensors (both types) can be shown to be invariant under isometries, meaning that if $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ is an isometry (i.e. $\Phi^* \widetilde{g} = g$), then we have $\Phi^* \widetilde{\mathrm{Rm}} = \mathrm{Rm}$. To prove this, we first show that the Levi-Civita connection is also isometric invariant.

Proposition 10.4. *Suppose $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ is an isometry, then we have*

$$\Phi_*(\nabla_X Y) = \widetilde{\nabla}_{\Phi_* X}(\Phi_* Y)$$

for any $X, Y \in \Gamma^\infty(TM)$. Here ∇ and $\widetilde{\nabla}$ are the Levi-Civita connections for g and $\widetilde{g}$ respectively.

Idea of Proof. While it is possible to give a proof by direct computations using local coordinates, there is a much smarter way of doing it. Here we outline the idea of proof and leave the detail for readers to fill in. The desired result is equivalent to:

$$\nabla_X Y = (\Phi^{-1})_* \big(\widetilde{\nabla}_{\Phi_* X}(\Phi_* Y) \big).$$

We define an operator $D : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ by:

$$D(X, Y) = (\Phi^{-1})_* \big(\widetilde{\nabla}_{\Phi_* X}(\Phi_* Y) \big).$$

Then, one can verify that such D is a connection on M satisfying conditions (8.7) and (8.8). This shows D must be the Levi-Civita connection of M by Proposition (8.14), completing the proof. □

Exercise 10.3. Complete the detail of the proof of Proposition 10.4.

Now given that the Levi-Civita connection is isometric invariant, it follows easily that Rm is too.

Proposition 10.5. *Suppose* $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ *is an isometry, then we have*

$$\Phi^*\widetilde{\mathrm{Rm}} = \mathrm{Rm}$$

where both Riemann curvature tensors are of $(4, 0)$*-type.*

Proof. Let X, Y, Z, W be four tangent vectors on M, then we have:

$$\begin{aligned}
&(\Phi^*\widetilde{\mathrm{Rm}})(X, Y, Z, W)\\
&= \widetilde{\mathrm{Rm}}(\Phi_*X, \Phi_*Y, \Phi_*Z, \Phi_*W)\\
&= \widetilde{g}\Big(\widetilde{\nabla}_{\Phi_*X}\widetilde{\nabla}_{\Phi_*Y}(\Phi_*Z) - \widetilde{\nabla}_{\Phi_*Y}\widetilde{\nabla}_{\Phi_*X}(\Phi_*Z) - \widetilde{\nabla}_{[\Phi_*X,\Phi_*Y]}(\Phi_*Z),\ \Phi_*W\Big)\\
&= \widetilde{g}\Big(\widetilde{\nabla}_{\Phi_*X}\Phi_*(\nabla_Y Z) - \widetilde{\nabla}_{\Phi_*Y}\Phi_*(\nabla_X Z) - \widetilde{\nabla}_{\Phi_*[X,Y]}(\Phi_*Z),\ \Phi_*W\Big)\\
&= \widetilde{g}\Big(\Phi_*(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - \nabla_{[X,Y]}Z,\ \Phi_*W\Big)\\
&= (\Phi^*\widetilde{g})\big(\mathrm{Rm}(X, Y)Z, W\big) = g\big(\mathrm{Rm}(X, Y)Z, W\big) = \mathrm{Rm}(X, Y, Z, W).
\end{aligned}$$

□

Example 10.6. As derivatives (hence curvatures) are local properties, the above results also hold if Φ is just an isometry locally between two open subsets of two Riemannian manifolds. The flat metric δ on $\mathbb{R}^n$ certainly has $\mathrm{Rm} = 0$. Now for the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, it has an induced metric g from the covering map $\pi : \mathbb{R}^n \to \mathbb{T}^n$ (see Proposition 8.11) which is a local isometry since $\pi^*g = \delta$. This metric also has a zero Riemann curvature tensor by Proposition 10.5. As such, we call this torus with such an induced metric the *flat torus*. □

Example 10.7. Consider the round sphere $\mathbb{S}^n$, defined as $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$, with Riemannian metric given by the first fundamental form $g = \iota^*\delta$. From standard Lie theory, one can show the symmetry group $\mathrm{SO}(n+1)$ acts transitively on $\mathbb{S}^n$, meaning that given any $p, q \in \mathbb{S}^n$, there exists $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $\Phi(p) = q$.

$$g(\Phi_*X, \Phi_*Y) = \delta(\iota_*\Phi_*X, \iota_*\Phi_*Y) = \Phi(X) \cdot \Phi(Y) = X \cdot Y = g(X, Y).$$

Hence $\Phi^*g = g$, and consequently $\Phi^*\mathrm{Rm} = \mathrm{Rm}$. Therefore, if one can find out the Riemann curvature tensor at one point $p \in \mathbb{S}^n$, then we can determine this tensor at all other points on $\mathbb{S}^n$. In the next section, we will focus on the input type $\mathrm{Rm}(X, Y, Y, X)$, known as *sectional curvatures*, and will use it to determine the exact form of R_{ijkl} of the sphere. □

10.2. Sectional Curvature

The Riemann curvature tensor is defined in a way so that in the two dimensional case it (essentially) gives the Gauss curvature. The $\mathrm{Rm}^{(3,1)}$-tensor essentially measures the defect of parallel transports along two different paths. Here we will exploit more about the geometric meanings of the $\mathrm{Rm}^{(4,0)}$-tensor in higher dimensions.

In this section, we will explain two geometric quantities associated with the Riemann curvature tensor, namely sectional curvatures and the curvature operator. The former is inspired from the formula of Gauss curvature two dimensional case. The latter makes good use of the symmetric properties of R_{ijkl} and has deep connections with holonomy groups and parallel transports which we discussed earlier.

10.2.1. Definition of Sectional Curvature.

Recall that for a regular surface $\Sigma^2 \in \mathbb{R}^3$, its Gauss curvature equals to

$$K = \frac{R_{1221}}{g_{11}g_{22} - g_{12}^2} = \frac{\mathrm{Rm}(\partial_1, \partial_2, \partial_2, \partial_1)}{|\partial_1|^2 |\partial_2|^2 - g(\partial_1, \partial_2)^2}.$$

Inspired by this, we define the sectional curvatures as follows:

Definition 10.8 (Sectional Curvature). On a Riemannian manifold (M, g), consider two linearly independent tangent vectors X and Y in T_pM. We define the *sectional curvature at p associated to $\{X, Y\}$* by

$$K_p(X, Y) := \frac{\mathrm{Rm}(X, Y, Y, X)}{|X|^2 |Y|^2 - g(X, Y)^2}$$

Although it appears that $K_p(X, Y)$ depends on both X and Y, we can show it is in fact independent of the choice of basis in $\mathrm{span}\{X, Y\}$:

Proposition 10.9. *Let Π_p be a 2-dimensional subspace in T_pM, then given any bases $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ for Π_p we have:*

$$K_p(X_1, Y_1) = K_p(X_2, Y_2).$$

Proof. Let a, b, c, d be real constants such that

$$\begin{aligned} X_1 &= aX_2 + bY_2 \\ Y_2 &= cX_2 + dY_2 \end{aligned}$$

Then, we have

$$\begin{aligned}
&\mathrm{Rm}(X_1, Y_1, Y_1, X_1) \\
&= \mathrm{Rm}(aX_2 + bY_2, cX_2 + dY_2, cX_2 + dY_2, aX_2 + bY_2) \\
&= ad\,\mathrm{Rm}(X_2, Y_2, cX_2 + dY_2, aX_2 + bY_2) \\
&\quad + bc\,\mathrm{Rm}(Y_2, X_2, cX_2 + dY_2, aX_2 + bY_2) \\
&= ad\left(ad\,\mathrm{Rm}(X_2, Y_2, Y_2, X_2) + bc\,\mathrm{Rm}(X_2, Y_2, X_2, Y_2)\right) \\
&\quad + bc\left(ad\,\mathrm{Rm}(Y_2, X_2, Y_2, X_2) + bc\,\mathrm{Rm}(Y_2, X_2, X_2, Y_2)\right) \\
&= (ad - bc)^2\,\mathrm{Rm}(X_2, Y_2, Y_2, X_2).
\end{aligned}$$

The last step follows from symmetric properties of Rm.

$$\begin{aligned}
&|X_1|^2 |Y_1|^2 - g(X_1, Y_1)^2 \\
&= \left(a^2 |X_2|^2 + 2ab g(X_2, Y_2) + b^2 |Y_2|^2\right)\left(c^2 |X_2|^2 + 2cd g(X_2, Y_2) + d^2 |Y_2|^2\right) \\
&\quad - \left(ac |X_2|^2 + (ad + bc) g(X_2, Y_2) + bd |Y_2|^2\right)^2 \\
&= (ad - bc)^2 \left(|X_2|^2 |Y_2|^2 - g(X_2, Y_2)^2\right).
\end{aligned}$$

The last step follows from direct computations.

It follows easily from the definition that $K_p(X_1, Y_1) = K_p(X_2, Y_2)$. $\square$

Remark 10.10. Therefore, one can also define the sectional curvature at p associated to a *plane* (i.e. 2-dimensional subspace) Π in T_pM:

$$K_p(\Pi) := K_p(X, Y)$$

where $\{X, Y\}$ is any basis for Π. $\square$

Remark 10.11. Note that $K_p(X, Y)$ is not a $(2, 0)$-tensor! It is evident from the above result that $K(2X, Y) = K(X, Y) \neq 2K(X, Y)$. $\square$

Although the sectional curvature is essentially the Riemann curvature tensor restricted on inputs of type (X, Y, Y, X), it is interesting that $\mathrm{Rm}(X, Y, Z, W)$ itself can also be expressed in terms of sectional curvatures:

Proposition 10.12. *The Riemann curvature tensor* $\mathrm{Rm}^{(4,0)}$ *is uniquely determined by its sectional curvatures. Precisely, given any tangent vectors* $X, Y, Z, W \in T_pM$*, we have:*

$$\begin{aligned}
&\mathrm{Rm}(X, Y, Z, W) \\
&= \mathrm{Rm}(X + W, Y + Z, Y + Z, X + W) - \mathrm{Rm}(X + W, Y, Y, X + W) \\
&\quad - \mathrm{Rm}(X + W, Z, Z, X + W) - \mathrm{Rm}(X, Y + Z, Y + Z, X) \\
&\quad - \mathrm{Rm}(W, Y + Z, Y + Z, W) + \mathrm{Rm}(X, Z, Z, X) + \mathrm{Rm}(W, Y, Y, W) \\
&\quad - \mathrm{Rm}(Y + W, X + Z, X + Z, Y + W) + \mathrm{Rm}(Y + W, X, X, Y + W) \\
&\quad + \mathrm{Rm}(Y + W, Z, Z, Y + W) + \mathrm{Rm}(Y, X + Z, X + Z, Y) \\
&\quad + \mathrm{Rm}(W, X + Z, X + Z, W) - \mathrm{Rm}(Y, Z, Z, Y) - \mathrm{Rm}(W, X, X, W).
\end{aligned}$$

Proof. Omitted. We leave it as an exercise for readers who need to wait for half an hour for a morning minibus back to HKUST.

It is worthwhile to note that the result holds true when Rm is replaced by any $(4, 0)$-tensor T satisfying all of the following:

- $T_{ijkl} = -T_{jikl} = -T_{ijlk} = T_{klij}$
- $T_{ijkl} + T_{jkil} + T_{kijl} = 0$

$\square$

10.2.2. Geometric Meaning of Sectional Curvature.

We are going to explain the geometric meaning of sectional curvatures. Let X, Y be two linearly independent tangent vectors in T_pM of a Riemannian manifold (M, g). Consider the 2-dimensional surface $\Sigma_p(X, Y)$ obtained by spreading out geodesics from p along all directions in the plane $\mathrm{span}\{X, Y\}$. The sectional curvature $K_p(X, Y)$ can then be shown to be the Gauss curvature of the surface $\Sigma_p(X, Y)$ with the induced metric. Precisely, we have

Proposition 10.13. *Let (M,g) be a Riemannian manifold, $p \in M$, and $\{X,Y\}$ be two linearly independent tangents in T_pM. Consider the surface:*

$$\Sigma_p(X,Y) := \{\exp_p(uX+vY) : u,v \in \mathbb{R} \text{ and } |uX+vY| < \varepsilon\}$$

where $\operatorname{inj}(p) > \varepsilon > 0$. *Denote $\overline{g}$ to be the induced Riemannian metric $\iota^* g$ where $\iota : \Sigma_p(X,Y) \to M$ is the inclusion map. Then, we have:*

$$K_p(X,Y) = \text{Gauss curvature of } \Sigma_p(X,Y) \text{ at } p \text{ with respect to the metric } \overline{g}.$$

Proof. Denote ∇ and Rm the Levi-Civita connection and the Riemann curvature tensor of (M,g), and denote $\overline{\nabla}$ and $\overline{\mathrm{Rm}}$ to be those of $(\Sigma,\overline{g})$. Here we denote $\Sigma := \Sigma_p(X,Y)$ for simplicity.

The Gauss curvature of $(\Sigma,\overline{g})$ is given by

$$\frac{\overline{\mathrm{Rm}}(X,Y,Y,X)}{\overline{g}(X,X)\overline{g}(Y,Y)-\overline{g}(X,Y)^2}.$$

As we have $g = \overline{g}$ when restricted on $T_p\Sigma$, it suffices to show:

$$\mathrm{Rm}(X,Y,Y,X) = \overline{\mathrm{Rm}}(X,Y,Y,X).$$

By Exercise 8.6, we know that for any vector fields X,Y on Σ, we have

$$\overline{\nabla}_X Y = (\nabla_X Y)^T$$

where T denotes the projection onto $T_p\Sigma$. Hence, one can decompose $\nabla_X Y$ as:

$$\nabla_X Y = \overline{\nabla}_X Y + h(X,Y)$$

where $h(X,Y) = \big(\nabla_X Y\big)^N$, the projection onto the normal space of Σ.

Consider the local parametrization $G(u,v) = \exp_p(uX+vY)$ of Σ. As in the proof of Proposition 9.14, we have

$$\frac{\partial G}{\partial u}(p) = X, \quad \frac{\partial G}{\partial v}(p) = Y,$$

and also that Γ^*_{uu}, Γ^*_{uv}, and Γ^*_{vv} (here $*$ means any index) all vanish at p, or equivalently, we have:

$$\nabla_u \partial_u = \nabla_u \partial_v = \nabla_v \partial_v = 0 \quad \text{at } p.$$

In particular, it implies $h(\partial_u,\partial_u) = h(\partial_u,\partial_v) = h(\partial_v,\partial_v) = 0$ at p.

