$\mathcal{L}_{EH}(g) : \mathrm{Riem}(M) \longrightarrow \mathbb{R}$ (M is compact without boundary)

$\mathcal{L}_{EH}(g) := \int_M R \, \underbrace{d\mu_g}$ $\sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n$.

Einstein-Hilbert

Goal:
Minimizer $\mathrm{Ric} - \frac{R}{2} g = 0$ $(\Rightarrow \mathrm{Ric} = 0)$.

$g(t)$ family of Riem. metrics $\frac{\partial g(t)}{\partial t} = v(t)$. ← Sym. 2-tensor.

$\frac{\partial g_{ij}(t)}{\partial t} = v_{ij}(t)$

If $g(0)$ is a minimizer for $\mathcal{L}_{EH}$

then $\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}_{EH}(g(t)) = 0$ ← $\int_M g(v, ---)\, d\mu_g$

$$\frac{d}{dt} \int_M R \sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n$$
$$= \int_M \underbrace{\frac{d}{dt}\left(R \sqrt{\det[g]}\right)} du^1 \wedge \cdots \wedge du^n.$$

Prop: $D\sqrt{\det[g]} = \frac{1}{2} g^{ij} (D g_{ij}) \cdot \sqrt{\det[g]}$

e.g. $\frac{\partial}{\partial t}\sqrt{\det(g)} = \frac{1}{2} g^{ij} \underbrace{\left(\frac{\partial}{\partial t} g_{ij}\right)}_{v_{ij}} \cdot \sqrt{\det[g]} = \frac{1}{2}\mathrm{Tr}_g v \sqrt{\det(g)}$

Idea of proof:

$\det[g] = \lambda_1 \cdots \lambda_n$

$\log \det[g] = \log \lambda_1 + \cdots + \log \lambda_n$

$2 D \log(\det[g])^{\frac{1}{2}} = \frac{D\lambda_1}{\lambda_1} + \cdots + \frac{D\lambda_n}{\lambda_n}$.

$2 \cdot \frac{D\sqrt{\det[g]}}{\sqrt{\det[g]}} = \frac{D\lambda_1}{\lambda_1} + \cdots + \frac{D\lambda_n}{\lambda_n}$.

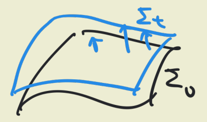
$D\sqrt{\det[g]} = \frac{1}{2}\sqrt{\det[g]}\left(\mathrm{Tr}_g Dg.\right)$

$\mathrm{Tr}_g Dg = g^{ij} D g_{ij} = \sum_i \frac{1}{\lambda_i} D\lambda_i$ $\qquad g_{ij} = \lambda_i \delta_{ij}$

If $\frac{\partial}{\partial t} g_{ij} = v_{ij} \Rightarrow \frac{\partial}{\partial t}\sqrt{\det g} = \frac{1}{2}\mathrm{Tr}_g v \sqrt{\det g}$.

$\Sigma_0^n \subset \mathbb{R}^{n+1}$ ↖ $F_0(u_1, \ldots, u_n)$

$$\boxed{\frac{\partial F(t)}{\partial t} = f(t)\, \nu(t)}$$
↑ unit normal.



$\Rightarrow \frac{\partial}{\partial t} g_{ij}(t) = \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle$

$\ldots = \left\langle \frac{\partial}{\partial u_i}\left(\frac{\partial F}{\partial t}\right), \frac{\partial F}{\partial u_j}\right\rangle + \left\langle \frac{\partial F}{\partial u_i}, \frac{\partial}{\partial u_j}\left(\frac{\partial F}{\partial t}\right)\right\rangle$

$= \left\langle \frac{\partial}{\partial u_i}(f\nu), \frac{\partial F}{\partial u_j}\right\rangle + \left\langle \frac{\partial F}{\partial u_i}, \frac{\partial}{\partial u_j}(f\nu)\right\rangle$

$\frac{\partial \nu}{\partial u_i} = -h_i^j \frac{\partial F}{\partial u_j}$ $\qquad = \left\langle f\frac{\partial \nu}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle + \left\langle \frac{\partial F}{\partial u_i}, f\frac{\partial \nu}{\partial u_j}\right\rangle$

