(M,g) sect $\le C$

const sect. $= C$
$\ddot u + Cu = 0 \iff \frac{d^2}{dt^2}|V(t)| + C|V(t)| = 0$

$E(t)$, $\gamma(t)$, E_0, $t=0$

Jacobi

$\nabla_T\nabla_T V + Rm(V,T)T = 0.$

$\frac{d^2}{dt^2}|V(t)|^2 = 2\frac{d}{dt} g(\nabla_T V, V) = 2g(\nabla_T\nabla_T V, V) + 2g(\nabla_T V, \nabla_T V)$

$2|V|\frac{d^2}{dt^2}|V| + 2\left(\frac{d}{dt}|V|\right)^2$

$\frac{d}{dt}|V| = g(\nabla_T V, \frac{V}{|V|})$

$\frac{d^2}{dt^2}|V| = \frac{1}{|V|} g(\nabla_T\nabla_T V, V) + g(\nabla_T V, \nabla_T V) - g(\nabla_T V, \frac{V}{|V|})^2$

$\ge \frac{1}{|V|} g(\nabla_T\nabla_T V, V) = -\frac{1}{|V|} g(Rm(V,T)T, V)$

$\ge -\frac{1}{|V|} C|V|^2 = -C|V|.$

$\frac{d^2}{dt^2}|V| + C|V| \ge 0.$

const. sec
$\ddot u + Cu = 0$

Sturm-Liouville comparison.

Given $\begin{cases} \ddot u + Cu = 0 \\ \ddot v + Cv \ge 0. \\ u(0) = v(0) = 0, \ u \ge 0 \\ \dot u(0) = \dot v(0) > 0. \end{cases}$

$\Rightarrow v(t) \ge u(t)$ whenever $u(t) > 0$.

$|V(t)| \ge u(t)$ as long as $u(t) > 0$.

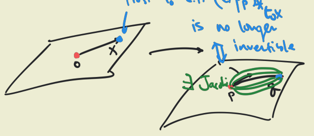
$|V(t)| \ge \begin{cases} \dot u(0)\,t & \text{if } C = 0 \\ \dot u(0)\sqrt{C}\sin\frac{t}{\sqrt{C}} & \text{if } C > 0 \\ & \text{if } C < 0. \end{cases}$

$\exp_p : T_pM \to M$

$(\exp_p)_{*X} : T_X(T_pM) \to T_qM$

$\cong T_pM$

first t s.t. $(\exp_p)_{*tX}$ is no longer invertible



$X+sW$, W, X, 0

$\exp_p(t(X+sW)) =: \gamma_s(t)$

$V := \frac{\partial\gamma_s}{\partial s}\Big|_{s=0} = \frac{\partial}{\partial s}\exp_p(tX + s(tW))\Big|_{s=0}$

directional der. of $\exp_p$ at tX along tW

WLOG assume $t_0 = 1$ $= (\exp_p)_{*tX}(tW).$

$V(t=1, s=0) = (\exp_p)_{*X}(W) \longleftarrow *$

$(\Rightarrow)$ $(\exp_p)_{*X}(W) = 0$ $\exists W \ne 0.$

$\gamma_s(t) = \exp_p(t(X+sW))$ gives $V = \frac{\partial\gamma_s}{\partial s}$

s.t. $V(q) = (\exp_p)_{*X}(W) = 0$
$t=1, s=0$

p, q

$(\Leftarrow)$

$\exists$ Jacobi field $V \ne 0$ s.t.
$V(p) = V(q) = 0$

but: $(\exp_p)_{*tX}$ is invertible. 0, tX

Choose $\{e_i\}$ o.n. basis for T_pM
$\gamma_s^{(i)}(t) := \exp_p(t(X + se_i))$

$V^{(i)}_{(t)} := \frac{\partial\gamma_s^{(i)}}{\partial s} = (\exp_p)_{*tX}(te_i)$

$\sum_i \lambda_i V^{(i)}(t) \equiv 0 \Rightarrow \sum_i \lambda_i (\exp_p)_{*tX}(te_i) = (\exp_p)_{*tX}\left(\sum_i \lambda_i te_i\right)$

