Constant sect. curvature $K_p(\Pi) = C$.

$\Rightarrow R_{ijkl} = C(g_{il}g_{jk} - g_{ik}g_{jl})$. $R_{ij} = g^{kl}R_{kijl}$

Einstein metric: $\underset{\text{constant.}}{Ric = cg}$ $R_{ij} = C(n-1)g_{ij}$

If $Ric = fg$ (f: function), then f = const. $(n\ge 3)$

$\Downarrow$

$Ric = \frac{R}{n}g$ $Ric - \frac{R}{2}g = 0$.

$R_{ij} = cg_{ij} \Rightarrow R = g^{ij}cg_{ij} = cn$ (constant scalar curvature metric)

Yamabe problem.

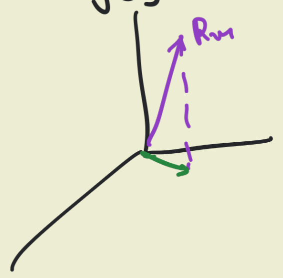
g metric with positive scalar curvature, M compact $\partial M = \emptyset$.

$\exists ?$ $f\in C^\infty(M,\mathbb{R})$ s.t. $\bar g = e^f g$ has constant scalar curvature.

$$Rm = -\frac{R}{2n(n-1)}\, g⊙g + \frac{1}{n-1}\left(Ric - \frac{R}{n}g\right)⊙g + Weyl.$$

$g(T,S) := g^{ik}g^{jl}T_{ij}S_{kl}$ (2-tensor)

$g(U,V) = g^{ip}g^{jq}g^{kr}g^{ls}U_{ijkl}V_{pqrs}$ (4-tensor)



$$Rm = -\frac{R}{2n(n-1)}\, g⊙g + f\left(Ric - \frac{R}{n}g\right)⊙g + W$$

WANT: $W \perp g⊙g$ and $W \perp (Ric - \frac{R}{n}g)⊙g$.

$Tr: \otimes^4 T^*M \to \otimes^2 T^*M$

$T_{ijkl} \mapsto g^{il}T_{ijkl}$

Exercise: 10.13

$g(\underset{\text{4-tensor}}{T}, \overset{\text{sym 2-tensor}}{h}⊙g) = -4g(\underbrace{Tr(T), h}_{\text{2-tensors}})$

4-tensor symmetric prop. and Bianchi

Need $tr(W) = 0$.

$$0 = g^{il}W_{ijkl} = g^{il}\left(R_{ijkl} + \frac{R}{2n(n-1)}(g⊙g)_{ijkl} - f\left(\left(Ric - \frac{R}{n}g\right)⊙g\right)_{ijkl}\right)$$

$\Rightarrow f = \frac{1}{n-2}$.

Ex: $W(e^f g) = e^f W(g)$

In $n=3$: $W \equiv 0$.

$g(\underset{\text{4-tensor}}{T}, \overset{\text{2-tensor}}{h}⊙g) = -4g(Tr(T), h)$

$B = \{$ 4-tensor, sym. properties, first Bianchi $\}$

- Any $T \in B$ (in dim = 3) can be written in the form $T = k⊙g$.
- $g(W,W) = g(k⊙g, W) = -4g(\underbrace{Tr(W)}_{0}, k) = 0$.

$\Rightarrow W = 0$.

$$Rm = -\frac{R}{12}g⊙g + \left(Ric - \frac{R}{3}g\right)⊙g \quad (\dim = 3)$$

$L: Sym^2 T^*M \to B$, $h \mapsto h⊙g$. injective $(n\ge 3)$.

If $h⊙g = 0$

$0 = g(h⊙g, h⊙g) \Rightarrow h_{ij} = 0 \quad \forall i,j$.

dim M = 3

dim $Sym^2 T^*M = 6$

dim $B = 6$

$b: \Lambda^2 T^*M \otimes_s \Lambda^2 T^*M \to \Lambda^4$

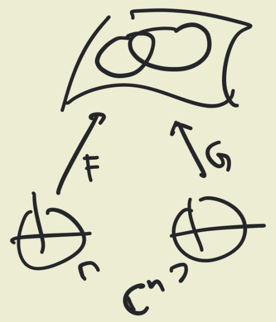
$b(T_{ijkl}) = \frac13(T_{ijkl} + T_{jkil} + T_{kijl})$.

b onto: $b((du^i\wedge du^j)\otimes_s(du^k\wedge du^l)) = \frac13 du^i\wedge du^j\wedge du^k\wedge du^l$.

$\dim B = \dim Ker(b) = \dim(\Lambda^2T^*M \otimes_s \Lambda^2T^*M) - \dim Im(b) = \frac{1}{12}n^2(n^2-1)$ (= 6 if n=3)

$\dim \Lambda^2 T^*M = C^n_2$, $\dim(\Lambda^2T^*M \otimes_s \Lambda^2T^*M) = C^{C^n_2}_2 + C^n_2$.

M complex manifold.



$G^{-1}\circ F$ is holomorphic

$F^{-1}\circ G$ is holomorphic.

$(z_1,\dots,z_n) = (x_i + \sqrt{-1}y_i, \dots)$

$(\bar z_1, \dots, \bar z_n)$

$dz^i = dx^i + \sqrt{-1}dy^i$

$d\bar z^i = dx^i - \sqrt{-1}dy^i$

$df = \frac{\partial f}{\partial x_i}dx^i + \frac{\partial f}{\partial y_i}dy^i = \frac{\partial f}{\partial z_i}dz^i + \frac{\partial f}{\partial \bar z_i}d\bar z^i$

$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}$; $\frac{\partial}{\partial z_i} = \frac12\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$

$g_{i\bar j} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_j}\right)$.

g is Hermitian metric $\overset{\text{def}}{\iff} g(JX,JY) = g(X,Y) \quad \forall X,Y\in TM$.

$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}$, $J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$

$J^2 = -id$.

$\omega(X,Y) = g(JX,Y)$.

$g_{ij} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = 0$

$g_{\bar i\bar j} = g\left(\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\right) = 0$.

$J\left(\frac{\partial}{\partial z_i}\right) = \pm\sqrt{-1}\frac{\partial}{\partial z_i}$

- $\nabla J = 0$
- $d\omega = 0$
- $\nabla\omega = 0$.

Kähler.

$J = \pm\sqrt{-1}\,dz^i\otimes\frac{\partial}{\partial z_i} \mp \sqrt{-1}\,d\bar z^i\otimes\frac{\partial}{\partial \bar z_i}$.

$\nabla_k dz^i = -\Gamma^i_{kj}dz^j - \Gamma^i_{k\bar j}d\bar z^j$

$\frac{\partial g_{i\bar j}}{\partial z_k} = \frac{\partial g_{k\bar j}}{\partial z_i}$ and $\frac{\partial g_{i\bar j}}{\partial \bar z_k} = \frac{\partial g_{i\bar k}}{\partial \bar z_j}$

$g_{i\bar j} = \partial_i\partial_{\bar j}(\quad)$

$\Gamma^k_{ij} = g^{k\bar l}\frac{\partial}{\partial z_i}g_{j\bar l}$, $R_{i\bar j k}{}^{l} = -\frac{\partial}{\partial \bar z_j}\Gamma^l_{ik}$

$R_{i\bar j} = -\partial_i\partial_{\bar j}\log\det[g_{k\bar l}]$.

Monge-Ampère.