$p \in M$, $\{X,Y\} \subset T_pM$ linearly indep., $\bar g := \iota^* g$, $\bar g(X,Y) = g(X,Y)$, $X,Y \in T_p\Sigma \subset T_pM$.

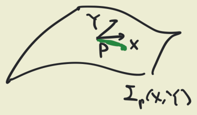
$\Sigma_p(X,Y) \xhookrightarrow{\iota} M$.

Need: $\underbrace{\frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}}_{K_p(X,Y)} = \frac{\overline{Rm}(X,Y,Y,X)}{|X|^2|Y|^2 - \bar g(X,Y)}$ — Gauss curvature of $\Sigma_p(X,Y)$ at p. $g(X,Y)$, $\bar g(X,Y)$

$\Leftrightarrow Rm(X,Y,Y,X) = \overline{Rm}(X,Y,Y,X)$

Define $G(u,v) := \exp_p(uX+vY)$

$\frac{\partial G}{\partial u}\Big|_{(0,0)} = X_p$, $\frac{\partial G}{\partial v}\Big|_{(0,0)} = Y_p$



$Rm(X,Y)Y = Rm(\partial_u,\partial_v)\partial_v$ at p.

$= \nabla_{\partial_u}\nabla_{\partial_v}\partial_v - \nabla_{\partial_v}\nabla_{\partial_u}\partial_v$

(M,∇,g), $(\Sigma,\bar\nabla,\bar g)$

$\nabla_u\partial_v(p) = \nabla_v\partial_u(p) = \nabla_v\partial_u(p) = \nabla_u\partial_u(p) = 0$.

$\bar\nabla_Z W = (\nabla_Z W)^{T\Sigma}$

$\nabla_Z W = \underbrace{\bar\nabla_Z W}_{T\Sigma} + \underbrace{h(Z,W)}_{N\Sigma}$

$\bar\nabla_u\partial_v + h(\partial_u,\partial_v) = 0 \Rightarrow \begin{cases}\bar\nabla_u\partial_v = 0\\ h(\partial_u,\partial_v)=0\end{cases}$

$= \nabla_{\partial_u}(\bar\nabla_{\partial_v}\partial_v + h(\partial_v,\partial_v)) - \nabla_{\partial_v}(\bar\nabla_{\partial_u}\partial_v + h(\partial_u,\partial_v))$

$= \bar\nabla_{\partial_u}(\bar\nabla_{\partial_v}\partial_v + h(\partial_v,\partial_v)) + h(\partial_u, \bar\nabla_{\partial_v}\partial_v + h(\partial_v,\partial_v))$

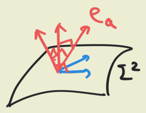
$- \bar\nabla_{\partial_v}(\bar\nabla_{\partial_u}\partial_v + h(\partial_u,\partial_v)) - h(\partial_v, \bar\nabla_{\partial_u}\partial_v + h(\partial_u,\partial_v))$

$= \overline{Rm}(\partial_u,\partial_v)\partial_v + \bar\nabla_{\partial_u}(h(\partial_v,\partial_v)) - \bar\nabla_{\partial_v}(h(\partial_u,\partial_v))$

Last step: $g(\bar\nabla_{\partial_u}(h(\partial_v,\partial_v)),\partial_u) = 0$ at p ✓

$g(\bar\nabla_{\partial_v}(h(\partial_u,\partial_v)),\partial_u) = 0$ ✓

$a = 1,2,\dots,n-2$, e_a



in M^5

Write $h(\partial_v,\partial_v) = h^a_{vv}e_a$

$g(\bar\nabla_{\partial_u}(h^a_{vv}e_a),\partial_u) = g((\partial_u h^a_{vv})e_a,\partial_u) + g(h^a_{vv}\bar\nabla_{\partial_u}e_a,\partial_u) = 0$ at p.

$h(\partial_u,\partial_v) = 0$ at p

Constant sect.:

$K_p(X,Y) = C \quad \forall p\in M,\ \forall\{X,Y\}\in Gr(2,T_pM)$.

Claim: $R_{ijkl} = C(g_{il}g_{jk} - g_{ik}g_{jl})$.

Kulkarni–Nomizu's product.

h,k: symmetric 2-tensor.

$h\owedge k$: 4-tensor

$(h\owedge k)(X,Y,Z,W) := h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W)$

$h\owedge k$ satisfies symmetric properties ($\sim Rm$) and first Bianchi identity.

$(h\owedge k)(X,Y,Y,X)$ determines $(h\owedge k)$.