Take small $t \Rightarrow \lambda_i = 0.$

$V(t) = \sum_i c_i V^{(i)}(t) = \sum_i c_i (\exp_p)_{*tX}(te_i) = (\exp_p)_{*tX}\left(\sum_i c_i te_i\right)$

Take $t = t_0 \leftrightarrow q.$

$0 = V(t_0) = (\exp_p)_{*t_0X}\left(\sum_i t_0 c_i e_i\right) \Rightarrow \sum_i t_0 c_i e_i = 0.$
$V(q)$ invertible

$\Rightarrow c_i = 0.$
$\Rightarrow V(t) \equiv 0.$

$\gamma_0(t)$, $t=a$, $t=b$, $\gamma_s(t)$

$V = \frac{\partial\gamma_s}{\partial s}$

$\frac{d}{ds}L(\gamma_s) = -\int_a^b g(V, \nabla_T(\frac{T}{|T|}))\,dt$

$= \int_a^b g(\nabla_T V, \frac{T}{|T|})\,dt$

$\nabla_V|T| = g(\nabla_V T, \frac{T}{|T|})$

$\frac{d^2}{ds^2}L(\gamma_s) = \int_a^b \frac{d}{ds} g(\nabla_T V, \frac{T}{|T|})\,dt$

$= \int g(\nabla_V\nabla_T V, \frac{T}{|T|}) + g(\nabla_T V, \nabla_V(\frac{T}{|T|}))\,dt$

$= \int_a^b g(\nabla_V\nabla_T V, \frac{T}{|T|}) + g(\nabla_T V, \frac{|T|\nabla_V T - T g(\nabla_V T, \frac{T}{|T|})}{|T|^2})\,dt$

$\overset{s=0}{=} \int_a^b g(\nabla_V\nabla_T V, T) + g(\nabla_T V, \nabla_V T - T g(\nabla_V T, T))\,dt$

$|T| = 1$
$\nabla_T T = 0$

$= \int_a^b g(\nabla_V\nabla_T V, T) + g(\nabla_T V, \nabla_V T) - g(\nabla_V T, T) g(\nabla_T V, T)$

$[V,T] = 0$

$g(\nabla_T V, \nabla_T V) - g(\nabla_T V, T)^2$

$= \int_a^b g(\nabla_V\nabla_T V, T) + |(\nabla_T V)^N|^2$

$\nabla_V\nabla_T V - \nabla_T\nabla_V V = Rm(V,T)V$

$\nabla_T V$, $(\nabla_T V)^N$, $g(\nabla_T V, T)$

$\nabla_T(V^N) = \nabla_T(V - g(V,T)T)$
$= \nabla_T V - g(\nabla_T V, T)T$
$= (\nabla_T V)^N$

$= \int_a^b g(Rm(V,T)V + \nabla_T\nabla_V V, T)$
$+ |(\nabla_T V)^N|^2$

$= \int_a^b |\nabla_T V^N|^2 - Rm(T,V,V,T) \longleftarrow$

$= -\int_a^b g(V^N, \nabla_T\nabla_T V^N) + g(Rm(V^N,T)T, V^N)$

$= -\int_a^b g(V^N, \nabla_T\nabla_T V^N + Rm(V^N,T)T)$

normal

$I(V,V) = \int_a^b |\nabla_T V|^2 - Rm(T,V,V,T)\,dt$

$I(V,W) = \int_a^b g(\nabla_T V, \nabla_T W) - Rm(T,V,W,T)\,dt$

index form. $\uparrow$ meaning?

$\gamma_{u,s}(t)$ $U = \frac{\partial\gamma}{\partial u}$. $S = \frac{\partial\gamma}{\partial s}$, $T = \frac{\partial\gamma}{\partial t}$.

$\gamma_{0,0}(t)$ geodesic, $\gamma_{u,s}(a) = p$, $\gamma_{u,s}(b) = q$.

$\frac{\partial^2 L(\gamma_{u,s})}{\partial u\partial s} = I(U,S)$ HW.

$\exists$ Jacobi field

p, q

$\Downarrow$

$(\exp_p)_{*t_0X}$ is sing.