Now consider the relation between the two Riemann curvature tensors:

$$\begin{aligned}
\mathrm{Rm}(\partial_u,\partial_v)\partial_v &= \nabla_u\nabla_v\partial_v - \nabla_v\nabla_u\partial_v \\
&= \nabla_u\big(\overline{\nabla}_v\partial_v + h(\partial_v,\partial_v)\big) - \nabla_v\big(\overline{\nabla}_u\partial_v + h(\partial_u,\partial_v)\big) \\
&= \overline{\nabla}_u\overline{\nabla}_v\partial_v + h(\partial_u,\overline{\nabla}_v\partial_v) + \nabla_u\big(h(\partial_v,\partial_v)\big) \\
&\quad - \overline{\nabla}_v\overline{\nabla}_u\partial_v - h(\partial_v,\overline{\nabla}_u\partial_v) - \nabla_v\big(h(\partial_u,\partial_v)\big) \\
&= \overline{\mathrm{Rm}}(\partial_u,\partial_v)\partial_v + h(\partial_u,\overline{\nabla}_v\partial_v) - h(\partial_v,\overline{\nabla}_u\partial_v) \\
&\quad + \nabla_u\big(h(\partial_v,\partial_v)\big) - \nabla_v\big(h(\partial_u,\partial_v)\big)
\end{aligned}$$

Recall that $h(X,Y) \perp T\Sigma$ for any $X,Y \in \Gamma^\infty(T\Sigma)$, so we have

$$\begin{aligned}
&\mathrm{Rm}(\partial_u,\partial_v,\partial_v,\partial_u) = g\big(\mathrm{Rm}(\partial_u,\partial_v)\partial_v,\, \partial_v\big) \\
&= g\big(\overline{\mathrm{Rm}}(\partial_u,\partial_v)\partial_v,\, \partial_u\big) + g\Big(\nabla_u\big(h(\partial_v,\partial_v)\big) - \nabla_v\big(h(\partial_u,\partial_v)\big),\, \partial_u\Big).
\end{aligned}$$

We are only left to show the terms involving h vanish at p. Take an orthonormal frame[1] $\{e_\alpha\}$ of normal vectors to the surface Σ. Write $h(\partial_u, \partial_v) = h^\alpha_{uv} e_\alpha$ (and similarly for h^α_{uu} and h^α_{vv}). Then, one can compute that:

$$g\Big(\nabla_u\big(h(\partial_v,\partial_v)\big),\, \partial_u\Big) = g\Big(\nabla_u\big(h^\alpha_{vv}e_\alpha\big),\, \partial_u\Big) = g\big(h^\alpha_{vv}\nabla_u e_\alpha,\, \partial_u\big).$$

Here we have used the fact that $e_\alpha \perp \partial_u$. Using this fact again, one can also show that

$$g\big(\nabla_u e_\alpha, \partial_u\big) = -g(e_\alpha, \nabla_u \partial_u) = 0 \text{ at } p.$$

Similarly, one can also show

$$g\Big(\nabla_v\big(h(\partial_u,\partial_v)\big),\, \partial_u\Big) = 0 \text{ at } p.$$

It completes our proof that

$$\mathrm{Rm}(\partial_u,\partial_v,\partial_v,\partial_u) = \overline{\mathrm{Rm}}(\partial_u,\partial_v,\partial_v,\partial_u) \text{ at } p,$$

and the desired result follows from the fact that $\partial_u = X$ and $\partial_v = Y$ at p. □

As we have seen in the proof above, the key idea is to relate Rm and $\overline{\mathrm{Rm}}$. In the above proof it suffices to consider inputs of type (X,Y,Y,X), yet it is not difficult to generalize the calculation to relate $\mathrm{Rm}(X,Y,Z,W)$ and $\overline{\mathrm{Rm}}(X,Y,Z,W)$ of any input types. This in fact gives a generalized Gauss's equation, which we leaves it as an exercise for readers.

Exercise 10.4. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇, and $(\Sigma,\overline{g})$ be a submanifold of M with induced metric $\overline{g}$ and Levi-Civita connection $\overline{\nabla}$. Given any vector field $X, Y \in T\Sigma$, we denote

$$h(X,Y) := (\nabla_X Y)^N = \text{normal projection of } \nabla_X Y \text{ onto the normal space } N\Sigma.$$

Prove that for any $X,Y,Z,W \in T\Sigma$, we have

$$\begin{aligned}&\mathrm{Rm}(X,Y,Z,W)\\ &= \overline{\mathrm{Rm}}(X,Y,Z,W) + h(X,Z)h(Y,W) - h(X,W)h(Y,Z).\end{aligned}$$

10.2.3. Constant Sectional Curvature Metrics. A Riemannian metric is said to have *constant sectional curvature* if $K_p(\Pi)$ is independent of both p and the choice of plane $\Pi \subset T_pM$. We will show that such metric has local components of $\mathrm{Rm}^{(4,0)}$ given by:

$$R_{ijkl} = C(g_{il}g_{jk} - g_{ik}g_{jl})$$

where C is a real constant. To begin, we first introduce a special product for a pair of 2-tensors, commonly known as the *Kulkarni-Nomizu's product*. Given two symmetric 2-tensors h and k, we define $h \owedge k$ as a 4-tensor given by:

$$\begin{aligned}&(h \owedge k)(X,Y,Z,W)\\ &:= h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W).\end{aligned}$$

It is straight-forward to show that whenever h and k are both **symmetric**, then the Kulkarni-Nomizu's product $h \owedge k$ satisfies symmetric properties and the first Bianchi identity like the $\mathrm{Rm}^{(4,0)}$-tensor:

- $(h \owedge k)_{ijkl} = -(h \owedge k)_{jikl} = -(h \owedge k)_{ijlk} = (h \owedge k)_{klij}$; and
- $(h \owedge k)_{ijkl} + (h \owedge k)_{jkil} + (h \owedge k)_{kijl} = 0$.

[1]"Frame" typically means a set of locally defined vector fields which are linearly independent.

Exercise 10.5. Prove the above symmetric properties and the first Bianchi identity for $h \owedge k$.

Therefore, the $(4,0)$-tensor $h \owedge k$ is completely determined by its values of type

$$(h \owedge k)(X, Y, Y, X)$$

(see Proposition 10.12). Now we consider the product $g \owedge g$, which is given by

$$(g \owedge g)(X, Y, Z, W) = 2g(X, Z)g(Y, W) - 2g(X, W)g(Y, Z),$$

or locally, $(g \owedge g)_{ijkl} = 2g_{ik}g_{jl} - 2g_{il}g_{jk}$.

One can easily see that

$$(g \owedge g)(X, Y, Y, X) = 2\big(g(X, Y)^2 - g(X, X)g(Y, Y)\big).$$

Now if (M, g) has constant sectional curvature, then there exists a constant C such that $K_p(X, Y) = C$ for any $p \in M$ and any linearly independent vectors $\{X, Y\} \subset T_pM$. In other words, we have

$$\mathrm{Rm}(X, Y, Y, X) = C\big(g(X, X)g(Y, Y) - g(X, Y)^2\big) = -\frac{C}{2}(g \owedge g)(X, Y, Y, X).$$

Since $\mathrm{Rm} + \frac{C}{2}(g \owedge g)$ is a $(4,0)$-tensor satisfying symmetric properties and the first Bianchi's identity, and it equals zero when acting on any (X, Y, Y, X), we can conclude that $\mathrm{Rm} + \frac{C}{2}(g \owedge g) \equiv 0$. To conclude, we have proved that if $K_p(X, Y) = C$ for any $p \in M$ and any linearly independent vectors $\{X, Y\} \subset T_pM$, then

$$\begin{aligned} \mathrm{Rm}(X, Y, Z, W) &= C\big(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\big), \\ R_{ijkl} &= C(g_{il}g_{jk} - g_{ik}g_{jl}). \end{aligned}$$

Example 10.14. The most straight-forward example of metrics with constant sectional curvature is the Euclidean space $\mathbb{R}^n$ with the flat metric δ. We have $\mathrm{Rm} \equiv 0$. Consequently, the n-torus $\mathbb{T}^n$ equipped with the quotient metric given by the covering $\pi : \mathbb{R}^n \to \mathbb{T}^n$ also has 0 sectional curvature. □

Example 10.15. The round sphere $\mathbb{S}^n$ of radius r, given by $x_1^2 + \cdots + x_{n+1}^2 = r^2$, has constant sectional curvature $\frac{1}{r^2}$. The key reason is that geodesics are great circles of radius r. Given any linearly independent vectors $X, Y \in T_p\mathbb{S}^n$, the sectional surface $\Sigma_p(X, Y)$ considered in Proposition 10.13 is formed by spreading out geodesics from p. As each geodesic has the same curvature $\frac{1}{r}$, the principal curvatures of $\Sigma_p(X, Y)$ at p are $\{\frac{1}{r}, \frac{1}{r}\}$, showing that $\Sigma_p(X, Y)$ has Gauss curvature $\frac{1}{r^2}$ at p.

This shows $R_{ijkl} = C(g_{il}g_{jk} - g_{ik}g_{jl})$ for some constant C. Next we try to find out what this C is. Consider the sectional curvature associated to $\{\partial_i, \partial_j\}$, then we have:

$$\frac{1}{r^2} = K(\partial_i, \partial_j) = \frac{R_{ijji}}{g_{ii}g_{jj} - g_{ij}g_{ji}}.$$

By the fact that $R_{ijji} = C(g_{ii}g_{jj} - g_{ij}g_{ji})$ for constant sectional curvature metrics, we must have $C = \frac{1}{r^2}$, and we conclude that:

$$R_{ijkl} = \frac{1}{r^2}(g_{il}g_{jk} - g_{ik}g_{jl}).$$

□

Example 10.16. The hyperbolic space $\mathbb{H}^n$ under the upper-half space model:

$$\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 > 0\}$$

can be easily shown to have constant sectional curvatures by direct computations. Recall that its Riemannian metric is given by:

$$g = \frac{\sum_{i=1}^n dx^i \otimes dx^i}{x_1^2}.$$

Under the global coordinates $\{x_i\}$, the metric components $g_{ij} = \frac{1}{x_1^2}\delta_{ij}$ forms a diagonal matrix, so $g^{ij} = x_1^2 \delta_{ij}$. It follows that $\partial_k g_{ij} = -2x_1^{-3}\delta_{k1}\delta_{ij}$.

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2} g^{kl}\big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\big) = \frac{1}{2} x_1^2 \big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big) \\
&= \frac{1}{2} x_1^2 \big(- 2x_1^{-3}\delta_{i1}\delta_{jk} - 2x_1^{-3}\delta_{j1}\delta_{ik} + 2x_1^{-3}\delta_{k1}\delta_{ij}\big) \\
&= \frac{1}{x_1}\big(\delta_{k1}\delta_{ij} - \delta_{i1}\delta_{jk} - \delta_{j1}\delta_{ik}\big).
\end{aligned}$$

By the local expression:

$$R_{ijkq} = g_{ql}\left(\frac{\partial \Gamma_{jk}^l}{\partial u_i} - \frac{\partial \Gamma_{ik}^l}{\partial u_j} + \Gamma_{jk}^p \Gamma_{ip}^l - \Gamma_{ik}^p \Gamma_{jp}^l\right),$$

it is straight-forward to verify that $R_{ijkl} = C(g \owedge g)_{ijkl}$ for some constant $C > 0$. This is equivalent to saying that the metric has negative constant sectional curvature. We leave it as an exercise for readers to complete the computations. $\square$

Exercise 10.6. Complete the computations of R_{ijkq} for the hyperbolic space under the upper-half plane model. Find out the $C > 0$ and the sectional curvature explicitly.

Also, consider the rescaled hyperbolic metric $\widetilde{g} := \alpha g$, where $\alpha > 0$ is a constant. What is the sectional curvature of $\widetilde{g}$?

The above examples of metrics on $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$, and their quotients (such as $\mathbb{T}^n$, $\mathbb{RP}^n$, and higher genus tori) in fact form a complete list of geodesically complete constant sectional curvature metrics (certainly, up to isometry). This is one major goal of the next chapter, which discusses second variations of arc-lengths, index form, etc. in order to establish such a classification result.

Brendle-Schoen's Differentiable Sphere Theorem

A typical type of questions that geometers and topologists would like to ask is given some conditions on curvatures, then what can we say about the topology of the manifold? In 1951, H.E. Rauch posted a question of whether a compact, simply-connected Riemannian manifold whose sectional curvatures are all bounded in $(\frac{1}{4}, 1]$ must be topologically a sphere. In 1960, M. Berger and W. Klingenberg gave an affirmative answer to the question. The result is sharp in a sense that $\mathbb{CP}^n$ with Fubini-Study metric has sectional curvature $\frac{1}{4}$ along holomorphic planes.

The results of Berger and Klingenberg are topological – they showed such a manifold must be *homeomorphic* to $\mathbb{S}^n$, but higher dimensional spheres can have many exotic differential structures (see Milnor's exotic spheres). It was a long-standing conjecture of whether the $\frac{1}{4}$-pinched condition would warrant the sphere must have the standard differential structure. In 2007, Simon Brendle and Richard Schoen (both at Stanford at that time) gave an affirmative answer to this conjecture using the Ricci flow.

10.2.4. Curvature Operator. Another interesting notion about the $\mathrm{Rm}^{(4,0)}$-tensor is the *curvature operator*. Recall that the tensor satisfies symmetric properties $R_{ijkl} = -R_{jikl} = R_{klij}$. If we group together i with j, and k with l, then one can regard $\mathrm{Rm}^{(4,0)}$ as a symmetric operator on $\wedge^2 TM$. Precisely, we define

$$\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$$
$$\mathcal{R}(X \wedge Y) := -\mathrm{Rm}\big(X, Y, g^{pk}\partial_p, g^{lq}\partial_q\big)\partial_k \wedge \partial_l$$

and extend tensorially to all of $\wedge^2 TM$. The minus sign is to make sure that positive curvature operator (to be defined later) would imply positive sectional curvature.

Furthermore, given the metric g, we can define an induced metric, still denoted by g, on $\otimes^2 TM$ and $\wedge^2 TM$ by the following way:

$$g(X \otimes Y, Z \otimes W) := g(X, Z)g(Y, W).$$

As $X \wedge Y := X \otimes Y - Y \otimes X$, one can easily verify that

$$g(X \wedge Y, Z \wedge W) = (g \owedge g)(X, Y, Z, W).$$

Then, one can also verify easily that $\mathcal{R}$ is a self-adjoint operator with respect to this induced metric g on $\wedge^2 TM$, meaning that

$$g\big(\mathcal{R}(X \wedge Y), Z \wedge W\big) = g\big(X \wedge Y, \mathcal{R}(Z \wedge W)\big).$$

Indeed, we have $g\big(\mathcal{R}(X \wedge Y), Z \wedge W\big) = -\mathrm{Rm}(X, Y, Z, W)$. By standard linear algebra, such an operator would be diagonalizable and that all its eigenvalues are real.

Exercise 10.7. Verify all the above claims, including

- $\mathcal{R}$ is well-defined;
- $g(X \wedge Y, Z \wedge W) = (g \owedge g)(X, Y, Z, W)$;
- $g\big(\mathcal{R}(X \wedge Y), Z \wedge W\big) = \mathrm{Rm}(X, Y, Z, W)$; and
- $\mathcal{R}$ is self-adjoint with respect the inner product g on $\wedge^2 TM$.

We say (M, g) has *positive curvature operator* if all eigenvalues of $\mathcal{R}$ are positive at every point in M. The curvature operator is related to sectional curvatures in a sense that the later are the "diagonals" of $\mathcal{R}$. As a matrix is positive-definite implies all its diagonal entries are positive (NOT vice versa), so positive curvature operator implies positive sectional curvatures. The converse is not true: $\mathbb{CP}^n$ with the Fubini-Study metric can be shown to have positive sectional curvatures (in fact $K(X, Y)$ is either 1 or 4), but does not have positive curvature operator.

It had been a long-standing conjecture (recently solved in 2006) that what topology a compact Riemannian manifold (M, g) must have if it has positive curvature operator. In dimension 3, positive curvature operator implies positive Ricci curvature (to be defined in the next section). In 1982, Richard Hamilton introduced the *Ricci flow* to show that such a 3-manifold must be diffeomorphic to $\mathbb{S}^3$ with round metric or its quotient. The key idea (modulo intensive technical detail) is to show that such a metric would evolves under the heat equation to the one which constant sectional curvature (compared with heat diffusion distributes temperature evenly in the long run), which warrants that such a manifold must be diffeomorphic to $\mathbb{S}^3$ or its quotient.

Several years after the celebrated 3-manifold paper, Hamilton used the Ricci flow again to show that if a compact 4-manifold (M, g) has positive curvature operator, then it must be diffeomorphic to $\mathbb{S}^4$ or its quotient. He conjectured that the result holds for any dimension. It was until 2006 that Christoph Boehm and Burkhard Wilking resolved this conjecture completely, again using the Ricci flow (combined with some Lie algebraic techniques).

10.3. Ricci and Scalar Curvatures

In this section we introduce the Ricci curvature and the scalar curvature. The former is essentially the *trace* of the Riemann curvature tensor, and the later is the trace of the former.

10.3.1. Ricci Curvature. We will define the Ricci curvature using both invariant (global) notations and local coordinates. Let's start with the invariant notations first (which give clearer geometric meaning):

Definition 10.17 (Ricci Curvature)**.** The *Ricci curvature* of a Riemannian manifold (M, g) is a 2-tensor defined as:

$$\text{Ric}(X, Y) := \sum_{i=1}^{n} \text{Rm}(e_i, X, Y, e_i)$$

where $\{e_i\}$ is an orthonormal frame of the tangent space.

Exercise 10.8. Show that $\text{Ric}(X, Y)$ is independent of the choice of orthonormal basis $\{e_i\}$ of the tangent space.

By the symmetry properties of Rm, it is clear that Ric is a symmetric tensor, i.e.

$$\text{Ric}(X, Y) = \text{Ric}(Y, X).$$

Given a unit vector $X \in T_pM$, one can pick an orthonormal basis $\{e_i\}$ for T_pM such that $e_1 = X$, then we have:

$$\text{Ric}(X, X) = \sum_{i=2}^{n} \text{Rm}(e_i, e_1, e_1, e_i) = \sum_{i=2}^{n} K(e_i, e_1).$$

Therefore, the quantity $\text{Ric}(X, X)$ is essentially the sum of sectional curvatures associated to all orthogonal planes containing X.