$(g\owedge g)(X,Y,Y,X) = 2g(X,Y)g(Y,X) - 2g(X,X)g(Y,Y) = -2(|X|^2|Y|^2 - g(X,Y)^2)$

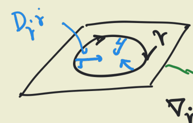
Suppose Rm has constant sect.

$Rm(X,Y,Y,X) = C(|X|^2|Y|^2 - g(X,Y)^2) = -\frac{C}{2}(g\owedge g)(X,Y,Y,X)$

$(Rm + \frac{C}{2}g\owedge g)(X,Y,Y,X) = 0 \Rightarrow Rm + \frac{C}{2}g\owedge g \equiv 0$.

$Rm = -\frac{C}{2}g\owedge g$.

$R_{ijkl} = -\frac{C}{2}(g\owedge g)_{ijkl} = -\frac{C}{2}(2g_{ik}g_{jl} - 2g_{il}g_{jk}) = C(g_{il}g_{jk} - g_{ik}g_{jl})$.



$\begin{cases} x_1^2+\dots+x_{n+1}^2 = r^2\\ a_1x_1+\dots+a_{n+1}x_{n+1} = 0\end{cases}$

$\nabla_{\dot\gamma}\dot\gamma = (D_{\dot\gamma}\dot\gamma)^{TS^n} = 0$

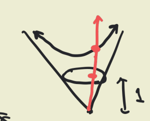
$(S^n_{(r)}, \iota^*\delta)$ Gauss curvature of $\Sigma_p(X,Y) = \frac{1}{r^2}$

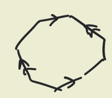
$\Rightarrow R_{ijkl} = \frac{1}{r^2}(g_{il}g_{jk} - g_{ik}g_{jl})$

$g = \frac{dx^1\otimes dx^1 + \dots + dx^n\otimes dx^n}{x_1^2}$

$M^n = \{(x_1,\dots,x_n): x_1 > 0\}$.

$g_{ij} = x_1^{-2}\delta_{ij}$





Given $\frac14 < K_p(X,Y) \le 1$, then ?

Berger, Klingenberg. M^n homeomorphic to S^n or S^n/G.

(2007) Brendle, Schoen

M^n is diffeo. to S^n/G (with standard diff. str.)

$R_{ijkl} = -R_{jikl}$

$Rm(X\wedge Y, Z\wedge W): \Lambda^2TM\times\Lambda^2TM \to \mathbb{R}$

$\mathcal{R}: \Lambda^2TM \longrightarrow \Lambda^2TM$

curvature operator.

$T_{ij} = T(\partial_i,\partial_j)$, $T: TM\times TM \to \mathbb{R}$.

$\tilde T_i^{\ j} = g^{jk}T_{ik}$ $\quad \tilde T: TM \to TM$

$\tilde T(\partial_i) := \underbrace{g^{jk}T_{ik}}_{g^{jk}T(\partial_i,\partial_k)}\partial_j$

$\mathcal{R}(\partial_i\wedge\partial_j) := -Rm(\partial_i,\partial_j,g^{pk}\partial_p,g^{ql}\partial_q)\,\partial_k\wedge\partial_l$

$\mathcal{R}: \Lambda^2TM \to \Lambda^2TM$

$Im(\mathcal{R}) < \Lambda^2TM \cong so(n)$ $\begin{pmatrix} 0 & & 1\\ & \ddots & \\ -1 & & 0\end{pmatrix}$

$T_I Hol_p(M,g) = hol_p(M,g)$

Ambrose–Singer

Ric: $TM\times TM \to \mathbb{R}$

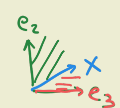
$Ric(X,Y) := \sum_{i=1}^n Rm(e_i,X,Y,e_i)$

$\{e_i\}$ o.n. frame of TM.

$Ric(X,X) = Rm(\cancel{e_1,e_1,e_1},e_1) + Rm(e_2,X,X,e_2) + \dots + Rm(e_n,X,X,e_n)$

$\{e_1,e_2,\dots,e_n\}$, $X = e_1$

$= K(e_2,X) + \dots + K(e_n,X)$



$R_{ij} = Ric(\partial_i,\partial_j)$, $\quad Ric = R_{ij}du^i\otimes du^j$.