$= -g^{jk} h_{ki} \frac{\partial F}{\partial u_j}$ $\qquad = f\left\langle -h_i^k \frac{\partial F}{\partial u_k}, \frac{\partial F}{\partial u_j}\right\rangle + f\left\langle \frac{\partial F}{\partial u_i}, -h_j^k \frac{\partial F}{\partial u_k}\right\rangle$

$= -f h_i^k g_{kj} - f h_j^k g_{ik} = -2f h_{ij}$

$\frac{d}{dt}\int_\Sigma \sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n = \int_\Sigma \frac{1}{2}\mathrm{Tr}_g(-2fh)\, \underbrace{\sqrt{\det[g]}\, du^1 \wedge \cdots \wedge du^n}_{d\mu_g}$

$\underline{\mathrm{Tr}_g h = g^{ij} h_{ij} = H}$ $\qquad = -\int_\Sigma f H \, d\mu_g$

$\frac{d}{dt}|\Sigma| = 0 \quad \forall f \iff H = 0$.

$\frac{\partial}{\partial t} R = \frac{\partial}{\partial t}\left(g^{ij} R_{ij}\right)$ $\qquad \frac{\partial g_{ij}}{\partial t} = v_{ij}$

$= \left(\frac{\partial}{\partial t} g^{ij}\right) R_{ij} + g^{ij} \frac{\partial}{\partial t} R_{ij}$

$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} \frac{\partial}{\partial t} g_{kl}$. $\Leftarrow$ $g^{ij} g_{kj} = \delta_k^i$

$= -g^{ik} g^{jl} v_{kl}$

$R_{ij} = R_{kij}{}^k = \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma * \Gamma - \Gamma * \Gamma$

$\frac{\partial}{\partial t} R_{ij} = \partial_k\left(\frac{\partial \Gamma_{ij}^k}{\partial t}\right) - \partial_i\left(\frac{\partial \Gamma_{kj}^k}{\partial t}\right)$ at p.

$= \underbrace{\frac{1}{2}\partial_k\left(g^{kl}(\nabla_i v_{jl}) + \cdots\right)}$

$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\nabla_i v_{jl} + \cdots)$ $\qquad$ Good luck!

$= \; - \; \text{—}$.

$\frac{\partial R}{\partial t} = -\Delta(\mathrm{Tr}_g v) - \underbrace{g(v, \mathrm{Ric})} + \nabla^l \nabla^i v_{il}$ $\qquad \nabla^j \nabla_j = g^{jk}\nabla_k \nabla_j = \Delta$

$\left[\text{If } \frac{\partial g}{\partial t} = -2\mathrm{Ric}\right]$ ← Ricci flow equation.

$\frac{\partial R}{\partial t} = -\Delta\left(-2\,\mathrm{Tr}_g \mathrm{Ric}\right) - 2g(-\mathrm{Ric}, \mathrm{Ric}) - 2\nabla^j \underbrace{\nabla^i R_{ij}}_{\frac{1}{2}\nabla_j R}$

$= 2\Delta R + 2\|\mathrm{Ric}\|^2 - \Delta R$

$= \Delta R + 2\|\mathrm{Ric}\|^2$. $\qquad$ Richard Hamilton.

$\frac{\partial}{\partial t}\int_M R\sqrt{\det g} = \int_M \left(-\Delta(\mathrm{Tr}_g v) - g(v, \mathrm{Ric}) + \nabla^l \nabla^i v_{il}\right)\sqrt{\det g}$
$+ R \cdot \underbrace{\frac{1}{2}\mathrm{Tr}_g(v) \cdot \sqrt{\det g}}$.

$= \int_M \Big(\cancel{-\Delta(\mathrm{Tr}_g v)} - g(v, \mathrm{Ric}) + \cancel{\nabla^l (\nabla^i v_{il})} + \frac{1}{2} R\, g(g, v)\Big) d\mu_g = \int_M -g(v, \mathrm{Ric}) + \frac{1}{2} R g(g, v)$

$\nabla^i(\nabla_i \mathrm{Tr}_g v)$, T_i $\qquad T_l$

$= \int_M -g\left(v, \mathrm{Ric} - \frac{1}{2} R g\right)$

(If $= 0 \; \forall v$, then $\mathrm{Ric} - \frac{1}{2} R g = 0$.)