We say that (M, g) has *positive Ricci curvature at* $p \in M$ if Ric is positive-definite at p, namely $\text{Ric}(X, X) \geq 0$ for any $X \in T_pM$ with equality holds if and only if $X = 0$. If it holds true at every $p \in M$, we may simply say (M, g) has positive Ricci curvature. Clearly, positive sectional curvature implies positive Ricci curvature. If one only has $\text{Ric}(X, X) \geq 0$ for any $X \in T_pM$ without the equality conditions, we say (M, g) has semi-positive (or non-negative) Ricci curvature at p.

Exercise 10.9. Suppose (M, g) has non-negative sectional curvature. Show that if $\text{Ric} \equiv 0$, then $\text{Rm} \equiv 0$.

In terms of local coordinates $\{u_i\}$, we denote

$$R_{ij} := \text{Ric}\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

so that $\text{Ric} = R_{ij}\, du^i \otimes du^j$. We next show that R_{ij} has the following local expression:

$$R_{ij} = \sum_{k} R^k_{kij} = g^{kl} R_{kijl}. \tag{10.1}$$

One nice way of proving it is to use geodesic normal coordinates. Fix a point $p \in M$, and pick an orthonormal basis $\{e_i\} \in T_pM$, then there exists a local coordinate system $\{u_i\}$ such that $\frac{\partial}{\partial u_i} = e_i$ at p, and $g_{ij} = 0$ at p. By Exercise 10.8, the definition of Ric is independent of such an orthonormal basis.

Now we consider:

$$\begin{aligned}R_{ij} &:= \mathrm{Ric}(\partial_i, \partial_j) = \sum_k \mathrm{Rm}(e_k, \partial_i, \partial_j, e_k)\\ &= \sum_k \mathrm{Rm}(\partial_k, \partial_i, \partial_j, \partial_k) = \sum_k R_{kijk}\\ &= \delta_{lk} R_{kijl} = g^{lk} R_{kijl}.\end{aligned}$$

The last line was to convert $\sum_k R_{kijk}$ into a tensorial quantity. Now that both R_{ij} and $g^{lk}R_{kijl}$ are tensorial (the latter is so because the summation indices k and l appear as top-bottom pairs), so the identity $R_{ij} = g^{lk}R_{kijl}$ holds under any local coordinates covering p. Again since p is arbitrary, we have proved (10.1) holds globally on M.

It is possible to give a less "magical" (but more transparent) proof of (10.1). Write $e_k = A_k^i \frac{\partial}{\partial u_i}$ where $\{e_k\}$ is an orthonormal basis of T_pM and $\{u_i\}$ is any local coordinate system. By orthogonality, we have

$$\delta_{ij} = g(e_i, e_j) = A_i^k g_{kl} A_j^l.$$

In matrix notations, we have $I = [A][g][A]^T$, where A_i^k is the (i,k)-th entry of $[A]$, then we get $[g] = [A]^{-1}([A]^T)^{-1}$ and $[g]^{-1} = [A]^T[A]$, i.e.

$$g^{ij} = \sum_k A_k^i A_k^j.$$

Now consider

$$\begin{aligned}R_{ij} &= \sum_k \mathrm{Rm}(e_k, \partial_i, \partial_j, e_k)\\ &= \sum_{k,p,q} \mathrm{Rm}(A_k^p \partial_p, \partial_i, \partial_j, A_k^q \partial_q)\\ &= \sum_{k,p,q} A_k^p A_k^q R_{pijq} = \sum_{p,q} g^{pq} R_{pijq}\end{aligned}$$

as desired.

10.3.2. Scalar Cuvature. The *scalar curvature* R is the trace of the Ricci tensor, in a sense that

$$R := \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal frame of the tangent space. It can be shown that R is independent of the choice of the orthonormal frame, and has the following local expression

$$R = g^{ij} R_{ij} \tag{10.2}$$

Exercise 10.10. Show that the definition of R does not depend on the choice of the orthonormal frame $\{e_i\}_{i=1}^n$, and prove (10.2).

Using the second Bianchi identity, one can prove the following identity relating the Ricci curvature and scalar curvature.

Proposition 10.18 (Contracted Bianchi Identity). *On a Riemannian manifold (M, g), we have the following identity:*

$$\mathrm{div}_g \mathrm{Ric} = \frac{1}{2} dR, \quad \text{or in local coordinates:} \quad \nabla^i R_{ij} = \frac{1}{2} \frac{\partial R}{\partial u_j}. \tag{10.3}$$

Proof. The second Bianchi identity asserts that

$$\nabla_i R_{jklp} + \nabla_j R_{kilp} + \nabla_k R_{ijlp} = 0.$$

Multiplying both sides by g^{jp} and keeping in mind that $\nabla g = 0$, we have

$$\nabla_i R_{kl} + \nabla^p R_{kilp} - \nabla_k R_{il} = 0.$$

Here we have used the fact that $R_{ijlp} = -R_{jilp}$ in the last term.

Next, multiply both sides by g^{kl} and using again the fact that $\nabla g = 0$, we get

$$\nabla_i R - \nabla^p R_{ip} - \nabla^l R_{il} = 0.$$

By rearrangement and change-of-indices, we get:

$$\nabla^i R_{ij} = \frac{1}{2}\nabla_i R = \frac{1}{2}\frac{\partial R}{\partial u_i}$$

as desired.

□

One immediate consequence of the contracted Bianchi identity is that the 2-tensor $\text{Ric} - \frac{R}{2}g$ is divergence-free:

$$\nabla^i \left(R_{ij} - \frac{R}{2}g_{ij}\right) = \nabla^i R_{ij} - \frac{1}{2}g_{ij}\nabla^i R = \frac{1}{2}\partial_i R - \frac{1}{2}\partial_i R = 0.$$

The 2-tensor $G := \text{Ric} - \frac{R}{2}g$ is called the *Einstein tensor*, named after Albert Einstein's famous equation:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = T_{\mu\nu}$$

where the tensor $T_{\mu\nu}$ described stress and energy. Here we use μ and ν for indices instead of i and j as physicists use the former. We will learn in the next subsection that a metric satisfying the Einstein's equation is a critical metric of the functional $\int_M R\sqrt{\det[g]}$, known as the Einstein-Hilbert's functional.

Inspired by the (vacuum) Einstein's equation (where $T = 0$), mathematicians called a Riemannian metric g satisfying the equation below an *Einstein metric*:

$$R_{ij} = cg_{ij}, \quad \text{where } c \text{ is a real constant.}$$

A metric g with constant sectional curvature must be an Einstein metric. To see this, let's suppose

$$R_{ijkl} = C(g_{il}g_{jk} - g_{ik}g_{jl}).$$

Then by tracing both sides, we get:

$$R_{jk} = g^{il}R_{ijkl} = C(g^{il}g_{il}g_{jk} - g^{il}g_{ik}g_{jl}) = C(ng_{jk} - \delta_{lk}g_{jl}) = C(n-1)g_{jk}.$$

When $\dim M \geq 3$, a connected Riemannian metric satisfying $R_{ij} = fg_{ij}$, where f is a smooth scalar function, must be an Einstein metric. To show this, we first observe that $f = \frac{R}{n}$:

$$R = g^{ij}R_{ij} = g^{ij}fg_{ij} = nf.$$

Then, the contracted Bianchi identity shows

$$\frac{1}{2}\partial_j R = \nabla^i R_{ij} = \nabla^i\left(\frac{R}{n}g_{ij}\right) = \frac{1}{n}g_{ij}\nabla^i R = \frac{1}{n}\partial_j R.$$

When $n > 2$, it implies $\partial_j R = 0$ for any j, and hence R is a constant.

The proof obviously fails in dimension 2. In contrast, it is *always* true that $R_{ij} = \frac{R}{2}g_{ij}$ when $n = 2$, for *any* Riemannian metric g. It can be seen easily by combinatoric inspections:

$$
\begin{aligned}
R_{11} &= g^{ij}R_{i11j} = g^{22}R_{2112} = g^{22}R_{1221} \\
R_{12} &= g^{21}R_{2121} = -g^{21}R_{1221} \\
R_{21} &= g^{12}R_{1212} = -g^{12}R_{1221} \\
R_{22} &= g^{11}R_{1221}.
\end{aligned}
$$

Hence, the scalar curvature is given by

$$R = g^{ij}R_{ij} = 2(g^{11}g^{22} - g^{12}g^{21})R_{1221} = 2\det[g]^{-1}R_{1221} = \frac{2R_{1221}}{\det[g]}.$$

In dimension 2, we have

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \frac{1}{\det[g]}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}.$$

It is then clear that $R_{ij} = \frac{R}{2}g_{ij}$ for any $i, j = 1, 2$.

One bonus result we have obtained from above is that $R = 2K$ in dimension 2, where K is the Gauss curvature. Recall from the proof of Gauss's Theorema Egregium that we have shown $\det[h] = R_{1221}$.

Exercise 10.11. Let (M, g) be a connected Riemannian manifold with $\dim M \geq 3$. Suppose for any fixed $p \in M$, the sectional curvature $K_p(\Pi)$ is independent of any 2-plane $\Pi \subset T_pM$, but not *assumed* to be independent of p. Show that in fact $K_p(\Pi)$ is independent of p as well.

An Einstein metric must have constant scalar curvature. The proof is easy: if $R_{ij} = cg_{ij}$, then $R = g^{ij}R_{ij} = nc$. It was a fundamental question of what manifolds admit a metric with constant scalar curvature. In dimension 2, it was the Uniformization Theorem. Any simply connected Riemann surface (i.e. complex manifold with real dimension 2) must be biholomorphic (i.e. conformal) to one of the standard models with constant curvature: disc, plane, sphere (corresponding to negative, zero, and positive curvatures respectively).

In higher dimensions (i.e. $\dim M \geq 3$), the positive curvature case was known as the Yamabe problem, named after Hidehiko Yamabe. The problem was progressively resolved by Neil Trudinger, Thierry Aubin, and finally by Richard Schoen in 1984, concluding that on any compact manifold (M, g) of dimension $n \geq 3$ with positive scalar curvature, there exists a smooth function f on M such that the conformally rescaled metric $\widetilde{g} = e^f g$ has constant positive scalar curvature.

10.3.3. Decomposition of Riemann Curvature Tensor. On a Riemannian manifold with dimension $n \geq 3$, the Riemann curvature $(4, 0)$-tensor admits an orthogonal decomposition into scalar part, Ricci part, and Weyl part:

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{n}g\right) \owedge g + W.$$

The Weyl tensor is defined as

$$W := \mathrm{Rm} - \frac{R}{2n(n-1)} g \owedge g - \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{n}g\right) \owedge g.$$

We will soon see that $W = 0$ when $n = 3$. When $n \geq 4$, the Weyl tensor is not necessarily 0, but if it is so, then one can show the metric g is locally conformal to a flat metric.

Let's first discuss what motivates such a definition of the Weyl tensor. Given any pair of tensor fields of the same type, it is possible to find an inner product between using the given Riemannian metric g. For instance, consider $S = S_{ij}^k \, du^i \otimes du^j \otimes \frac{\partial}{\partial u_k}$, and $T = T_{ij}^k \, du^i \otimes du^j \otimes \frac{\partial}{\partial u_k}$, then we define:

$$g(S,T) := g^{ip} g^{jq} g_{kr} S_{ij}^k T_{pq}^r.$$

It is a globally defined scalar function on M, and such a definition is independent of local coordinates. In the Euclidean case where $g_{ij} = \delta_{ij}$, then $\delta(S,T) = \sum_{i,j,k} S_{ij}^k T_{ij}^k$ which appears like the dot product.

We can then make sense of two tensor fields (of the same type) being orthogonal to each other. For example, one can check:

$$g\left(\text{Ric} - \frac{R}{n} g, g\right) = g^{ik} g^{jl} \left(R_{ij} - \frac{R}{n} g_{ij}\right) g_{kl} = g^{ij} R_{ij} - \frac{R}{n} g^{ij} g_{ij} = R - \frac{R}{n} \cdot n = 0,$$

so $\text{Ric} - \frac{R}{n} g$ and g are orthogonal.

Exercise 10.12. Show that $g \owedge g$ and $(\text{Ric} - \frac{R}{n} g) \owedge g$ are orthogonal to each other (with respect to the inner produced inherited from g).

Recall that for a constant sectional curvature metric, we have

$$R_{ijkl} = C(g_{il} g_{jk} - g_{ik} g_{jl}) = \frac{C}{2} (g \owedge g)_{ijkl},$$

and this implies $R_{ij} = C(n-1) g_{ij}$, and hence it is necessary that $C = \frac{R}{n(n-1)}$. Therefore, the difference

$$R_{ijkl} - \frac{R}{2n(n-1)} (g \owedge g)_{ijkl}$$

measures how close is the metric from having constant sectional curvature.

The tensor $\text{Ric} - \frac{R}{n} g$ measures how close the metric is from being Einstein. Since an Einstein metric may not have constant sectional curvature, we want to append a term above such that it captures how much the metric deviate from being Einstein:

$$\text{Rm} - \frac{R}{2n(n-1)} (g \owedge g) - C \left(\text{Ric} - \frac{R}{n} g\right) \owedge g$$

where C is a constant to be determined. Denote

$$W := \text{Rm} - \frac{R}{2n(n-1)} (g \owedge g) - C \left(\text{Ric} - \frac{R}{n} g\right) \owedge g,$$

then one can find there exists a unique constant C such that W has zero trace, i.e.

$$g^{il} W_{ijkl} = 0,$$

and this constant C is in fact $\frac{1}{n-2}$.

We are going to show that $W \equiv 0$ in dimension 3 using some elementary linear algebra (dimension counting). We present the proof this way because it is easier for us to see why dimension 3 is special.

By the symmetric properties of Rm, this $(4,0)$-tensor can be regarded as a section of the bundle $\wedge^2 T^* M \otimes_S \wedge^2 T^* M$, where $\otimes_S$ denotes the symmetric tensor product. We denote

$$\mathcal{E} := \wedge^2 T^* M \otimes_S \wedge^2 T^* M,$$
$$\mathcal{B} := \left\{ T \in \mathcal{E} : T_{ijkl} + T_{jkil} + T_{kijl} = 0 \text{ for any } i,j,k,l. \right\}.$$

That is, $\mathcal{B}$ is a sub-bundle of $\mathcal{E}$ consisting of $(4,0)$-tensors satisfying the Bianchi identity in addition to the symmetric properties like Rm. We begin by finding the dimensions of $\mathcal{E}$ and $\mathcal{B}$:

Lemma 10.19. *On a Riemannian manifold (M, g) of dimension $n \geq 2$, we have*

$$\dim \mathcal{E} = \frac{n(n-1)(n^2-n+2)}{8}$$
$$\dim \mathcal{B} = \frac{1}{2}n^2(n^2-1).$$

Proof. Each two-form is spanned by $du^i \wedge du^j$ with $i < j$. Hence we have

$$\dim \wedge^2 T^*M = C_2^n = \frac{n(n-1)}{2}.$$

Similarly, given a vector space V of dimension N, we have

$$\dim(V \otimes_S V) = C_2^N + N = \frac{N(N+1)}{2}.$$

Hence, we have

$$\dim \mathcal{E} = C_2^{C_2^n} = \frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right)}{2},$$

as desired.

The dimension of $\mathcal{B}$ can be found in a trickier way. We consider a map known *Bianchi symmetrization*:

$$b : \mathcal{E} \to \otimes^4 T^*M$$
$$b(T)_{ijkl} := \frac{1}{3}(T_{ijkl} + T_{jkil} + T_{kijl}).$$

Then, $\mathcal{B} = \ker(b)$, hence $\dim \mathcal{B} = \dim \mathcal{E} - \dim \operatorname{Im}(b)$ by elementary linear algebra.

One can show that the image $\operatorname{Im}(b)$ is in fact $\wedge^4 T^*M$. To show $b(T) \in \wedge^4 T^*M$, we observe that $T_{ikjl} + T_{kjil} = -T_{jkil} - T_{kijl}$, so that

$$3b(T)_{jikl} = T_{jikl} + T_{ikjl} + T_{kjil} = -T_{ijkl} - T_{jkil} - T_{kijl} = -3b(T)_{ijkl},$$

and similarly for other switching of indices. Conversely, to show $\wedge^4 T^*M \subset \operatorname{Im}(b)$, we observe by direct computations that:

$$b\big((du^i \wedge du^j) \otimes_S (du^k \wedge du^l)\big) = \frac{1}{3}\, du^i \wedge du^j \wedge du^k \wedge du^l.$$

By rank-nullity theorem, we conclude that

$$\dim \mathcal{B} = \dim \mathcal{E} - \dim \wedge^4 T^*M = \frac{n(n-1)(n^2-n+2)}{8} - C_4^n = \frac{1}{12}n^2(n^2-1).$$

□

Next, recall that $h \owedge k \in \mathcal{B}$ whenever h and k are symmetric 2-tensors. We define the following linear map:

$$L : S^2(T^*M) \to \mathcal{B}$$
$$h \mapsto h \owedge g$$

Here $S^2(T^*M)$ denotes the space of symmetric 2-tensors.

