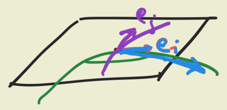
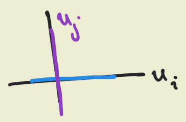


$e_i \in T_pM$
$\{e_i\}$ o.n.



$E: \mathbb{R}^n \to T_pM$
$(u_1,\dots,u_n) \mapsto \sum_{i=1}^n u_i e_i$

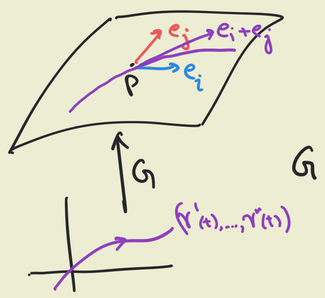
$G = \exp_p \circ E : \mathbb{R}^n \to M$
local diffeo.

① $\frac{\partial G}{\partial u_i}(p) = G_*\left(\frac{\partial}{\partial u_i}\right) = (\exp_p)_* \circ E_*\left(\frac{\partial}{\partial u_i}\right) = (\exp_p)_*(e_i)$, $\quad \frac{\partial E}{\partial u_i} = e_i$, $\quad T_{E(x)=0}$

$g_{ij} = g\left(\frac{\partial G}{\partial u_i}, \frac{\partial G}{\partial u_j}\right) = g(e_i,e_j) = \delta_{ij}$ at p.

② $\Gamma^k_{ij}(p) = 0 \Leftrightarrow \partial_k g_{ij} = 0$ $\quad \Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \cdots)$ $\quad \frac{\partial g_{ij}}{\partial u_k} = \Gamma^?_{??} g_{??} + \Gamma^?_{??} g_{??}$

③



$\gamma(t) := \exp_p(t(e_i+e_j))$
$= G(\gamma^1(t),\dots,\gamma^n(t))$
$= \exp_p \circ E(\gamma^1(t),\dots,\gamma^n(t))$ $\quad 0<t\ll 1$

$\gamma^1(t)e_1 + \cdots + \gamma^n(t)e_n = t(e_i+e_j)$
$\Rightarrow \gamma^i(t) = \gamma^j(t) = t, \quad \gamma^k(t) = 0 \ (k \neq i,j)$

Geodesic equation:
$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0.$$
$\Rightarrow \Gamma^k_{ij} = 0 \quad \forall k$ along $\gamma(t) \ni p$.
$\Rightarrow \Gamma^k_{ij}(p) = 0 \quad \forall i,j,k$.

Gauss equation:
$$\nabla_i\left(\nabla_j \frac{\partial F}{\partial u_k}\right) - \nabla_j\left(\nabla_i \frac{\partial F}{\partial u_k}\right) = g^{ql}(h_{jk}h_{li} - h_{ik}h_{lj})\frac{\partial}{\partial u_q}$$
$\Gamma^q_{jk}\frac{\partial F}{\partial u_q}$

WANT: $T^{(3,1)}(\partial_i,\partial_j,\partial_k) = \nabla_i\left(\nabla_j\frac{\partial}{\partial u_k}\right) - \nabla_j\left(\nabla_i\frac{\partial}{\partial u_k}\right)$.

LHS $= T^q_{ijk}\frac{\partial}{\partial u_q}$. $\quad K \overset{\dim 2}{=} \frac{\pm g_{1q}T^q_{212}}{\det[g]}$

$T(X,Y,Z) \overset{?}{:=} \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)$
gives $T(\partial_i,\partial_j,\partial_k) = \cdots$

Not tensorial!
$T(fX,Y,Z) = \nabla_{fX}(\nabla_Y Z) - \nabla_Y(\nabla_{fX}Z)$
$= f\nabla_X(\nabla_Y Z) - \nabla_Y(f\nabla_X Z)$
$= f\nabla_X\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z$
$= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - Y(f)\nabla_X Z.$

$\nabla_{Y(f)X}Z$

$[fX,Y]\varphi = fXY\varphi - Y(fX\varphi)$
$= fXY\varphi - Y(f)\cdot X\varphi - fYX\varphi$
$= f[X,Y]\varphi - Y(f)X\varphi$

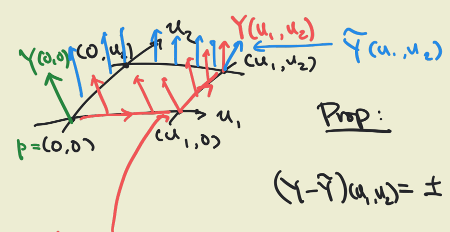
$\Rightarrow [fX,Y] = f[X,Y] - Y(f)X$

$\nabla_{f[X,Y]}Z \quad \nabla_{[fX,Y]}Z$

$$\nabla_{fX}(\nabla_Y Z) - \nabla_Y(\nabla_{fX}Z) - \nabla_{[fX,Y]}Z = f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)$$

