MATH 6250I • Fall 2018 • Riemannian Geometry
Problem Set #3 • Due Date: 09/12/2018

1. Exercises 11.1 and 11.2

2. Consider a Riemannian manifold (M, g) with $\nabla \mathrm{Rm} \equiv 0$. Let $\gamma(t) : [a, b] \to M$ be a unit-speed geodesic. Consider the operator at each point $\gamma(t)$:

$$\mathfrak{R}_{\gamma(t)} : T_{\gamma(t)}M \to T_{\gamma(t)}M$$
$$\mathfrak{R}_{\gamma(t)}(X) := \mathrm{Rm}(X, \dot\gamma)\dot\gamma$$

[For LaTeX-typer: $\mathfrak{R}$ is `\mathfrak{R}`]

   (a) Show that $\mathfrak{R}$ is self-adjoint with respect to g (and hence its diagonalizable with real eigenvalues).

   (b) Show that the set of eigenvalues of $\mathfrak{R}$ is independent of t, and there exists an orthonormal eigen-basis $\{e_i(t)\}_{i=1}^n$ of $\mathfrak{R}$ so that each $e_i(t)$ is parallel along γ.

   (c) Denote $\{\lambda_i\}_{i=1}^n$ the set of eigenvalues of $\mathfrak{R}$. Show that if $V(t) = \sum_{i=1}^n V^i(t)e_i(t)$ is a Jacobi field along γ, then each $V^i(t)$ satisfies the following "hand-solvable" ODE:

$$\frac{d^2}{dt^2}V^i(t) + \lambda_i V^i(t) = 0.$$

3. Let $\gamma_{u,s}(t)$, with $u, s \in (-\varepsilon, \varepsilon)$ and $t \in [a, b]$, be a smooth 2-parameter family of smooth curves with $\gamma_{0,0}(t)$ being a unit-speed geodesic on a Riemannian manifold (M, g). Suppose $\gamma_{u,s}(a) = p$ and $\gamma_{u,s}(b) = q$ for any u, s. Denote:

$$U := \frac{\partial \gamma_{u,s}(t)}{\partial u},\ S := \frac{\partial \gamma_{u,s}(t)}{\partial s},\ T := \frac{\partial \gamma_{u,s}(t)}{\partial t}.$$

Prove the second variation formula of the length functional $L(\gamma_{u,s})$:

$$\left.\frac{\partial^2 L(\gamma_{u,s})}{\partial u \partial s}\right|_{(u,s)=(0,0)} = \int_a^b g(\nabla_T U^N, \nabla_T S^N) - \mathrm{Rm}(U, T, T, S)\, dt =: I(U, S).$$

4. Consider a smooth family of Riemannian metrics $g(t)$ to the Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$, where $t \in [0, T)$, on a closed manifold M. We have shown that the Ricci tensor (in any dimension of M) satisfies a heat-type equation below (in fact we have calculated the variation of Ric under any variation $\frac{\partial g_{ij}}{\partial t} = v_{ij}$):

$$\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2g^{pk}g^{ql}R_{kijl}R_{pq} - 2g^{kl}R_{ik}R_{jl}.$$

   (a) Show that when $\dim M = 3$, we have:

$$\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 3RR_{ij} - 6g^{kl}R_{ik}R_{jl} + \big(2\,|\mathrm{Ric}|^2 - R^2\big)g_{ij}.$$

   (b) Hence, using Hamilton's tensor maximum principle, show that each of the following conditions is preserved under the Ricci flow:

      i. $\frac{1}{2}Rg_{ij} - R_{ij} \geq 0$

      ii. $R_{ij} \geq \varepsilon R g_{ij} > 0$ where $\varepsilon \in (0, 1/3)$ is a constant.

5. In the finishing part of Hamilton's 3-manifold paper, the Bonnet-Myers' Theorem was used. The author applied the theorem by observing that $\mathrm{Ric}(g(t)) \geq \varepsilon(1-\eta)R_{\max}(t)g(t)$ on the geodesic ball $B_{g(t)}\big(p_t, \eta^{-1}R_{\max}(t)^{-1/2}\big)$, and showed that for $\eta > 0$ small, the geodesic ball is the entire M. However, the statement of Bonnet-Myers' Theorem requires a uniform Ricci lower bound on the whole M, not just on a ball.

   Explain why Hamilton's argument is still valid.