This map can be shown to be injective whenever $\dim M \geq 3$. Whenever we have $h \owedge g = 0$, one can check that in geodesic normal coordinates:

$$\begin{aligned} 0 &= g(h \owedge g, h \owedge g) \\ &= \sum_{i,j,k,l} (h_{il}\delta_{jk} + h_{jk}\delta_{il} - h_{ik}\delta_{jl} - h_{jl}\delta_{ik})(h_{il}\delta_{jk} + h_{jk}\delta_{il} - h_{ik}\delta_{jl} - h_{jl}\delta_{ik}) \\ &= 4(n-2)\sum_{i,j} h_{ij}^2 + 4\left(\sum_i h_{ii}\right)^2. \end{aligned}$$

Hence, we must have $h_{ij} = 0$ for any i, j. As h is tensorial, this holds true under any local coordinate system.

In fact, the map L is adjoint to the trace map. Define a map:

$$\mathrm{Tr} : \mathcal{B} \to S^2(T^*M)$$
$$T_{ijkl}\,(du^i \wedge du^j) \otimes_S (du^k \otimes du^l) \mapsto g^{il} T_{ijkl}\, du^j \otimes_S du^k$$

Then one can prove the following useful identity (whose proof is left as an exercise):

Exercise 10.13. Given any symmetric 2-tensor h, and a 4-tensor $T \in \mathcal{B}$, we have:

$$g(T, L(h)) = g(4\,\mathrm{Tr}(T), h).$$

As a result, the Weyl tensor is orthogonal to the image of L.

Now the key observation is that L is in fact bijective when $\dim M = 3$. It is because $S^2(T^*M)$ and $\mathcal{B}$ both have the same dimension (equal to 6). Since $W \in \mathcal{B}$, there exists a symmetric 2-tensor h such that $L(h) = W$. Then, the fact that $W = 0$ follows easily from Exercise 10.13:

$$g(W, W) = g(W, L(h)) = 4g(\mathrm{Tr}(W), h) = 0$$

by the fact that $\mathrm{Tr}(W) = 0$. This clearly shows $W = 0$.

Consequently, in dimension 3, the Riemann curvature tensor can be expressed as:

$$\mathrm{Rm} = \frac{R}{12} g \owedge g + \left(\mathrm{Ric} - \frac{R}{3} g\right) \owedge g.$$

An immediate corollary is that in dimension 3, the Ricci tensor Ric determines Rm. Also, any Einstein metric (in dimension 3 only!) must have constant sectional curvature.

In dimension 4 or above, the Weyl tensor may not be 0. But being trace-free, it is orthogonal to all 4-tensors of type $h \owedge g$ (where h is symmetric 2-tensor). In particular, it is orthogonal to both

$$\frac{R}{2n(n-1)}(g \owedge g) \quad \text{and} \quad \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{n} g\right) \owedge g,$$

making the following an orthogonal decomposition:

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{n} g\right) \owedge g + W.$$

The Weyl tensor can also be a conformal invariant: consider $\widetilde{g} = e^f g$ where f is a smooth scalar function. It can be shown that the Weyl tensors are related by $W(\widetilde{g}) = e^f W(g)$. The proof is somewhat computational (either using geodesic normal coordinates or Cartan's notations). One needs to find out the how the Rm, Ric and R of $\widetilde{g}$ are related to those of g.

10.4. Variation Formulae of Curvatures

The ultimate goal of this section is to show that the Euler-Lagrange's equation of the following Einstein-Hilbert's functional:

$$\mathcal{L}_{EH}(g) := \int_M R\sqrt{\det[g_{ij}]}\, du^1 \wedge \cdots \wedge du^n$$

is in fact the vacuum Einstein equation

$$\mathrm{Ric} - \frac{R}{2}g = 0.$$

We will assume throughout that g is Riemannian, i.e. positive-definite, although almost all parts of the proof remains valid with a Lorzentian g (having $(-,+,+,+)$ signature).

We need to understand what metric g minimizes the functional $\mathcal{L}_{EH}$. Mathematically, it means that if $g(t)$ is a 1-parameter family of Riemannian metrics with $g(0) = g$ and $\frac{\partial}{\partial t}g(t) = v(t)$ where $v(t)$ is a 1-parameter family of symmetric 2-tensors, then $\mathcal{L}_{EH}(g(t))$ has achieves minimum at $t = 0$. It is then necessary that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}_{EH}(g(t)) = 0$$

for *any* variation directions $v(t)$.

In order to differentiate $\mathcal{L}_{EH}$, we need to know the variation formulae for R and $\det[g_{ij}]$. We have already derived using geodesic normal coordinates the variation formula in Section 9.3: given that $\frac{\partial}{\partial t}g(t) = v(t)$, then we have

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right).$$

We will use this result to derive the variation formulae for curvatures.

10.4.1. Variation Formula for Ricci Tensor. Let's first start with the Ricci tensor. Given that $\frac{\partial}{\partial t}g_{ij}(t) = v_{ij}(t)$. We want to find a nice formula for $\frac{\partial}{\partial t}R_{ij}$ where $R_{ij} := \mathrm{Ric}_{g(t)}(\partial_i, \partial_j)$. Again we use geodesic normal coordinates at a fixed point p. Recall that

$$R_{ij} = R_{kij}^k = \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \text{ quadratic terms of } \Gamma_{ij}^k\text{'s}.$$

Although $\Gamma_{ij}^k(p) = 0$ does not warrant its derivative is 0 at p, we still have

$$\frac{\partial}{\partial t}(\text{quadratic terms of } \Gamma_{ij}^k\text{'s}) = 0$$

at p by the product rule. Hence, we may focus on the the first two terms when computing $\frac{\partial}{\partial t}R_{ij}$. We consider:

$$\begin{aligned}
\frac{\partial}{\partial t}\partial_k\Gamma_{ij}^k &= \frac{\partial}{\partial u_k}\frac{\partial}{\partial t}\Gamma_{ij}^k \\
&= \frac{\partial}{\partial u_k}\left\{\frac{1}{2}g^{kl}\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right)\right\} \\
&= \frac{1}{2}g^{kl}\frac{\partial}{\partial u_k}\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right).
\end{aligned}$$

Here we have use the fact that $\partial_k g^{kl} = 0$ at p for geodesic normal coordinates. Next note that

$$\nabla_k\nabla_i v_{jl} = \partial_k(\nabla_i v_{jl}) + \text{ terms with Christoffel symbols,}$$

so under geodesic normal coordinates we have

$$\nabla_k\nabla_i v_{jl} = \partial_k(\nabla_i v_{jl}).$$

This shows at p under geodesic normal coordinates we have:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial u_k}\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\nabla_k\nabla_i v_{jl} + \nabla_k\nabla_j v_{il} - \nabla_k\nabla_l v_{ij}).$$

Similarly, we can also get at p:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial u_i}\Gamma_{kj}^k = \frac{1}{2}g^{kl}(\nabla_i\nabla_k v_{jl} + \nabla_i\nabla_j v_{kl} - \nabla_i\nabla_l v_{kj}).$$

Combining these, we get:

$$\frac{\partial}{\partial t}R_{ij} = \frac{1}{2}g^{kl}(\nabla_k\nabla_i v_{jl} + \nabla_k\nabla_j v_{il} - \nabla_k\nabla_l v_{ij} - \nabla_i\nabla_k v_{jl} - \nabla_i\nabla_j v_{kl} + \nabla_i\nabla_l v_{kj}).$$

Although the above holds true for one particular coordinate system and at one point only, yet since both sides are tensorial, it holds for all local coordinate systems and at every point.

We can further simplify the expression a bit. For instance, we can write $g^{kl}\nabla_k = \nabla^l$, and $g^{kl}\nabla_k\nabla_l = \Delta$ (the tensor Laplacian). Noting that $\nabla g = 0$, we can also write $g^{kl}\nabla_i\nabla_j v_{kl} = \nabla_i\nabla_j(g^{kl}v_{kl})$. It is common to denote the trace of v (with respect to g) by $\mathrm{Tr}_g v := g^{kl}v_{kl}$. Note that $\mathrm{Tr}_g v = g(v, g)$ where $g(\cdot,\cdot)$ is the induced inner product on 2-tensors. After all these make-over, we can simplify the variation formula for Ric as:

$$\frac{\partial}{\partial t}R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}\nabla_i\nabla_j(\mathrm{Tr}_g v) + \frac{1}{2}\nabla^l\nabla_i v_{jl} + \frac{1}{2}\nabla^l\nabla_j v_{il}.$$

10.4.2. Variation Formula for Scalar Curvature. Next we derive the variation formula for the scalar curvature. Recall that $R = g^{ij}R_{ij}$, and we already have the variation formula for R_{ij}. We need the formula for g^{ij}. Using the fact that

$$g_{ij}g^{jk} = \delta_{ik},$$

by taking the time derivative on both sides we get

$$\left(\frac{\partial}{\partial t}g_{ij}\right)g^{jk} + g_{ij}\left(\frac{\partial}{\partial t}g^{jk}\right) = 0.$$

By rearrangement, we get

$$\frac{\partial}{\partial t}g^{jk} = -g^{jp}g^{kq}\frac{\partial}{\partial t}g_{pq}.$$

Given that $\frac{\partial}{\partial t}g_{ij} = v_{ij}$, we conclude that

$$\frac{\partial}{\partial t}g^{ij} = -g^{ip}g^{jq}v_{pq}$$

Now we are ready to derive the variation formula for R:

$$\begin{aligned}\frac{\partial}{\partial t}R &= \frac{\partial}{\partial t}(g^{ij}R_{ij})\\ &= -g^{ip}g^{jq}v_{pq}R_{ij} + g^{ij}\left(-\frac{1}{2}\Delta v_{ij} - \frac{1}{2}\nabla_i\nabla_j(\mathrm{Tr}_g v) + \frac{1}{2}\nabla^l\nabla_i v_{jl} + \frac{1}{2}\nabla^l\nabla_j v_{il}\right)\\ &= -g(v,\mathrm{Ric}) - \frac{1}{2}\Delta(g^{ij}v_{ij}) - \frac{1}{2}\Delta(\mathrm{Tr}_g v) + \frac{1}{2}\nabla^l\nabla^j v_{jl} + \frac{1}{2}\nabla^l\nabla^i g_{il}\\ &= -\Delta(\mathrm{Tr}_g v) - g(v,\mathrm{Ric}) + \nabla^l\nabla^i v_{il}.\end{aligned}$$

In particular, if the metric $g(t)$ deforms along the direction of $-2\mathrm{Ric}(g(t))$, i.e.

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij},$$

then the scalar curvature evolves by

$$\frac{\partial}{\partial t}R = -\Delta(-2R) - g(-2\mathrm{Ric},\mathrm{Ric}) - 2\nabla^i\nabla^j R_{ij} = \Delta R + 2\left|\mathrm{Ric}\right|^2$$

where we have used the contracted Bianchi identity $\nabla^j R_{ij} = \frac{1}{2}\partial_i R$. Using parabolic maximum principle, one can then show on a compact manifold if $R \geq C$ at $t = 0$, then it remains so for all $t > 0$ under the evolution $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$. This deformation of the metric $g(t)$ is called the Ricci flow, which is the major tool for resolving the Poincaré conjecture.

10.4.3. Variation Formula for Determinants. Next we want to compute the variation formula for $\det[g_{ij}]$. The following general result is very useful:

Lemma 10.20. *Let $A(t)$ be a time-dependent invertible $n \times n$ matrix with entries denoted by A_{ij}, then we have:*

$$\frac{\partial}{\partial t} \det A(t) = \mathrm{Tr}\left(A^{-1}\frac{\partial A}{\partial t}\right) \det A(t), \tag{10.4}$$

where A^{ij} is the (i,j)-th entry of A^{-1}.

Proof. It is best be done using differential forms. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. Then we have:

$$Ae_1 \wedge \cdots \wedge Ae_n = \det(A)\, e_1 \wedge \cdots \wedge e_n.$$

We then differentiate both sides by t using the product rule:

$$\sum_{i=1}^n Ae_1 \wedge \cdots \wedge Ae_{i-1} \wedge \frac{\partial A}{\partial t} e_i \wedge Ae_{i+1} \wedge \cdots \wedge Ae_n = \frac{\partial}{\partial t} \det(A)\, e_1 \wedge \cdots \wedge e_n.$$

Next we prove the following general result then the proof is completed. For any invertible $n \times n$ matrix A, and any $n \times n$ matrix B, we have:

$$\sum_{i=1}^n Ae_1 \wedge \cdots \wedge Ae_{i-1} \wedge Be_i \wedge Ae_{i+1} \wedge \cdots \wedge Ae_n = \mathrm{Tr}(A^{-1}B) \det(A)\, e_1 \wedge \cdots \wedge e_n.$$

To show this, we first prove the above holds for diagonalizable matrices A, which is dense in the set of invertible matrices. Let $P = [P_{ij}]$ be an invertible matrix such that $A = PDP^{-1}$ where $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Then, we have

$$\begin{aligned}
&\sum_{i=1}^n Ae_1 \wedge \cdots \wedge Ae_{i-1} \wedge Be_i \wedge Ae_{i+1} \wedge \cdots \wedge Ae_n \\
&= \sum_{i=1}^n PDP^{-1}e_1 \wedge \cdots \wedge P(P^{-1}BP)P^{-1}e_i \wedge \cdots \wedge PDP^{-1}e_n \\
&= \det(P) \sum_{i=1}^n DP^{-1}e_1 \wedge \cdots \wedge (P^{-1}BP)P^{-1}e_i \wedge \cdots \wedge DP^{-1}e_n \\
&= \det(P)\det(D) \sum_{i=1}^n P^{-1}e_1 \wedge \cdots \wedge D^{-1}(P^{-1}BP)P^{-1}e_i \wedge \cdots \wedge P^{-1}e_n
\end{aligned}$$

Write $f_i := P^{-1}e_i$, then we have

$$\begin{aligned}
&\sum_{i=1}^{n} P^{-1}e_1 \wedge \cdots \wedge D^{-1}(P^{-1}BP)P^{-1}e_i \wedge \cdots \wedge P^{-1}e_n \\
&= \sum_{i=1}^{n} f_1 \wedge \cdots \wedge D^{-1}P^{-1}BPf_i \wedge \cdots \wedge f_n \\
&= \sum_{i,j=1}^{n} f_1 \wedge \cdots \wedge (D^{-1}P^{-1}BP)_{ji} f_j \wedge \cdots \wedge f_n \\
&= \sum_{i=1}^{n} f_1 \wedge \cdots \wedge (D^{-1}P^{-1}BP)_{ii} f_i \wedge \cdots \wedge f_n \\
&= \mathrm{Tr}(D^{-1}P^{-1}BP)\, f_1 \wedge \cdots \wedge f_n \\
&= \mathrm{Tr}(D^{-1}P^{-1}BP)\det(P^{-1})\, e_1 \wedge \cdots \wedge e_n.
\end{aligned}$$

Recall that $A = PDP^{-1}$, so we have $D^{-1}P^{-1}BP = P^{-1}A^{-1}BP$, and so

$$\mathrm{Tr}(D^{-1}P^{-1}BP) = \mathrm{Tr}(A^{-1}B).$$

Combining with the results above, we get

$$\sum_{i=1}^{n} Ae_1 \wedge \cdots \wedge Ae_{i-1} \wedge Be_i \wedge Ae_{i+1} \wedge \cdots \wedge Ae_n = \det(D)\mathrm{Tr}(A^{-1}B)\, e_1 \wedge \cdots \wedge e_n$$

as desired. Note that $\det(D) = \det(A)$.