Key: $\int_M \nabla^i T_i \, d\mu_g = 0$ $\qquad (\partial M = \emptyset)$.

$= \int_M \nabla_i T^i \, d\mu_g$ $\qquad T = \text{tangent} = T^i \frac{\partial}{\partial u_i}$

$= \int_M d(i_T d\mu_g)$ $\qquad (i_T d\mu_g)(X_1, \ldots, X_{n-1})$
↑ interior product $\qquad := d\mu_g(T, X_1, \ldots, X_{n-1})$

$\int_M d\omega = \int_{\partial M} \omega$

$\int_M \nabla^i X_i \, d\mu_g = 0$ $\qquad \int_M \nabla_i X^i \, d\mu_g = 0$.

↖ compact $\partial M = \emptyset$

$\int_M \nabla^i (S_{ji} T_k^j)\, d\mu_g = 0$

$\int_M (\nabla^i S_{ji}) T_k^j + S_{ji} \nabla^i T_k^j \, d\mu_g = 0$

$\Rightarrow \int_M (\nabla^i S_{ji}) T_k^j \, d\mu_g = -\int S_{ji} \nabla^i T_k^j \, d\mu_g$.

$\overline{\partial M \neq \emptyset}$

$\mathcal{I}_{EH} := \int_M R\, d\mu_g + \underbrace{\int_{\partial M} ?}$

Gibbons-Hawking-York's boundary term.

Bonnett-Myers'

If (M^n, g) complete Riem. mfd.

and $\mathrm{Ric} \geq k(n-1) g$
↑ constant > 0

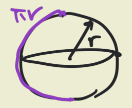
then: $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{k}}$ — (*)

(Cor: M is compact)
$|\pi_1(M)| < \infty$

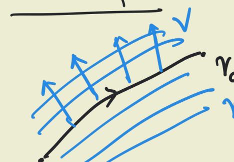
(*): equality holds $\iff$ $(M^n, g) \overset{\text{iso}}{\cong} (S^n, g_{\text{standard}})$ with radius $\frac{1}{\sqrt{k}}$.

(Shiu-Yuen Cheng).

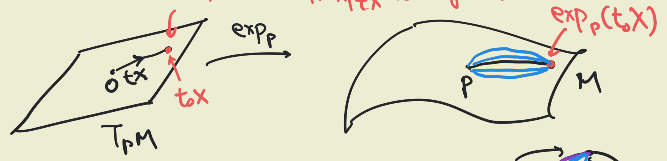
Tools:
- Jacobi field
- Index form



Jacobi field:



$V := \frac{\partial \gamma_s}{\partial s}$
↑ Jacobi field.

①


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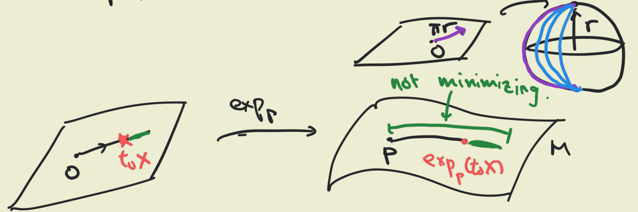


$V = \frac{\partial \gamma_s}{\partial s} = (\gamma_s)_* \frac{\partial}{\partial s}$
$T = \frac{\partial \gamma_s}{\partial t}$ $\qquad (\gamma_s)_*\left(\frac{\partial}{\partial t}\right)$

$\nabla_T T = 0 \quad \forall s$.

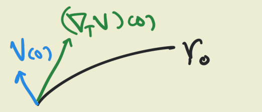
$\mathrm{Rm}(T, V)T = \nabla_T \nabla_V T - \cancel{\nabla_V \underbrace{\nabla_T T}_{=0 \; \forall s}} - \cancel{\nabla_{[T,V]} T}$

$= \nabla_T \nabla_T V$

$$\boxed{\nabla_T \nabla_T V + \mathrm{Rm}(V, T)T = 0}$$ ← Jacobi equation.