Definition: Riemann curvature (3,1)-tensor
$Rm(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$
$Rm(\partial_i,\partial_j)\partial_k = \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k$ ($[\partial_i,\partial_j]=0$)
$= R^q_{ijk}\frac{\partial}{\partial u_q}$ $\quad R_{ijk}{}^q \quad R_{kij}{}^q \quad R_{kji}{}^q$

$Rm\left(\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2}\right)Y$ $\quad$ geometric meaning?



Prop:
$$(Y-\tilde Y)(u_1,u_2) = \pm Rm(\partial_1,\partial_2)Y\big|_{(0,0)}\cdot u_1u_2 + O(|\vec u|^3)$$

$Y^k(u_1,0) = Y^k(0,0) + \frac{\partial Y^k}{\partial u_1}(0,0)\cdot u_1 + \frac{\partial^2 Y^k}{\partial u_1^2}(0,0)\cdot u_1^2 + O(u_1^3)$

$\nabla_{\partial_1}Y = 0 \Rightarrow \frac{\partial Y^k}{\partial u_1} + \Gamma^k_{1i}Y^i = 0$ along $(0,0) \to (u_1,0)$
$\frac{\partial^2 Y^k}{\partial u_1^2} + (\partial_1\Gamma^k_{1i})Y^i + \Gamma^k_{1i}\frac{\partial Y^i}{\partial u_1} = 0$

$Y^k(u_1,u_2) = Y^k(u_1,0) + \frac{\partial Y^k}{\partial u_2}(u_1,0)\cdot u_2 + \frac{1}{2!}\frac{\partial^2 Y^k}{\partial u_2^2}(u_1,0)\cdot u_2^2 + \cdots$

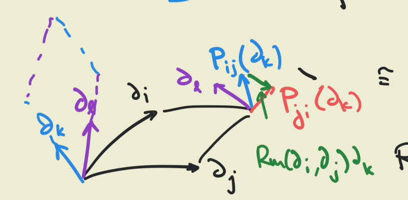
$Rm^{(3,1)}(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$
$Rm^{(4,0)}(X,Y,Z,W) := g(Rm^{(3,1)}(X,Y)Z, W).$

$\det[h] = \pm g_{2q}R^q_{121}$
$= \pm R_{1212}$

$Rm(\partial_1,\partial_2,\partial_1,\partial_2)$
$= g(Rm(\partial_1,\partial_2)\partial_1, \partial_2)$
$= g(R_{121}{}^q\partial_q, \partial_2)$
$= g_{2q}R_{121}{}^q$

$R_{ijkl} = Rm^{(4,0)}(\partial_i,\partial_j,\partial_k,\partial_l)$

- $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$
- $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (Bianchi identity).



$\approx R_{ijk}{}^q\partial_q$. $\quad R_{ijkl} = g(R_{ijk}{}^q\partial_q, \partial_l) = R_{ijk}{}^q g_{ql}$.

$R_{ijkl} = R_{klij}$:
$R_{ijkl} = g_{pl}R_{ijk}{}^p = g_{pl}\left(\partial_i\Gamma^p_{jk} - \partial_j\Gamma^p_{ik} + \cancel{\Gamma^q_{jk}\Gamma^p_{iq}} - \cancel{\Gamma^q_{ik}\Gamma^p_{jq}}\right)$
$= \delta_{pl}\left(\frac{\partial}{\partial u_i}\left(\frac{1}{2}g^{pq}(\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk})\right) \cdots\right)$
$= \delta_{pl}\left(\frac{1}{2}g^{pq}(\partial_i\partial_j g_{kq} + \partial_i\partial_k g_{jq} - \partial_i\partial_q g_{jk}) - \frac{1}{2}g^{pq}(\partial_j\partial_i g_{kq} + \partial_j\partial_k g_{iq} - \partial_j\partial_q g_{ik})\right)$ $\quad \delta_{pq}$
$= \frac{1}{2}(\cancel{\partial_i\partial_j g_{kl}} + \partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \cancel{\partial_j\partial_i g_{kl}} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik})$ $\quad$ i and j cancel

$R_{klij} = \frac{1}{2}(\cancel{\partial_k\partial_l g_{ij}} + \partial_k\partial_i g_{jl} - \partial_k\partial_j g_{li} - \cancel{\partial_l\partial_k g_{ij}} - \partial_l\partial_i g_{kj} + \partial_l\partial_j g_{ki})$

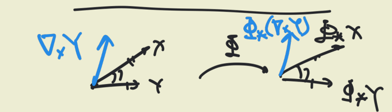
$R_{ijkl} - R_{klij} = 0.$ $\quad T(X,Y,Z,W) := Rm(X,Y,Z,W) - Rm(Z,W,X,Y)$

$R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (Exercise) $\quad T(X,Y,Y,X)$
first Bianchi identity.