□

As a corollary, given that $g(t)$ satisfies $\frac{\partial}{\partial t} g_{ij} = v_{ij}$, then we have

$$\frac{\partial}{\partial t}\det[g_{ij}] = \mathrm{Tr}([g]^{-1}[v])\det[g] = g^{ij}v_{ij}\det[g] = (\mathrm{Tr}_g v)\det[g].$$

One immediate consequence of the above variation formula is that the mean curvature H of a Euclidean hypersurface Σ (with boundary C) must be 0 if it minimizes the area among all smooth hypersurface with the boundary C. To see this, we let Σ_0 be such a hypersurface, and Σ_t is a 1-parameter family of hypersurfaces such that $\mathrm{Area}(\Sigma_0) \leq \mathrm{Area}(\Sigma_t)$ for any t.

Denote $F_t(u_1, \cdots, u_n)$ to be the local parametrization of Σ_t, and $g_{ij}(t)$ be the first fundamental form. Suppose $\frac{\partial F}{\partial t} = f\nu$ where $\nu := \nu_t$ is the unit normal for Σ_t, then one can check that

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle = \left\langle f\frac{\partial \nu}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle + \left\langle \frac{\partial F}{\partial u_i}, f\frac{\partial \nu}{\partial u_j}\right\rangle.$$

Here we have used the fact that ν and $\frac{\partial}{\partial u_i}$ are orthogonal. Recall that

$$\frac{\partial \nu}{\partial u_i} = -h_i^k \frac{\partial F}{\partial u_k},$$

we conclude that

$$\frac{\partial g_{ij}}{\partial t} = -2fh_{ij},$$

and hence

$$\frac{\partial}{\partial t}\sqrt{\det[g]} = \frac{1}{2\sqrt{\det[g]}}(-2fg^{ij}h_{ij})\,\det[g] = -fH\sqrt{\det[g]}.$$

Therefore, the variation formula for the area is given by:

$$\frac{d}{dt}\text{Area}(\Sigma_t) = \int_{\Sigma_t} \frac{\partial}{\partial t}\sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n = -\int_{\Sigma_t} fH\sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n.$$

Consequently, if we need Σ_0 to minimize the area, then we must have

$$\int_{\Sigma_0} fH\sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n = 0$$

for *any* normal variation $f\nu$. It is then necessary that $H = 0$ on Σ_0.

10.4.4. Deriving the vacuum Einstein equation. Finally (after a short detour to minimal surfaces), we get back on deriving the Einstein equation. The essential task is to derive the variation formula for $R\sqrt{\det[g]}$. Suppose $\frac{\partial}{\partial t}g_{ij} = v_{ij}$, then

$$\begin{aligned}\frac{\partial}{\partial t}R\sqrt{\det[g]} &= \left(-\Delta \text{Tr}_g v - g(v, \text{Ric}) + \nabla^i\nabla^j v_{ij}\right)\sqrt{\det[g]} + R\cdot\frac{1}{2}\text{Tr}_g v \cdot \sqrt{\det[g]} \\ &= \left(-\Delta \text{Tr}_g v - g(v, \text{Ric}) + \nabla^i\nabla^j v_{ij} + \frac{R}{2}g(v,g)\right)\sqrt{\det[g]} \\ &= \left(-\Delta \text{Tr}_g v - g\big(v, \text{Ric} - \frac{R}{2}g\big) + \nabla^i\nabla^j v_{ij}\right)\sqrt{\det[g]}.\end{aligned}$$

Consequently, we have

$$\frac{d}{dt}\mathcal{L}_{EH}(g(t)) = \int_M \left(-\Delta \text{Tr}_g v - g\big(v, \text{Ric} - \frac{R}{2}g\big) + \nabla^i\nabla^j v_{ij}\right)\sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n.$$

Assuming M has no boundary, any divergence terms such as $\nabla^i T_{ij}$ or $\nabla_i T^{ij}$, etc. must have integral 0 by the Stokes' Theorem. This is can be shown by the following nice observation:

Exercise 10.14. Suppose X is a vector field on a Riemannian manifold (M, g) without boundary. Denote the volume element by

$$d\mu_g := \sqrt{\det[g_{ij}]}\, du^1 \wedge \cdots \wedge du^n.$$

Show that

$$d(i_X d\mu_g) = \nabla_i X^i\, d\mu_g.$$

Hence, by Stokes' Theorem, we get

$$\int_M \nabla_i X^i\, d\mu_g = \int_M d(\text{something}) = 0.$$

One can then write $\nabla^i\nabla^j v_{ij} = \nabla_k(g^{ik}\nabla^j v_{ij})$ and letting

$$X = g^{ik}\nabla^j v_{ij}\frac{\partial}{\partial u_k} =: X^k\frac{\partial}{\partial u_k},$$

then $\nabla^i\nabla^j v_{ij} = \nabla_i X^i$, and so

$$\int_M \nabla^i\nabla^j v_{ij}\, d\mu_g = 0.$$

Similarly, for any scalar function f, we have

$$\Delta f = g^{ij}\nabla_i\nabla_j f = \nabla_i(g^{ij}\nabla_j f) = \nabla_i\nabla^i f.$$

By letting $X = \nabla f = \nabla^i f \partial_i$, one can also show

$$\int_M \Delta f\, d\mu_g = 0$$

provided that M has no boundary.

Therefore, the variation formula for the $\mathcal{L}_{EH}$-functional can be simplified as:

$$\frac{d}{dt}\mathcal{L}_{EH}(g(t)) = -\int_M g\left(v, \mathrm{Ric} - \frac{R}{2}g\right) d\mu_g.$$

If we need $g(0)$ to minimize $\mathcal{L}_{EH}$ under *any* variations v, then it is necessary that

$$\mathrm{Ric} - \frac{R}{2}g \equiv 0$$

which is exactly the vacuum Einstein's equation.

By taking the trace with respect to g on both sides, we get:

$$R - \frac{R}{2}n = 0.$$

If $n > 2$, we then have $R = 0$, and so $\mathrm{Ric} \equiv 0$. Therefore, the solution to the vacuum Einstein's equation is necessarily Ricci-flat (but may not be Riemann-flat).

Exercise 10.15. Consider the following modified Einstein-Hilbert's functional:

$$\widetilde{\mathcal{L}}_{EH}(g) := \int_M \big(R - 2\Lambda + \varphi(g)\big)\, d\mu_g$$

where Λ is a real constant, and $\varphi(g)$ is a scalar function depending on the metric g.

Show that is g minimizes this functional among all variations of g, then it is necessary that

$$\mathrm{Ric} - \frac{R}{2}g + \Lambda g = T$$

where T is a symmetric 2-tensor depending on φ and its derivatives.

Curvatures and Topology

A typical kind of questions that geometers and topologists would like to address is that given some curvature conditions (note that curvatures are *local* properties), what can one say about the *global* properties of the manifold (such as diameters, compactness, topological type, etc.)?

There are two fundamental analytical tools for addressing these kind of questions, namely the existence of Jacobi fields and the second variations of arc-lengths. The Riemann curvature tensors play an important role in these two tools. They are used to transfer local information about curvatures to some more global information (such as diameter of a manifold).

One neat result is the following theorem due to Bonnet and Myers: which says that if (M^n, g) is a complete manifold with $\mathrm{Ric} \geq k(n-1)g$ on M where $k > 0$ is a positive constant, then one has $\mathrm{diam}(M, g(t)) \leq \frac{\pi}{\sqrt{k}}$, and consequently M is compact and has finite fundamental group $\pi_1(M)$. We will prove the theorem in Section 11.2. The equality case of Bonnet-Myers' Theorem was addressed by S.-Y. Cheng (Professor Emeritus of HKUST Math) in 1975, who proved the equality holds if and only if (M^n, g) is a isometric to the round sphere with constant sectional curvature k.

In laymen terms, a Jacobi field J is a vector field whose integral curves are all geodesics. If there are infinitely many geodesics connecting p and q, then there would exist a Jacobi field J such that $J(p) = J(q) = 0$. An intuitive example is the round 2-sphere with p and q being the north and south poles. There is a family of great semi-circles connecting them and the Jacobi field is $\frac{\partial}{\partial \varphi}$ (or $\frac{\partial}{\partial \theta}$ in PHYS convention).

Another fundamental result relating Jacobi fields and geodesics is that the existence question of non-trivial Jacobi field J with $J(p) = J(q) = 0$ is (roughly) equivalent to whether $(\exp_p)_*$ is singular at the vector tX corresponding to q. In this case, if one connects p and q by a unit-speed geodesic $\gamma(t)$ (say $\gamma(0) = p$ and $\gamma(L) = q$), then one can use the above-mentioned equivalence to show that $\{\gamma(t)\}_{t>0}$ is no longer a minimizing geodesic when $t > L$. This can give an upper bound L on the distance $d(p, q)$, and by repeating the argument on arbitrary p and q, we can estimate the diameter (maximal distance) of the manifold (M, g).

11.1. Jacobi Fields

11.1.1. Definition. As mentioned in the introduction, a Jacobi field is the variation field of a family of geodesics. Let $\gamma_s(t) : (-\varepsilon, \varepsilon) \times [0, T] \to M$ be such a family, with s as the parameter of the family, and t as the arc-length parameter of each curve. Denote

$$V := \frac{\partial}{\partial s}\gamma_s(t) \text{ and } T := \frac{\partial}{\partial t}\gamma_s(t).$$

We want to derive an equation for the variation vector field V. First, observe that since each γ_s is a geodesic, the tangent vector field T satisfies

$$\nabla_T T = 0 \text{ for any } (s,t).$$

Consider the Riemann curvature tensor:

$$\mathrm{Rm}(V,T)T = \nabla_V\nabla_T T - \nabla_T\nabla_V T - \nabla_{[V,T]}T.$$

We have $\nabla_V\nabla_T T = \nabla_{\frac{\partial\gamma}{\partial s}}\nabla_T T = 0$ since $\nabla_T T = 0$ holds for any s. Also, by $V = \gamma_*\frac{\partial}{\partial s}$ and $T = \gamma_*\frac{\partial}{\partial t}$, we can see that $[V,T] = 0$. The only survival term is the second one, and further observe that $\nabla_V T - \nabla_T V = [V,T] = 0$, so we get

$$\mathrm{Rm}(V,T)T = -\nabla_T\nabla_T V.$$

This is so-called the *Jacobi field equation*.

Definition 11.1 (Jacobi Fields)**.** Let (M^n, g) be a complete Riemannian manifold, and $\gamma : [0,T] \to M$ be an arc-length parametrized geodesic. Then, a vector field V defined on γ is said to be a *Jacobi field* along γ if the following holds:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V + \mathrm{Rm}(V,\dot\gamma)\dot\gamma = 0. \tag{11.1}$$

Note that (11.1) is a second-order ODE. Once we prescribed $V(0)$ and $\nabla_{\dot\gamma}V$ at $t = 0$, then there exists a unique Jacobi field along γ with these initial data. The solution space is a vector space by the linearity of the equation (11.1). The solution space is parametrized by $V(0)$ and $\nabla_{\dot\gamma}V\big|_{t=0}$, hence is $2n$-dimensional.

Example 11.2. Two easy examples of Jacobi fields are $V = \dot\gamma$, and $W = t\dot\gamma$. The first one is obvious by the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$, and $\mathrm{Rm}(\dot\gamma,\dot\gamma)\dot\gamma = 0$ by the alternating property of Rm. For $W = t\dot\gamma$, we also have $\mathrm{Rm}(W,\dot\gamma)\dot\gamma = 0$ by the same reason. We can then show

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}(t\dot\gamma) = \nabla_{\dot\gamma}(\dot\gamma + \underbrace{t\nabla_{\dot\gamma}\dot\gamma}_{=0}) = \nabla_{\dot\gamma}\dot\gamma = 0.$$

□

The Jacobi fields $\dot\gamma$ and $t\dot\gamma$ are merely tangent vector fields so that their integral curves are just reparametrization of the geodesic. The shape of the geodesic along these variations is unchanged, and so they are not interesting examples. However, they are essentially the "only" tangential Jacobi fields, since we can show any Jacobi field V along γ can be decomposed into:

$$V = V^\perp + a\dot\gamma + bt\dot\gamma$$

for some constants a and b, $V^\perp \perp \dot\gamma$, and still we have $V^\perp$ being a Jacobi field. The argument is as follows:

Consider the inner product $g\big(V(t), \dot\gamma(t)\big)$, we will show that it is a linear polynomial of t using the fact that $\nabla_{\dot\gamma}\dot\gamma = 0$:

$$\begin{aligned}
&\frac{d^2}{dt^2} g\big(V(t), \dot\gamma(t)\big) \\
&= g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, \dot\gamma\big) \\
&= -g\big(\mathrm{Rm}(V, \dot\gamma)\dot\gamma, \dot\gamma\big) \\
&= -\mathrm{Rm}(V, \dot\gamma, \dot\gamma, \dot\gamma) = 0.
\end{aligned}$$

This shows $g(V(t), \dot\gamma(t)) = a + bt$ for some constants a and b. Then, it is easy to show

$$g\big(V - a\dot\gamma - bt\dot\gamma, \dot\gamma\big) = 0$$

and hence $V - a\dot\gamma - bt\dot\gamma \perp \dot\gamma$. Here we assume for simplicity that γ is arc-length parametrized. By linearity of (11.1), $V^\perp$ is still a Jacobi field along γ, and it is normal to the curve γ.

Consequently, the space of Jacobi fields has an orthogonal decomposition into tangential and normal subspaces. The tangential subspace has dimension 2 spanned by $\dot\gamma$ and $t\dot\gamma$ (note that they are linearly independent as functions of t, even thought they are parallel vectors pointwise), and the normal subspace has dimension $2n - 2$.

Exercise 11.1. Suppose a Jacobi field V along a geodesic γ is normal to the curve at two distinct points. What can you say about V?

11.1.2. Jacobi Fields on Spaces with Constant Curvatures. Since the Jacobi field equation (11.1) involves the curvature term, it is expected that one can find some explicit solutions if the Riemann curvature term is nice. Consider a complete Riemannian manifold (M, g) with constant sectional curvature C, i.e.

$$\mathrm{Rm}(X, Y, Y, X) = C\big(\,|X|^2\,|Y|^2 - g(X, Y)^2\big)$$

for any $X, Y \in T_pM$.

Now given an arc-parametrized geodesic $\gamma(t) : [0, T] \to M$, and fix a unit vector $E_0 \in T_{\gamma(0)}M$ such that $E_0 \perp \dot\gamma(0)$, and extend this vector by parallel transport along γ, i.e.

$$\nabla_{\dot\gamma}E(t) = 0 \;\text{ and }\; E(0) = E_0.$$

We want to find a Jacobi field along γ that is the form of $V(t) = u(t)E(t)$ where $u(t)$ is a scalar function. It turns out that $u(t)$ will satisfy a hand-solvable ODE if (M, g) has constant sectional curvature.

Consider the Jacobi equation:

$$\begin{aligned}
0 &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}(uE) + \mathrm{Rm}(uE, \dot\gamma)\dot\gamma \\
&= \nabla_{\dot\gamma}\big(\dot uE\big) + u\mathrm{Rm}(E, \dot\gamma)\dot\gamma \\
&= \ddot uE + u\mathrm{Rm}(E, \dot\gamma)\dot\gamma.
\end{aligned}$$

Here we have used the fact that $\nabla_{\dot\gamma}E = 0$. Taking inner product with E on both sides, we get

$$0 = \ddot u + ug\big(\mathrm{Rm}(E, \dot\gamma)\dot\gamma, E\big).$$

Here we have used the fact that $|E(t)|^2 = 1$ by the property of parallel transport. Note also that $g(E(t), \dot\gamma(t)) = 0$ since it is so at $t = 0$, we have $\mathrm{Rm}(E, \dot\gamma)\dot\gamma, E) = C$ and we get that

$$\ddot u + Cu = 0.$$

It is a hand-solvable ODE, and the general solution is given by:

$$u(t) = \begin{cases} \frac{a}{\sqrt{C}}\sin(t\sqrt{C}) + \frac{b}{\sqrt{C}}\cos(t\sqrt{C}) & \text{if } C > 0 \\ at + b & \text{if } C = 0 \\ \frac{a}{\sqrt{-C}}\sinh(t\sqrt{-C}) + \frac{b}{\sqrt{-C}}\cosh(t\sqrt{-C}) & \text{if } C < 0 \end{cases}$$

where a and b are any constants.

Now each $E_0 \in T_{\gamma(0)}M$ with $E_0 \perp \dot{\gamma}(0)$ gives a 2-dimensional subspace of Jacobi fields. On the other hand, there are $(n-1)$ dimensions of vectors at $\gamma(0)$ that are orthogonal to $\dot{\gamma}(0)$. The above solutions of Jacobi fields form a $(2n-2)$ dimensional subspace, and therefore these form all possible Jacobi fields normal to $\dot{\gamma}$.