$q = \exp_p(t_0X).$

$\int_{a_i}^{a_{i+1}} g(\nabla_T V, \nabla_T W) - Rm(T,V,W,T)\,dt$

$= \int_{a_i}^{a_{i+1}} \frac{d}{dt} g(V, \nabla_T W) - g(V, \nabla_T\nabla_T W)$
$- Rm(V,T,T,W)$

$= \left[g(V, \nabla_T W)\right]_{a_i^+}^{a_{i+1}^-} - \int_{a_i}^{a_{i+1}} g(V, \nabla_T\nabla_T W + Rm(W,T)T)$

Prop:

Given V, $\gamma_0 \varphi p$, $\gamma(b)$, q, $\gamma(a)$, $p = \gamma(0)$

then $\gamma_0(p,q)$ is not minimizing.

Key idea of proof: show $\exists X$ s.t. $X(p) = X(q) = 0$ but $I(X,X) < 0$.

Proof:

$\tilde V(t) = \begin{cases} V(t) & \text{if } t \in [0,a] \\ 0 & \text{if } t \in [a,b]. \end{cases}$

$I(\tilde V + \epsilon W, \tilde V + \epsilon W) = \cancel{I(\tilde V, \tilde V)} + 2\epsilon I(\tilde V, W) + \epsilon^2 I(W,W)$

WANT: $\ominus$

$I(\tilde V, \tilde V) = \int_0^a \cdots + \int_a^b \cdots$

$= \left[g(\tilde V, \nabla_T\tilde V)\right]_{\cancel{0^+}}^{\cancel{a^-}} + \left[g(\tilde V, \nabla_T\tilde V)\right]_{\cancel{a^+}}^{\cancel{b^-}} = 0.$

$I(\tilde V, W) = \left[g(\nabla_T\tilde V, W)\right]_{\cancel{0^+}}^{a^-} + \left[g(\nabla_T\tilde V, W)\right]_{a^+}^{\cancel{b^-}}$

Need:
$W(\underset{p}{t=0}) = 0$, $W(\underset{q}{t=b}) = 0$

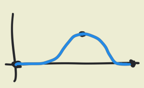
a

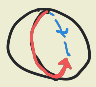
& $I(\tilde V, W) = g(\nabla_T\tilde V(a^-), W(a)) - g(\cancel{\nabla_T\tilde V(a^+)}, W(a))$

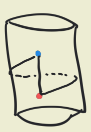
$\boxed{W(a) = -\nabla_T\tilde V(a^-)}$

$E(t)$ parallel transport
s.t. $E(a) = -\nabla_T\tilde V(a^-)$.

$W(t) = \rho(t)E(t).$







Bonnet Thm:

(M,g) complete Riem. mfd.
$\sec(M) \ge \frac{1}{R^2} > 0.$ $\quad K_p(X,Y) \ge \frac{1}{R^2}$
$\forall p,\ X,Y \in T_pM.$

$\Rightarrow \operatorname{diam}(M,g) \le \pi R.$

Proof: $E(t)$, p, q, γ min. geodesic

WANT: $L(\gamma) \le \pi R.$

Assume $L = L(\gamma) > \pi R$, want: construct $W(t)$
s.t. $I(W,W) < 0.$

$W(t) := \left(\sin\frac{\pi t}{L}\right)E(t)$ any normal parallel unit vector along γ.

$W(0) = 0 = W(L)$ ✓

$I(W,W) = \int_0^L |\nabla_T W|^2 - Rm(W,T,T,W)\,dt$

$= \int_0^L -g(W, \nabla_T\nabla_T W) - Rm(W,T,T,W)\,dt.$

$\nabla_T W = \left(\frac{\pi}{L}\cos\frac{\pi t}{L}\right)E(t).$, $\nabla_T\nabla_T W = -\frac{\pi^2}{L^2}\sin\frac{\pi t}{L}\cdot E(t).$

$= \int_0^L -g\Big(\underbrace{\sin\frac{\pi t}{L}\cdot E(t)}_{W}, -\frac{\pi^2}{L^2}\sin\frac{\pi t}{L}\cdot E(t)\Big) - \sin^2\frac{\pi t}{L}\cdot Rm(E,T,T,E)$

$\le \int_0^L +\frac{\pi^2}{L^2}\sin^2\frac{\pi t}{L} - \frac{1}{R^2}\sin^2\frac{\pi t}{L}\,dt < 0$
if $L > \pi R.$