$= \sum_{k=1}^n Rm(e_k,\partial_i,\partial_j,e_k)$ $\quad e_k = A_k^i\frac{\partial}{\partial u_i}$

$= \sum_{p,q,k=1}^n Rm(A_k^p\partial_p,\partial_i,\partial_j,A_k^q\partial_q)$ $\quad g(e_k,e_l) = A_k^iA_l^jg_{ij}$, $\delta_{kl} = A_k^ig_{ij}(A^T)^j_l$

$= \sum_{p,q=1}^n g^{pq}R_{pijq}$ $\quad I = [A][g][A^T]$, $[g] = A^{-1}(A^T)^{-1}$, $[g]^{-1} = A^TA$

$= \sum_{p=1}^n R_{pij}^{\ \ \ \ p}$ $\quad \Rightarrow \sum_k A_k^iA_k^j = g^{ij}$

$R = \sum_{i=1}^n Ric(e_i,e_i) = Ric(A_i^k\partial_k, A_i^l\partial_l) = g^{kl}R_{kl}$

scalar curvature (MATH)

Ricci scalar (PHYS)

(contracted Bianchi identity)

$\nabla^iR_{ij} = \frac12\frac{\partial R}{\partial u_j}$, $\quad \nabla^i R_{ij} = g^{ik}\nabla_kR_{ij}$

$\vec F = a^i\frac{\partial}{\partial x_i}$, $\quad div(\vec F) = \sum_i\frac{\partial a^i}{\partial x_i}$

$a^i\frac{\partial}{\partial x_i} = \underbrace{a^i\frac{\partial y_j}{\partial x_i}}_{\tilde a^j}\frac{\partial}{\partial y_j}$

$\sum_j\frac{\partial\tilde a^j}{\partial y_j} = \frac{\partial}{\partial y_j}\left(a^i\frac{\partial y^j}{\partial x_i}\right) \neq \sum\frac{\partial a^i}{\partial x_i}$

$div F = \nabla_i a^i$

$\nabla_iR_{jklp} + \nabla_jR_{kilp} + \nabla_kR_{ijlp} = 0$

$\times g^{jp}$: $\nabla_i(g^{jp}R_{jklp}) + \nabla_j(g^{jp}R_{kilp}) + \nabla_k(g^{jp}R_{ijlp}) = 0$

$\Rightarrow \nabla_iR_{kl} + \nabla^pR_{kilp} - \nabla_kR_{il} = 0$

$\cdot g^{kl}$: $\nabla_iR - \nabla^pR_{ip} - \nabla^lR_{il} = 0$

$\nabla_iR = 2\nabla^jR_{ij}$

$\frac{\partial R}{\partial u_i}$ $\quad \Rightarrow \nabla^jR_{ij} = \frac12\frac{\partial R}{\partial u_i}$.

Cor: $\underbrace{\nabla^j\left(R_{ij} - \frac R2 g_{ij}\right)}_{} = \cancel{\frac12\frac{\partial R}{\partial u_i}} - \cancel{\frac12 g_{ij}\nabla^jR} = 0$ $\quad g_{ij}g^{kj}\nabla_kR = \nabla_iR$

Einstein's tensor. $G_{\mu\nu} = R_{\mu\nu} - \frac R2 g_{\mu\nu} = 0$.

$\boxed{G_{\mu\nu} = T_{\mu\nu}}$

Given $R_{ij} = \frac R2 g_{ij}$. $\quad g^{ij}g_{ij} = n$.

$g^{ij}R_{ij} = \frac R2 g^{ij}g_{ij}$

$R = \frac R2\sum_i\delta_i^i = \frac R2 n \xRightarrow{n\neq 2} R = 0 \xRightarrow{Ric = \frac R2 g} Ric \equiv 0$.

(Kähler & $Ric\equiv 0$) ← Calabi–Yau manifold.

Given $Ric = fg$ $\quad (n\ge 3)$

$R_{ij} = fg_{ij} \Rightarrow g^{ij}R_{ij} = fg^{ij}g_{ij}$

$\Rightarrow R = fn \Rightarrow f = \frac Rn$.

$\Rightarrow Ric = \frac Rn g$. $\quad R_{ij} = \frac Rn g_{ij}$

$\nabla^iR_{ij} = \nabla^i\left(\frac Rn g_{ij}\right)$

$\frac12\frac{\partial R}{\partial u_j} = \frac1n g_{ij}\nabla^iR = \frac1n\nabla_jR = \frac1n\frac{\partial R}{\partial u_j}$.

$n\ge 3 \Rightarrow \frac{\partial R}{\partial u_j} \equiv 0 \Rightarrow R = C$.

g is Einstein $\overset{def}{\Leftrightarrow}$ $Ric = cg$, c constant.

Aubin, Yau showed $c_1 < 0$, K.E. exists.

Yau $c_1 = 0$, K.E. exists. $Ric \equiv 0$

$c_1 > 0$ when K.E. exists? (Yau–Tian–Donaldson conjecture)