11.1.3. Comparison Theorem of Jacobi Fields. In the method of solving for Jacobi fields for constant sectional curvature metrics, the factor $u(t)$ is equal to $|V(t)|$. Hence, $|V(t)|$ satisfies the second-order ODE:

$$\frac{d^2}{dt^2}|V(t)| + C|V(t)| = 0.$$

Generally speaking, on an arbitrary complete Riemannian manifold (M, g), it is possible to derive a differential *inequality* of a similar form.

Suppose the sectional curvature of (M, g) is bounded above by a constant C (which can be positive, zero, or negative). Given a unit speed geodesic $\gamma(t)$ and a normal Jacobi field $V(t)$ on $\gamma(t)$, we can compute that

$$\begin{aligned} \frac{d^2}{dt^2}|V|^2 &= 2\frac{d}{dt}g\big(\nabla_{\dot{\gamma}}V, V\big) \\ &= 2g\big(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}V, V\big) + 2g\big(\nabla_{\dot{\gamma}}V, \nabla_{\dot{\gamma}}V\big) \\ &= -2g\big(\mathrm{Rm}(V, \dot{\gamma})\dot{\gamma}, V\big) + 2|\nabla_{\dot{\gamma}}V|^2 \\ &\geq -2C|V|^2 + 2|\nabla_{\dot{\gamma}}V|^2. \end{aligned}$$

Hence, we have

$$\underbrace{2|V|\frac{d^2}{dt^2}|V| + 2\left(\frac{d}{dt}|V|\right)^2}_{\frac{d^2}{dt^2}|V|^2} \geq -2C|V|^2 + 2|\nabla_{\dot{\gamma}}V|^2.$$

Note that

$$|\nabla_{\dot{\gamma}}V|^2 - \left(\frac{d}{dt}|V|\right)^2 = |\nabla_{\dot{\gamma}}V|^2 - g\left(\nabla_{\dot{\gamma}}V, \frac{V}{|V|}\right)^2 \geq 0$$

by Cauchy-Schwarz's inequality. We can conclude that as long as $|V| > 0$, we have

$$\frac{d^2}{dt^2}|V| \geq -C|V|.$$

Therefore, we can then derive comparison inequality between Jacobi fields of a constant sectional curvature metric and a general metric with sectional curvature bounded from above.

The key idea is that for any non-negative function $f(t)$ that satisfies the inequalities

$$f''(t) + Cf(t) \geq 0 \ \text{ for } t > 0\,, f(0) = 0, \ \text{ and } f'(0) > 0,$$

then one can claim show that

$$f(t) \geq \begin{cases} \frac{f'(0)}{\sqrt{C}}\sin(t\sqrt{C}) \ \text{ for any } t \in [0, \frac{\pi}{\sqrt{C}}] & \text{if } C > 0 \\ f'(0)t \ \text{ for any } t \geq 0 & \text{if } C = 0\,. \\ \frac{f'(0)}{\sqrt{-C}}\sinh(t\sqrt{-C}) \ \text{ for any } t \geq 0 & \text{if } C < 0 \end{cases}$$

To prove this, let $u(t)$ be the solution to $\ddot{u} + Cu = 0$ with $u(0) = 0$ and $u'(0) = f'(0)$, i.e.

$$u(t) = \begin{cases} \frac{f'(0)}{\sqrt{C}} \sin(t\sqrt{C}) & \text{if } C > 0 \\ f'(0)t & \text{if } C = 0 \\ \frac{f'(0)}{\sqrt{-C}} \sinh(t\sqrt{-C}) & \text{if } C < 0 \end{cases}.$$

Sturm-Liouville's comparison theorem of ODEs then asserts that $f(t) \geq u(t)$ as long as $u(t) > 0$. Now let $f(t) = |V(t)|$ where $V(t)$ is a Jacobi field along γ such that $V(0) = 0$, if one can argue that $f'(0) > 0$, then we conclude that

$$|V(t)| \geq \begin{cases} \frac{f'(0)}{\sqrt{C}} \sin(t\sqrt{C}) \text{ for any } t \in [0, \frac{\pi}{\sqrt{C}}] & \text{if } C > 0 \\ f'(0)t \text{ for any } t \geq 0 & \text{if } C = 0 \\ \frac{f'(0)}{\sqrt{-C}} \sinh(t\sqrt{-C}) \text{ for any } t \geq 0 & \text{if } C < 0 \end{cases}. \tag{11.2}$$

As a corollary, if the sectional curvature is non-positive, any non-trivial Jacobi field $V(t)$ along $\gamma(t)$ with $V(0) = 0$ would not vanish again for any $t > 0$.

We are left to show $f'(0) > 0$. Since $V(0) = 0$, we cannot compute $f'(0)$ directly. Consider the limit quotient:

$$f'(0) = \lim_{t \to 0^+} \frac{|V(t)| - |V(0)|}{t} = \lim_{t \to 0^+} \left| \frac{V(t)}{t} \right|.$$

We leave it as an exercise for readers to show that for $t > 0$ sufficiently small, we in fact have $V(t) = t\big(\nabla_{\dot{\gamma}} V\big|_{\gamma(0)}\big)$, then it completes the proof.

Exercise 11.2. Suppose V is a Jacobi field along an arc-length parametrized curve $\gamma_0(t)$ with $V(0) = 0$. Denote $\exp := \exp_{\gamma(0)}$ and $W_0 := \nabla_{\dot{\gamma}} V\big|_{\gamma(0)}$. Show that the family of curves

$$\gamma_s(t) := \exp\big(t(\dot{\gamma}_0(0) + sW_0)\big)$$

is a geodesic family that gives the prescribed Jacobi field $V(t)$. Furthermore, show that, under geodesic normal coordinates $(u_1, \cdots, u_n)$ at p with $\frac{\partial}{\partial u_1} = \dot{\gamma}_0(0)$, we have

$$V(t) = tW^i \frac{\partial}{\partial u_i}\bigg|_{\gamma_0(t)}$$

for $t > 0$ sufficiently small, where $W_0 = W^i \frac{\partial}{\partial u_i}$ at $T_{\gamma(0)}M$.

11.1.4. Conjugate Points and Jacobi Fields.

One fundamental theorem about Jacobi fields and the exponential maps is that the existence of Jacobi field $V(t)$ along a curve $\gamma(t) : [0, T] \to M$ with $V(0) = V(T) = 0$, is equivalent to $(\exp_{\gamma(0)})_*$ being singular at the point corresponding to $\gamma(T)$. In this subsection, let's make this relation more precise. We first introduce:

Definition 11.3 (Conjugate Points)**.** Let γ be a geodesic connecting points p and q on a Riemannian manifold (M, g). We say q is *conjugate* to p along γ if there exists a non-zero Jacobi field V along γ such that $V(p) = V(q) = 0$.

Example 11.4. The vector field $(\sin\varphi)\frac{\partial}{\partial\theta}$ (in MATH spherical coordinates) on the standard 2-sphere is a Jacobi field vanishing at the north and south poles. Therefore, the north and south poles are conjugate point to each other along any great circle passing through them. □

The relation between exponential maps and conjugate points is as follows:

Proposition 11.5. *Consider an arc-parametrized geodesic γ on a Riemannian manifold (M, g). Denote $p = \gamma(0)$, $T = \dot{\gamma}$, and $q = \exp_p(t_0 T)$ where $t_0 > 0$. Then, q is conjugate to p along γ if and only if the tangent map $(\exp_p)_{*t_0 T} : T_{t_0 T}(T_p M) \to T_p M$ is singular.*

Proof. Before giving the proof, we first observe the following key fact. Let $\gamma_s(t)$ be a family of geodesic defined by:

$$\gamma_s(t) := \exp_p\big(t(T + sW)\big)$$

where W is any vector in $T_p M$.

Then, $\gamma_0 = \gamma$. The Jacobi field generated by this family is given by:

$$V(t) := \left.\frac{\partial \gamma_s(t)}{\partial s}\right|_{s=0} = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p\big(tT + s(tW)\big)\big) = (\exp_p)_{*tT}(tW). \tag{11.3}$$

The last step follows from the fact that the derivative is the directional derivative of $\exp_p$ at tT along tW. In particular, $V(q) = (\exp_p)_{*t_0 T}(t_0 W)$.

To show the ($\Longrightarrow$)-part, we consider a basis $\{E_i\}_{i=1}^n$ of $T_p M$, and consider the family of geodesics $\gamma_s^i(t) := \exp_p\big(t(T + sE_i)\big)$ which generates the Jacobi fields $V_i(t) = (\exp_p)_{*tT}(tE_i)$ according to the above computation. These Jacobi fields are linearly independent because $(\exp_p)_{*tT}$ is invertible for small $t \geq 0$: suppose there are constants λ_i's such that

$$\sum_{i=1}^n \lambda_i V_i(t) \equiv 0,$$

then we have

$$(\exp_p)_{*tT}\left(t\sum_{i=1}^n \lambda_i E_i\right) \equiv 0.$$

Pick a small $\tau > 0$ such that $(\exp_p)_{*\tau T}$ is invertible, we have

$$\tau\sum_{i=1}^n \lambda_i E_i = 0 \quad \Longrightarrow \quad \sum_{i=1}^n \lambda_i E_i = 0.$$

By linear independence of E_i's, we get $\lambda_i = 0$ for any i. Therefore, $\{V_i(t)\}_{i=1}^n$ is a basis of Jacobi fields that vanish at p (which is an n-dimensional vector space).

To prove the desired claim, we suppose otherwise that there exists a non-zero Jacobi field V along γ such that $V(p) = V(q) = 0$ but $(\exp_p)_{*t_0 T}$ is invertible. From above, $V(t)$ must be spanned by $\{V_i(t)\}_{i=1}^n$:

$$V(t) \equiv \sum_{i=1}^n c_i V_i(t).$$

However, it would show

$$0 = V(q) = \sum_{i=1}^n c_i V_i(t_0) = (\exp_p)_{*t_0 T}\left(\sum_{i=1}^n t_0 c_i E_i\right),$$

and consequently

$$\sum_{i=1}^n t_0 c_i E_i = 0 \quad \Longrightarrow \quad c_i = 0 \text{ for any } i.$$

It is a contradiction to the fact that $V(t) \not\equiv 0$.

The ($\Longleftarrow$)-part is easier: suppose $(\exp_p)_{*_{t_0 T}}$ is singular, and in particular, there exists $W \neq 0$ in $T_p M$ such that

$$(\exp_p)_{*_{t_0 T}}(W) = 0.$$

Then, the Jacobi field $V(t)$ defined in (11.3) satisfies:

$$V(q) = (\exp_p)_{*_{t_0 T}} = 0.$$

By invertibility of $(\exp_p)_{*_{tT}}$ for $t > 0$ small, $V(t)$ is non-zero Jacobi field. It completes the proof. $\square$

11.2. Index Forms

11.2.1. Second Variations of Arc-Length. In Chapter 9, we have computed the first variation formula of the arc-length functional. If $\gamma_s(t) : (-\varepsilon, \varepsilon) \times [a, b]$ is a 1-parameter family of curves with the same end-points $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for any $s \in (-\varepsilon, \varepsilon)$, then we have shown that

$$\frac{d}{ds} L(\gamma_s) = -\int_a^b g\left(S, \nabla_T \frac{T}{|T|}\right) dt = \int_a^b g\left(\nabla_T S, \frac{T}{|T|}\right) dt$$

where $S = \frac{\partial \gamma_s}{\partial s}$ and $T = \frac{\partial \gamma_s}{\partial t}$. One necessary condition for γ_0 to minimize the arc-length is that $\nabla_T \frac{T}{|T|} = 0$, and if we assume γ_0 has constant speed, then $\nabla_T T = 0$ along γ.

However, to determine whether it is a local minimum, we need to consider the second derivative. In this section, we will derive the second variation formula of arc-length and discuss its applications.

Proposition 11.6 (Second Variations of Arc-Length). *Let $\gamma_s(t) : (-\varepsilon, \varepsilon) \times [a, b]$ be a 1-parameter family of curves with the same end-points $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for any $s \in (-\varepsilon, \varepsilon)$, γ_0 is a unit-speed geodesic. Then, we have:*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L(\gamma_s) = \int_a^b \left|\nabla_T S^N\right|^2 - \operatorname{Rm}(S, T, T, S)\, dt \tag{11.4}$$

where $S = \frac{\partial \gamma_s}{\partial s}$, $T = \frac{\partial \gamma_s}{\partial t}$, and $S^N = S - g(S, T)T$ is the normal projection of S.

Proof. We differentiate the first derivative of L:

$$\begin{aligned}
&\frac{d^2}{ds^2} L(\gamma_s)\\
&= \frac{d}{ds}\int_a^b g\left(\nabla_T S, \frac{T}{|T|}\right) dt\\
&= \int_a^b g\left(\nabla_S \nabla_T S, \frac{T}{|T|}\right) + g\left(\nabla_T S, \nabla_S \frac{T}{|T|}\right) dt\\
&= \int_a^b g\left(\nabla_S \nabla_T S, \frac{T}{|T|}\right) + g\left(\nabla_T S, \frac{|T|\, \nabla_S T - g\big(\nabla_S T, \frac{T}{|T|}\big)T}{|T|^2}\right) dt.
\end{aligned}$$

Now take $s = 0$, we have $|T| = 1$ and $\nabla_T T = 0$ (since γ_0 is a geodesic), then we have

$$\begin{aligned}
&\left.\frac{d^2}{ds^2}\right|_{s=0} L(\gamma_s)\\
&= \int_a^b g(\nabla_S \nabla_T S, T) + g(\nabla_T S, \nabla_S T) - g(\nabla_T S, T) g(\nabla_S T, T)\, dt\\
&= \int_a^b g(\nabla_S \nabla_T S, T) + |\nabla_T S|^2 - g(\nabla_T S, T)^2\, dt\\
&= \int_a^b g(\nabla_T \nabla_S S, T) + g(\operatorname{Rm}(S, T)S, T) + |\nabla_T S|^2 - g(\nabla_T S, T)^2\, dt
\end{aligned}$$

where we have used the fact that $[S, T] = 0$. By Pythagoreas' Theorem, we have

$$|\nabla_T S|^2 - g(\nabla_T S, T)^2 = \left|(\nabla_T S)^N\right|^2,$$

and as $\nabla_T T = 0$, we can also show

$$\nabla_T S^N = \nabla_T (S - g(S, T)T) = \nabla_T S - g(\nabla_T S, T)T = (\nabla_T S)^N.$$

For the first term in the integrand, we have

$$\int_a^b g(\nabla_T \nabla_S S, T)\, dt = \int_a^b \frac{d}{dt} g(\nabla_S S, T) - g(\nabla_S S, \nabla_T T)\, dt = 0$$

since $S(a) = S(b) = 0$, and $\nabla_T T \equiv 0$ along γ.

Summing all up, we have proved the desired formula (11.4). $\square$

Remark 11.7. The $\mathrm{Rm}(S, T, T, S)$ term in (11.4) also equals $\mathrm{Rm}(S^N, T, T, S^N)$ since $\mathrm{Rm}(T, T, T, T) = 0$. $\square$

Inspired by (11.4), we define:

Definition 11.8 (Index Forms)**.** Let $\gamma : [a, b] \to M$ be a geodesic on a Riemannian manifold (M, g). The *index form* $I : \mathcal{V}_\gamma \times \mathcal{V}_\gamma \to \mathbb{R}$ of γ is a bilinear form on the following space of vector fields:

$$\mathcal{V}_\gamma := \{V(t) \mid V \text{ is a vector field on } \gamma \text{ such that } V(a) = V(b) = 0 \text{ and } g(V, \dot\gamma) \equiv 0\},$$

and is defined as

$$I(V, W) := \int_a^b g(\nabla_T V, \nabla_T W) - \mathrm{Rm}(V, T, T, W)\, dt$$

where $T := \dot\gamma$.

In particular, for any $V \in \mathcal{V}_\gamma$, the second variation of $L(\gamma)$ along the variation field V is given by $I(V, V)$. When γ is minimizing geodesic, it is necessary that $I(V, V) \geq 0$ for any $V \in \mathcal{V}_\gamma$. In other words, if we want to show a certain geodesic γ is *not* minimizing, one needs to construct a vector field $V \in \mathcal{V}_\gamma$ such that $I(V, V) < 0$.