↑ 2nd-order ODE

Given $V(0)$, $(\nabla_T V)(0)$



$\Rightarrow$ existence and uniqueness

$\exists!$ $V(t)$ along γ_0 satisfies $\nabla_T \nabla_T V + \mathrm{Rm}(V, T)T = 0$.

e.g. $\nabla_T \underbrace{\nabla_T T}_{=0} + \mathrm{Rm}(T, T)T$

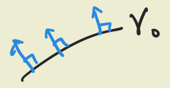
$= 0 + 0$

e.g. $V = tT$

$\nabla_T \nabla_T (tT) + \cancel{\mathrm{Rm}(tT, T)T}$

$= \nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial t}}\left(t\frac{\partial \gamma}{\partial t}\right) = \cancel{\nabla_{\frac{\partial \gamma}{\partial t}}}\left(\frac{\partial \gamma}{\partial t} + t\,\cancel{\nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial t}}\right) = 0$.

Lemma: V Jacobi field along γ_0
$\exists\, a, b \in \mathbb{R}$ s.t.
$V - aT - btT \perp \dot{\gamma}_0$



Proof: $\frac{d^2}{dt^2} g(V, \dot{\gamma}_0(t)) = \frac{d}{dt}\left[g(\nabla_T V, T) + \cancel{g(V, \nabla_T T)}\right]$

$= g(\underbrace{\nabla_T \nabla_T V}, T) + g(\nabla_T V, \cancel{\nabla_T T})$

$= g(\underline{-\mathrm{Rm}(V, T)T}, T)$

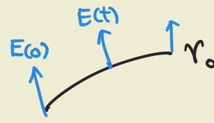
$= -\mathrm{Rm}(V, T, \underbrace{T, T}) = 0$.

$g(V, \dot{\gamma}) = a + bt$ ←

$g(V - aT - btT, \underset{\dot{\gamma}}{T}) = g(V, T) - a - bt = 0$

(M^n, g) constant sect. $= C$.

$\mathrm{Rm}(X, Y, Y, X) = C\left(\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2\right)$.



$\nabla_{\dot{\gamma}_0} E(t) = 0$.

WANT: $V(t) = u(t) E(t)$. s.t. satisfies Jac. eqt.

$\nabla_T \nabla_T V + \mathrm{Rm}(V, T)T = 0$.

$\iff \nabla_T \nabla_T (uE) + \mathrm{Rm}(uE, T)T = 0$

$\iff \nabla_T (\dot{u} E) + \mathrm{Rm}(uE, T)T = 0$

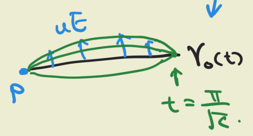
$\iff \ddot{u} E + u\, \mathrm{Rm}(E, T)T = 0$ $\qquad g(\cdot, E)$

$\Rightarrow \ddot{u} + u \underbrace{\mathrm{Rm}(E, T, T, E)}_{=C} = 0$ $\qquad$ Assume $|E|^2 = 1$, $g(E, T) = 0$.

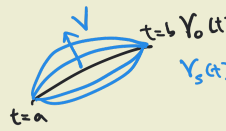
$\boxed{\ddot{u} + Cu = 0.}$ $\qquad u(0) = 0$. $u'(0) = 1$.

$$u(t) = \begin{cases} t & \text{if } C = 0 \\ \frac{1}{\sqrt{C}} \sin\sqrt{C}\, t & \text{if } C > 0 \\ \frac{1}{\sqrt{-C}} \sinh\sqrt{-C}\, t & \text{if } C < 0 \end{cases}$$

$V = uE$.



Thm 10.12 (Lee)



$V = \frac{\partial \gamma_s}{\partial s}$
$T = \frac{\partial \gamma_s}{\partial t}$

$\left.\frac{d^2}{ds^2}\right|_{s=0} L(\gamma_s) = \int_a^b |\nabla_T V|^2 - \mathrm{Rm}(V, T, T, V)\, dt$

$\langle \nabla_T V, \nabla_T V\rangle$ $\qquad = -\int_a^b g(\nabla_T \nabla_T V + \mathrm{Rm}(V, T)T, V)\, dt$