When $V \in \mathcal{V}_\gamma$ is a C^∞ Jacobi field and $W \in \mathcal{V}_\gamma$ is any C^∞ vector field, one can use integration by parts and show:

$$I(V, W) = \big[g(\nabla_T V, W)\big]_{t=a^+}^{t=b^-} - \int_a^b g\big(\nabla_T \nabla_T V + \mathrm{Rm}(V, T)T, W\big)\, dt = 0.$$

where we have used the fact that $W(a) = W(b) = 0$ and $\nabla_T \nabla_T V + \mathrm{Rm}(V, T)T = 0$.

Note that if V and W are merely piecewise smooth on $[a, b]$, say they are smooth on $[a, c)$ and $(c, b]$, then we have

$$I(V, W) = \big[g(\nabla_T V, W)\big]_{t=a^+}^{t=c^-} + \big[g(\nabla_T V, W)\big]_{t=c^+}^{t=b^-}.$$

11.2.2. Geodesics Beyond Conjugate Points.

One fundamental fact about geodesics is that it is never minimizing if γ has an interior point conjugate to the starting point.

Proposition 11.9. *Suppose $\gamma : [a, b] \to M$ is a unit-speed geodesic and there exists $c \in (a, b)$ such that $\gamma(c)$ is conjugate to $\gamma(a)$, then γ is not a minimizing geodesic.*

Proof. By Proposition 11.5, the given condition implies there exists a non-zero Jacobi field $V(t)$ defined on $[a, c]$ such that $V(a) = V(c) = 0$. We can assume that V is normal to $\dot\gamma$.

Then, we extend V to the whole curve $\gamma : [a, b] \to M$ as follows:

$$\widetilde{V}(t) := \begin{cases} V(t) & \text{if } t \in [a, c] \\ 0 & \text{if } t \in (c, b] \end{cases}.$$

Note that $\widetilde{V}(t)$ is not smooth. We are going to search for a C^∞ vector field W defined on $[a,b]$ and a small $\varepsilon > 0$ such that

$$I(\widetilde{V} + \varepsilon W, \widetilde{V} + \varepsilon W) < 0.$$

Consider that

$$I(\widetilde{V} + \varepsilon W, \widetilde{V} + \varepsilon W) = I(\widetilde{V}, \widetilde{V}) + 2\varepsilon I(\widetilde{V}, W) + \varepsilon^2 I(W, W).$$

Let's compute each term one-by-one:

$$\begin{aligned} I(\widetilde{V}, \widetilde{V}) &= \left[g(\nabla_T \widetilde{V}, \widetilde{V})\right]_{t=a^+}^{t=c^-} + \left[g(\nabla_T \widetilde{V}, \widetilde{V})\right]_{t=c^+}^{t=b^-} \\ &= g(\nabla_T V(c), V(c)) - g(\nabla_T V(a), V(a)) + g(\nabla_T 0, 0) - g(\nabla_T 0, 0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} I(\widetilde{V}, W) &= \left[g(\nabla_T \widetilde{V}, W)\right]_{t=a^+}^{t=c^-} + \left[g(\nabla_T \widetilde{V}, W)\right]_{t=c^+}^{t=b^-} \\ &= g(\nabla_T V(c), W(c)) - g(\nabla_T W(a), W(a)) \end{aligned}$$

In order for $W \in \mathcal{V}_\gamma$, we need $W(a) = W(b) = 0$. If we can construct $W \in \mathcal{V}_\gamma$ such that $W(c) = -\nabla_T V(c)$, then we will have $I(\widetilde{V}, W) < 0$. Regardless of the sign of $I(W, W)$, we will have

$$I(\widetilde{V} + \varepsilon W, \widetilde{V} + \varepsilon W) = 2\varepsilon I(\widetilde{V}, W) + \varepsilon^2 I(W, W) < 0$$

for sufficiently small $\varepsilon > 0$.

One can use a bump function to construct such a vector field W. Let $\rho : [a,b] \to [0,1]$ be a C^∞ function such that $\rho(a) = \rho(b) = 0$ and $\rho(c) = 1$. Take any parallel unit vector field $E(t)$ which is normal to $\dot{\gamma}(t)$. This vector field exists by parallel transporting $-\nabla_T V(c)$ along the curve γ. Let $X(t)$ be this parallel transport, then $W(t) := \rho(t) X(t)^N$ is a smooth vector field in $\mathcal{V}_\gamma$ such that $W(c) = -\nabla_T V(c)^N$ and so $I(\widetilde{V}, W) = -\left|\nabla_T V(c)^N\right|^2 < 0$, completing the proof. □

11.2.3. Bonnet-Myers' Theorem. We are ready to prove the Bonnet-Myers' Theorem mentioned in the introduction. The key ingredient is the use of index forms to show that any minimizing geodesic cannot exceed certain length. The theorem was first due to Bonnet who assumed a lower bound on the sectional curvature. It was then later extended by Myers who only required a Ricci lower bound.

Theorem 11.10 (Bonnet-Myers). *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$ such that there exists a constant $k > 0$ such that*

$$\mathrm{Ric} \geq (n-1)kg,$$

then the diameter of (M, g) is bounded above by $\frac{\pi}{\sqrt{k}}$.

As a corollary, M is compact and has finite fundamental group.

Proof. We claim that no minimizing geodesic has length greater than $\frac{\pi}{\sqrt{k}}$. Suppose otherwise and $\gamma(t) : [0, L] \to M$ is a unit-speed geodesic with $L > \frac{\pi}{\sqrt{k}}$.

Construct a parallel orthonormal frame $\{E_i(t)\}_{i=1}^n$ such that $E_1(t) = \dot{\gamma}(t)$, then for each i we define

$$V_i(t) := \left(\sin \frac{\pi}{L} t\right) E_i(t),$$

and consider its index form $I(V_i, V_i)$. Since each $E_i(t)$ is parallel, we have

$$\nabla_{\dot{\gamma}} V_i = \left(\frac{\pi}{L} \cos \frac{\pi}{L} t\right) E_i(t) \ \text{ and } \ \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V_i = -\left(\frac{\pi^2}{L^2} \sin \frac{\pi}{L} t\right) E_i(t).$$

Hence, we have

$$\begin{aligned} I(V_i, V_i) &= -\int_0^L g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V_i, V_i) + \mathrm{Rm}(V_i, \dot\gamma, \dot\gamma, V_i)\,dt \\ &= \int_0^L \frac{\pi^2}{L^2}\sin^2\frac{\pi}{L}t - \left(\sin^2\frac{\pi}{L}t\right)\mathrm{Rm}(E_i, E_1, E_1, E_i)\,dt. \end{aligned}$$

Summing up i over 2 to n, we get

$$\sum_{i=2}^n I(V_i, V_i) = \int_0^L \left(\sin^2\frac{\pi}{L}t\right)\left(\frac{(n-1)\pi^2}{L^2} - \sum_{i=2}^n \mathrm{Rm}(E_i, E_1, E_1, E_i)\right)dt.$$

Note that by the given condition about the Ricci curvature, we have

$$\sum_{i=2}^n \mathrm{Rm}(E_i, E_1, E_1, E_i) = \mathrm{Ric}(E_1, E_1) \geq (n-1)kg(E_1, E_1) = (n-1)k.$$

Hence, we have

$$\sum_{i=2}^n I(V_i, V_i) \leq \int_0^L \left(\sin^2\frac{\pi}{L}t\right)\left(\frac{(n-1)\pi^2}{L^2} - (n-1)k\right)dt < 0$$

since we assumed $L > \frac{\pi}{\sqrt{k}}$. At least one of the V_i's gives $I(V_i, V_i) < 0$, and this shows γ cannot be a minimizing geodesic. It leads to a contradiction.

(M, g) is compact because $\exp_p$ now maps $\overline{B(0, \pi/\sqrt{k})}$ onto M. To argue that it has finite fundamental group, we consider its universal cover $\pi : \widetilde{M} \to M$. Recall that $\widetilde{M}$ admits a Riemannian metric $\pi^* g$ which is locally isometric to g, so $(\widetilde{M}, \pi^* g)$ also satisfies the same Ricci curvature lower bound. This shows $\widetilde{M}$ is compact too, and consequently $\pi^{-1}(p)$ is a finite set for any $p \in M$. This shows $\pi_1(M)$ is a finite group. □

Remark 11.11. The Ricci condition cannot be relaxed to Ric > 0. Any non-compact regular surface in $\mathbb{R}^3$ with positive Gauss curvature serves as a counter-example, and there are plenty of them! Recall that for regular surfaces, we have $K = 2R$ and Ric $= \frac{R}{2}g$. □

Remark 11.12. As the circle $\mathbb{S}^1$ has an infinite π_1 (isomorphic to $\mathbb{Z}$) and so is $\mathbb{S}^1 \times M$ for any complete Riemannian manifold M, by the Bonnet-Myers' Theorem it is impossible for $\mathbb{S}^1 \times M$ to admit a Riemannian metric whose Ricci curvature has a uniform positive lower bound. If M is compact, it is even impossible for $\mathbb{S}^1 \times M$ to admit a Riemannian metric whose positive Ricci curvature. □

The equality of Bonnet-Myers' Theorem was proved by Cheng:

Theorem 11.13 (Cheng, 1975). *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$. Suppose there exists a constant $k > 0$ such that* Ric $\geq (n-1)kg$ *on M and the diameter of (M, g) equals $\frac{\pi}{\sqrt{k}}$, then (M^n, g) is isometric to the round sphere of radius $\frac{1}{\sqrt{k}}$.*

11.2.4. Synge's Theorem.

Another application of index forms is the proof of Synge's Theorem, which is about the dimension, orientability, and simply-connectedness of a Riemannian manifold with positive sectional curvature.

Theorem 11.14 (Synge). *Let (M, g) be a compact Riemannian manifold with positive sectional curvature. Then,*

- *if* $\dim M$ *is even and M orientable, then M is simply-connected;*
- *if* $\dim M$ *is odd, then it must be orientable.*

The proof of the theorem is based on two lemmas, one analytic and another linear algebraic.

Lemma 11.15. *Let (M, g) be a compact Riemannian manifold, then in every free homotopy class $[\gamma]$ of smooth closed curves , there exists a closed smooth geodesic $\widetilde{\gamma}$ (smooth at based point too) that minimizes the arc-length among all curves in $[\gamma]$.*

The proof of the lemma is by some convergence and compactness argument. We omit the proof here. Interested readers may consult J. Jost's book Theorem 1.5.1.

Another lemma is the following observation on orthogonal matrices.

Lemma 11.16. *Any orthogonal matrix $A \in O(n)$ with $\det(A) = (-1)^{n-1}$ must have 1 as one of its eigenvalues.*

Proof. Any real eigenvalue of $A \in O(n)$ is either 1 or -1. It can be shown by considering $Av = \lambda v$, so that

$$\|v\|^2 = v^T A^T A v = (Av)^T (Av) = Av \cdot Av = \|Av\|^2 = \lambda^2 \|v\|^2$$

If n is even, then $\det(A) = -1$. Since complex eigenvalues occur as conjugate pairs, the product of all complex (non-real) eigenvalues is positive, and hence the product of real eigenvalues must be negative. There are even many real eigenvalues counting multiplicity, so at least one of the real eigenvalue is 1. If n is odd, then $\det(A) = 1$. There are odd many real eigenvalues counting multiplicity. We have at least one of the real eigenvalue must be 1. $\square$

Proof of Theorem 11.14. We first prove the first statement. Suppose $\dim M$ is even and M is orientable, but M is not simply-connected. Take a non-trivial homotopy class $[\gamma]$ of closed curves with γ being the minimizer of arc-length within the class (such γ exists thanks to Lemma 11.15). Consider the parallel transport map $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(0)}M$ along γ which has determinant 1 by orientability. Note that $P_\gamma(\dot{\gamma}(0)) = \dot{\gamma}(0)$ as γ is a smooth closed geodesic. Consider the orthogonal complement E of span$\{\dot{\gamma}(0)\}$ in $T_{\dot{\gamma}(0)}M$ so that E is invariant under P_γ and $\det(P_\gamma\big|_E) = 1$. Note that E has odd dimension, by Lemma 11.16, there exists an eigenvector $X_0 \in E$ such that

$$P_\gamma(X_0) = X_0.$$

Extend X_0 by parallel transport along γ so that $\nabla_{\dot{\gamma}} X(t) = 0$ and $X(0) = X_0$, then we get:

$$I(X, X) = \int_\gamma \underbrace{|\nabla_{\dot{\gamma}} X|^2}_{=0} - \underbrace{\mathrm{Rm}(X, \dot{\gamma}, \dot{\gamma}, X)}_{>0} < 0,$$

hence γ is not a minimizing geodesic[1]. It leads to a contradiction, completing the proof of the first statement.

The second statement can be proved in a similar way. Suppose $\dim M$ is odd, but M is not orientable. Then, one can find a closed curve $\widetilde{\gamma}$ such that $\det P_{\widetilde{\gamma}} = -1$. Let $\gamma \in [\widetilde{\gamma}]$ be a smooth closed minimizing geodesic in the free homotopy class $[\widetilde{\gamma}]$, then we still have $\det P_\gamma = -1$ by continuity. Since $P_\gamma(\dot{\gamma}(0)) = \dot{\gamma}(0)$, and the orthogonal complement E of span$\{\dot{\gamma}(0)\}$ has odd dimension. By Lemma 11.16, there exists an eigenvector $X_0 \in E$ such that $P_\gamma(X_0) = X_0$. The rest of the proof goes exactly as in the even dimension case. $\square$

[1] Note that the index form we have discussed before require the variation vector field to vanish at end points, which is not the case here. However, as the curve γ and vector field $X(t)$ is smooth everything including the base point, the boundary terms of integration by parts also vanish. Therefore, the second variation formula of $L(\gamma)$ along X is also given by the index form $I(X, X)$.

11.3. Spaces of Constant Sectional Curvatures

The goal of this last section is to classify all simply-connected, complete Riemannian manifolds with constant sectional curvatures. We will show that they are either $\mathbb{S}^n$, $\mathbb{R}^n$, or $\mathbb{H}^n$, depending on whether the sectional curvature is positive, zero, or negative.

We will first prove a topological result, called the Cartan-Hadamard's Theorem, concerning spaces with non-positive sectional curvatures (not necessarily constant). The theorem pinpoints the topological type of such a manifold. Then, we will use Jacobi fields to express these metrics explicitly, and show that they are isometric to one of the standard metrics of $\mathbb{S}^n$, $\mathbb{R}^n$, and $\mathbb{H}^n$.

11.3.1. Cartan-Hadamard's Theorem. We first determine the topological type of manifolds with non-positive sectional curvatures. Here, non-positive sectional curvature means $K_p(X, Y) \leq 0$ for any $p \in M$ and any linearly independent vectors $X, Y \in T_pM$.

Theorem 11.17 (Cartan-Hadamard). *Let (M^n, g) be a complete, connceted Riemannian manifold with non-positive sectional curvatures. Then, for any $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a covering map.*

Consequently, M^n is diffeomorphic to a quotient manifold of $\mathbb{R}^n$. If in addition M^n is simply-connected, then M^n is diffeomorphic to $\mathbb{R}^n$.

Proof. One key ingredient of the proof is that the existence of a Jacobi field $V(t)$ along a geodesic $\gamma(t) : [a, b] \to M$ with $V(a) = V(b) = 0$ is equivalent to the non-invertibility of $(\exp_{\gamma(a)})_*$ at the point corresponding to $\gamma(b)$ (see Proposition 11.5). On the other hand, such a Jacobi field does not exist if the metric has non-positive sectional curvature according to (11.2). Hence, $(\exp_p)_*$ is always invertible, and by the inverse function theorem, it is a local diffeomorphism everywhere on T_pM. In particular, $\widetilde{g} := (\exp_p)^*g$ defines a Riemannian metric on T_pM, and $(T_pM, \widetilde{g})$ and (M, g) are locally isometric through the map $\exp_p$.

We first show that $\exp_p$ is surjective. Given a point $q \in M$, we let γ be the minimizing unit-speed geodesic from p to q. Suppose $d(p, q) = r > 0$, then the geodesic γ is given by

$$\gamma(t) = \exp_p(t\dot{\gamma}(0)).$$

Hence, we have $q = \gamma(r) = \exp_p(r\dot{\gamma}(0))$. This shows $\exp_p$ is surjective.

Next, we show that for each $q \in M$, there exists $\varepsilon > 0$ such that

$$\{B_\varepsilon(Q)\}_{Q \in \exp_p^{-1}(q)}$$

is a disjoint collection of open sets in T_pM. We pick such an $\varepsilon > 0$ so that $B_\varepsilon(q)$ is a geodesic ball such that all geodesics from q leaves the ball through $\partial B_\varepsilon(q)$ (before they come back to the ball, if ever). Index the set $\exp_p^{-1}(q)$ by $\{Q_\alpha\}$. For any distinct pair of Q_α and Q_β, we connect them through a minimizing geodesic $\widetilde{\gamma}$ (with respect to the metric $\widetilde{g}$). Then, the curve $\gamma := \exp_p \circ \widetilde{\gamma}$ is a geodesic on M from q to q. However, by our choice of ε, such a geodesic γ must go outside the geodesic ball $B_\varepsilon(q)$, and hence has length $> 2\varepsilon$. This shows Q_α and Q_β must be more than 2ε apart, so $B_\varepsilon(Q_\alpha)$ and $B(\varepsilon q_\beta)$ are disjoint.

Then we argue that

$$\exp_p^{-1}\left(B_\varepsilon(q)\right) = \bigsqcup_\alpha B_\varepsilon(Q_\alpha).$$

The $\supset$-part is easy. Suppose $X \in B_\varepsilon(Q_\alpha)$ for some Q_α. As $\exp_p$ is a local isometry and $d(X, Q_\alpha) < \varepsilon$, so we have $d\big(\exp_p(X), \exp_p(Q_\alpha)\big) < \varepsilon$ too. It shows $X \in \exp_p^{-1}\left(B_\varepsilon(q)\right)$.

Conversely, if $Y \in \exp_p^{-1}\big(B_\varepsilon(q)\big)$, then consider the points $y := \exp_p(Y)$ and q in M. Let γ be a minimizing geodesic from y to q, and $\widetilde{\gamma}$ be the geodesic lifted on T_pM, i.e. $\widetilde{\gamma}$ is geodesic on $(T_pM, \widetilde{g})$ such that $\widetilde{\gamma}(0) = Y$, and $(\exp_p)_*(\dot{\widetilde{\gamma}}(0)) = \dot{\gamma}(0)$. Let r be the length of γ so that $d(y,q) = r < \varepsilon$, then we have $\exp_p\big(\widetilde{\gamma}(r)\big) = \gamma(r) = q$. Hence $Q := \widetilde{\gamma}(r) \in \exp_p^{-1}(q)$. Note that $d(Q,Y) = d(q,y) = r < \varepsilon$ by isometry, so $Y \in B_\varepsilon(Q)$. This proves $\subset$-part of our claim. It completes the proof that $\exp_p$ is a covering map.

$\square$

11.3.2. Gauss Lemma. We next prove a fundamental result about radial tangent vectors from 0 in T_pM and its image curve under the exponential map.

Lemma 11.18 (Gauss). *Let (M,g) be a complete Riemannian manifold, $p \in M$, and consider the exponential map $\exp_p : T_pM \to M$. Denote the radial vector in T_pM by $r\frac{\partial}{\partial r} = r\partial_r$ (where r is the distance from the origin). Then, for any tangent vector $X \in T_{r\partial_r}(\partial B_r(0)) \subset T_{r\partial_r}(T_pM) \cong T_pM$, we have*

$$g\big((\exp_p)_{*_{r\partial_r}}(r\partial_r), (\exp_p)_{*_{r\partial r}}(X)\big) = 0.$$

Here we identify $T_{r\partial_r}(T_pM)$ with T_pM, so that the $r\partial_r$ in blue is regarded as in $T_{r\partial_r}(T_pM)$.

Proof. Consider a curve $\sigma(s) \subset \partial B_r(0) \subset T_pM$ such that $\sigma(0) = r\frac{\partial}{\partial r}$ and $\sigma'(0) = X$, and define a family of geodesics

$$\gamma_s(t) := \exp_p(t\sigma(s)).$$

For each fixed s, the curve $\gamma_s(t)$ is a geodesic by the definition of $\exp_p$. Next we compute

$$\begin{aligned}
\frac{\partial}{\partial s}\gamma_s(t)\Big|_{(s,t)=(0,0)} &= \frac{d}{ds}\Big|_{s=0} \exp_p(0\cdot\sigma(s)) = 0\\
\frac{\partial}{\partial t}\gamma_s(t)\Big|_{(s,t)=(0,0)} &= \frac{d}{dt}\Big|_{t=0} \exp_p(t\sigma(0))\\
&= (\exp_p)_{*_0}(\sigma(0)) = \sigma(0)\\
\frac{\partial}{\partial s}\gamma_s(t)\Big|_{(s,t)=(0,1)} &= \frac{d}{ds}\Big|_{s=0} \exp_p(\sigma(s))\\
&= (\exp_p)_{*_{\sigma(0)}}(\sigma'(0))\\
&= (\exp_p)_{*_{r\partial r}}(X)\\
\frac{\partial}{\partial t}\gamma_s(t)\Big|_{(s,t)=(0,1)} &= \frac{d}{dt}\Big|_{t=1} \exp_p(t\sigma(0))\\
&= \frac{d}{dt}\Big|_{t=0} \exp_p(\sigma(0)+t\sigma(0))\\
&= (\exp_p)_{*_{\sigma(0)}}(\sigma(0))\\
&= (\exp_p)_{*_{r\partial_r}}(r\partial r)
\end{aligned}$$

Therefore, to prove our claim, it suffices to show

$$g\left(\frac{\partial\gamma_s(t)}{\partial s}, \frac{\partial\gamma_s(t)}{\partial t}\right)\Big|_{(s,t)=(0,1)} = 0.$$

As before, we denote $S = \frac{\partial \gamma_s(t)}{\partial s}$ and $T = \frac{\partial \gamma_s(t)}{\partial t}$ for simplicity. Since we already know $g(S,T) = 0$ when $(s,t) = (0,0)$, we then calculate:

$$\begin{aligned}\frac{d}{dt}g(S,T) &= g(\nabla_T S, T) + g(S, \underbrace{\nabla_T T}_{=0}) \\ &= g(\nabla_S T, T) \\ &= \frac{1}{2}\frac{d}{ds}g(T,T).\end{aligned}$$

We argue that $|T|^2 = g(T,T)$ is independent of s as follows. With a fixed s, the length of the curve segment $\gamma_s(t)$ on $[t, t+\tau]$ is

$$L\big(\gamma_s\big|_{[t,t+\tau]}\big) = \int_t^{t+\tau} |T|\,.$$

Recall that the geodesic $\gamma_s(t)$ is given by $\exp_p(t\sigma(s))$, so that

$$L\big(\gamma_s\big|_{[t,t+\tau]}\big) = \tau\,|\sigma(s)|$$

by the definition of $\exp_p$. This shows

$$|T| = \frac{d}{d\tau}\bigg|_{\tau=0}\int_t^{t+\tau}|T| = \frac{d}{d\tau}\bigg|_{\tau=0} L\big(\gamma_s\big|_{[t,t+\tau]}\big) = |\sigma(s)| = r$$

since the curve $\sigma(s) \subset \partial B_r(0) \subset T_pM$. This completes proof that

$$\frac{d}{dt}g(S,T) = \frac{1}{2}\frac{d}{ds}|T|^2 = 0,$$

and conclude that $g(S,T) = 0$ when $(s,t) = (0,1)$ as desired. □

In short, the Gauss's Lemma asserts that radial lines and round spheres in T_pM remain to be orthogonal under the image of $\exp_p$. We will use this lemma in the next subsection to classify spaces with constant sectional curvatures.

11.3.3. Classification of Space Forms. Now we are ready to give a complete classification of (simply-connected) spaces of constant sectional curvatures. The key ingredient is to make use of the Jacobi fields (which are explicitly solvable) to give a fairly explicit expression of the Riemannian metric. We will see that such a Riemannian metric is uniquely determined by the sectional curvature, hence proving uniqueness of such a metric up to isometry.

Theorem 11.19 (Uniformization Theorem of Constant Sectional Curvature Spaces)**.** *Any simply-connected, complete Riemannian manifold of constant sectional curvatures must be isometric to one of the standard models:* $\mathbb{S}^n$*,* $\mathbb{R}^n$*, or* $\mathbb{H}^n$*.*

Consequently, any complete Riemannian manifold of constant sectional curvatures must be isometric to a quotient manifold of either $\mathbb{S}^n$*,* $\mathbb{R}^n$*, or* $\mathbb{H}^n$*.*

Proof. Let (M^n, g) be a simply-connected, complete Riemannian manifold with constant sectional curvature C. By Cartan-Hadamard's Theorem, we already know such (M,g) is diffeomorphic to $\mathbb{R}^n$ in case of $C = 0$ or $C < 0$, and the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism.

We will first deal with the cases $C = 0$ and $C < 0$. Let $(\widetilde{M}^n, \widetilde{g})$ be $\mathbb{R}^n$ with the Euclidean metric (in case of of $C = 0$), or $\mathbb{H}^n$ with the hyperbolic metric of constant sectional curvature C (in case of $C < 0$). Pick any point $p \in M$ and $\widetilde{p} \in \widetilde{M}$, and we identify T_pM and $T_{\widetilde{p}}\widetilde{M}$. Denote the exponential maps by $\exp_p : T_pM \to M$ and $\widetilde{\exp}_p : T_{\widetilde{p}}\widetilde{M} \to \widetilde{M}$.

We will show that

$$g\big((\exp_p)_*(X), (\exp_p)_*(X)\big) = \widetilde{g}\big((\widetilde{\exp}_p)_*(X), (\widetilde{\exp}_p)_*(X)\big)$$

for any $X \in T_{r\partial_r}(T_pM) \cong T_pM$. This is implies $\exp_p^* g = \widetilde{\exp}_p^* \widetilde{g}$ so that $\widetilde{\exp}_p \circ \exp_p^{-1}$ is an isometry from M to $\widetilde{M}$. Note that $\exp_p$ is invertible in case of $C = 0$ and $C < 0$ by Cartan-Hadamard's Theorem, and simply-connectedness of M.

We first assume that $X \in T_{r\partial_r}(\partial B_r(0)) \subset T_{r\partial r}(T_pM)$, and will then handle an arbitrary $X \in T_{r\partial r}(T_pM)$ using Gauss's Lemma. Consider the family of geodesics:

$$\gamma_s(t) := \exp_p\left(t\left(\frac{\partial}{\partial r} + s\frac{X}{r}\right)\right).$$

Note that use $\frac{\partial}{\partial r}$ for the radial vector instead of $r\frac{\partial}{\partial r}$ so that $\gamma_0(t)$ is a unit-speed geodesic. Many results about Jacobi fields that we derived required γ_0 to be unit-speed.

The family of geodesics generates a Jacobi field

$$V(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s(t) = (\exp_p)_{*t\partial_r}\left(\frac{tX}{r}\right).$$

In particular, we have $V(r) = (\exp_p)_{*r\partial_r}(X)$. From the above discussion, we need to find out $|V(r)|_g = \left|(\exp_p)_{*r\partial r}(X)\right|_g$. Since $V(0) = 0$, such a Jacobi field has been solved explicitly in page 269, in which we found:

$$|V(t)| = u(t) = \begin{cases} \frac{u'(0)}{\sqrt{C}}\sin(t\sqrt{C}) & \text{if } C > 0 \\ u'(0)t & \text{if } C = 0 \\ \frac{u'(0)}{\sqrt{-C}}\sinh(t\sqrt{-C}) & \text{if } C < 0 \end{cases}.$$

To find $u'(0)$, we consider the results from Exercise 11.2, which shows

$$V(t) = t\frac{X^i}{r}\left.\frac{\partial}{\partial u_i}\right|_{\gamma(t)}$$

under geodesic normal coordinates $(u_1, \cdots, u_n)$ at p. Hence, we have

$$u'(0) = \left.\frac{d}{dt}\right|_{t=0}|V(t)| = \left|\frac{X^i}{r}\left.\frac{\partial}{\partial u_i}\right|_{\gamma(0)}\right| = \frac{|X|_g}{r}.$$

Under geodesic normal coordinates at p, we also have $|X|_g = \|X\|$, where $\|\cdot\|$ is the standard Euclidean norm of $T_pM \cong \mathbb{R}^n$. Therefore, we conclude that $|V(r)|_g$ depends only on r, C, and $\|X\|$.

By exactly the same argument, we can get the same result for $\left|(\widetilde{\exp}_p)_*(X)\right|_{\widetilde{g}}$. Therefore, we have:

$$\left|(\widetilde{\exp}_p)_{*r\partial r}(X)\right|_{\widetilde{g}} = \left|(\exp_p)_{*r\partial r}(X)\right|_g = |V(r)| = \begin{cases} \frac{\|X\|}{r\sqrt{C}}\sin(r\sqrt{C}) & \text{if } C > 0 \\ \|X\| & \text{if } C = 0 \\ \frac{\|X\|}{r\sqrt{-C}}\sinh(r\sqrt{-C}) & \text{if } C < 0 \end{cases}$$

for any $X \in T_{r\partial_r}(\partial B_r(0))$.

Now given an arbitrary $X \in T_{r\partial_r}(T_pM)$, one can decompose it into

$$X = X_{\text{rad}} + X_{\text{sph}}$$

where X_{rad} is the radial component, and X_{sph} is tangential to $\partial B_r(0)$. By Gauss's Lemma, we have

$$g\big((\exp_p)_{*r\partial r}(X_{\text{rad}}), (\exp_p)_{*r\partial r}(X_{\text{sph}})\big) = 0.$$

Therefore, one can conclude that

$$\left|(\exp_p)_{*r\partial r}(X)\right|_g^2 = \left|(\exp_p)_{*r\partial r}(X_{\text{rad}})\right|_g^2 + \left|(\exp_p)_{*r\partial r}(X_{\text{sph}})\right|_g^2.$$

By the definition of $\exp_p$ (and a similar arc-length argument as in the proof of the Gauss's Lemma), one can argument easily that

$$\left|(\exp_p)_{*r\partial r}(X_{\text{rad}})\right|_g^2 = \|X_{\text{rad}}\|^2.$$

By the previous computations, we also have that $\left|(\exp_p)_{*r\partial r}(X_{\text{sph}})\right|_g^2$ depends only on r, C, and $\left\|X_{\text{sph}}\right\|$. One can repeat these arguments on $\widetilde{\exp}_p$ and yield the same result.

Therefore, we conclude that

$$\left|(\widetilde{\exp}_p)_{*r\partial r}(X)\right|_{\widetilde{g}} = \left|(\exp_p)_{*r\partial r}(X)\right|_g$$

for any $r > 0$, $X \in T_{r\partial_r}(T_pM)$ in the case of $C = 0$ and $C < 0$. Hence, g and $\widetilde{g}$ are isometric in these cases.

Finally, we deal with the case $C > 0$. Let $(\mathbb{S}^n, \widetilde{g})$ be the round sphere with sectional curvature C. With the same notations as the above, one can also argument in the same way that g is isometric to $\widetilde{g}$ locally (in the region on which $\exp_p$ is a diffeomorphism). Therefore, by compactness of M (guaranteed by Bonnet-Myers' Theorem), one can cover M by finitely many open geodesic balls $\{B_\alpha\}$ each of which is isometric to another geodesic ball $\{\widetilde{B}_\alpha\}$ on $\mathbb{S}^n$ via the map, say, $\varphi_\alpha : B_\alpha \subset M \to \widetilde{B}_\alpha \subset \mathbb{S}^n$.

We want to glue these local isometries φ_α's to form a global isometry. However, it is not a priori true that any pair φ_α and φ_β of local isometries must agree on the overlap. However, by the transitive action of $SO(n)$ acting on $\mathbb{S}^n$, one can compose an isometry $\Phi_{\alpha\beta} : \mathbb{S}^n \to \mathbb{S}^n$ that maps $\varphi_\alpha(B_\alpha \cap B_\beta)$ to $\varphi_\beta(B_\alpha \cap B_\beta)$ isometrically and $\Phi_{\alpha\beta} \circ \varphi_\alpha$ agrees with φ_β on the overlap. Replace φ_α by $\Phi_{\alpha\beta} \circ \varphi_\alpha$. Repeat this replacement process for each overlap (there are finitely many), one can construct a global isometry $\varphi : M \to \mathbb{S}^n$. It completes the proof of the case $C > 0$. □

By the above uniformization theorem, we say that $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ are *standard models* in Riemannian geometry. These three models and their quotient manifolds are called *space forms*.

Therefore, to prove a certain Riemannian manifold is diffeomorphic to a sphere (or its quotient), one can show that it admits a Riemannian metric with a constant positive sectional curvature. The Ricci flow, and also other geometric flows as well, is a very effective tool to produce such a metric by "distributing" curvatures uniformly across the manifold like heat diffusion.

**** This is the end of this lecture notes. ****

*** The course MATH 6250I will continue on the introduction to the Ricci flow. ***

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