$\nabla_i R_{jklp} + \nabla_j R_{kilp} + \nabla_k R_{ijlp} = 0.$ (HW).
second Bianchi identity

$R_{ijkl} = \partial\Gamma - \partial\Gamma + \Gamma*\Gamma - \Gamma*\Gamma$
$\partial_p R_{ijkl} = \partial\partial\Gamma - \partial\partial\Gamma + \partial\Gamma*\Gamma - \Gamma*\partial\Gamma$

Lemma: Given $\Phi: (M,g) \to (\tilde M,\tilde g)$ isometry. (u_i, v_α)
$\Phi^*\tilde g = g \Leftrightarrow \tilde g(\Phi_*X, \Phi_*Y) = g(X,Y)$



then $\Phi_*(\nabla_X Y) = \tilde\nabla_{\Phi_*X}\Phi_*Y.$
$\Updownarrow$
$\nabla_X Y = (\Phi^{-1})_*(\tilde\nabla_{\Phi_*X}\Phi_*Y)$ $\quad$ HW
- connection on M
- satisfies torsion-free, metric comp.

$\Phi(u_i) = (\Phi^1(u),\dots,\Phi^n(u))$
$\Phi_*(\partial_i) = \frac{\partial\Phi^j}{\partial u_i}\frac{\partial}{\partial v_j}$

Prop: $(M,g) \xrightarrow{\Phi} (\hat M,\hat g)$ isometry.
then $\Phi^*\widetilde{Rm}^{(4,0)} = Rm^{(4,0)} \to \widetilde{Rm}^{(4,0)}(\Phi_*X,\Phi_*Y,\Phi_*Z,\Phi_*W) = Rm^{(4,0)}(X,Y,Z,W).$

$\widetilde{Rm}^{(4,0)}(\Phi_*X,\Phi_*Y,\Phi_*Z,\Phi_*W)$
$= \tilde g\left(\tilde\nabla_{\Phi_*X}\tilde\nabla_{\Phi_*Y}\Phi_*Z - \tilde\nabla_{\Phi_*Y}\tilde\nabla_{\Phi_*X}\Phi_*Z - \tilde\nabla_{[\Phi_*X,\Phi_*Y]}\Phi_*Z, \Phi_*W\right)$
$\Phi_*(\nabla_Y Z)$ $\quad \tilde\nabla_{\Phi_*[X,Y]}\Phi_*Z$
$\Phi_*(\nabla_X\nabla_Y Z) - \Phi_*(\nabla_Y\nabla_X Z) - \Phi_*(\nabla_{[X,Y]}Z)$
$= \underbrace{(\Phi^*\tilde g)}_{g}(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$
$= Rm(X,Y,Z,W).$

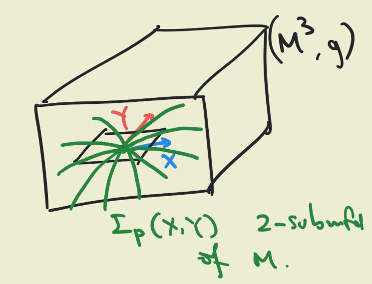
$\widetilde{Rm}^{(3,1)}(\Phi_*X,\Phi_*Y)\Phi_*Z = \Phi_*(Rm^{(3,1)}(X,Y)Z)$
on $\tilde M$ $\quad$ on M

$\tilde g(\widetilde{Rm}^{(3,1)}(\Phi_*X,\Phi_*Y)\Phi_*Z, \Phi_*W) = g(Rm^{(3,1)}(X,Y)Z, W)$
$= \tilde g(\Phi_*Rm^{(3,1)}(X,Y)Z, \Phi_*W)$

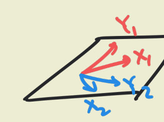
$\Sigma^2 \subset \mathbb{R}^3$
$$K = \frac{\pm g_{1q}R_{121}{}^q}{g_{11}g_{22} - g_{12}^2} = \frac{\pm R_{1221}}{g_{11}g_{22} - g_{12}^2}$$
$\det[h]$ $\quad g_{11} = \langle\partial_1,\partial_1\rangle = |\partial_1|^2$

(M^n,g). $\quad X,Y$ linearly indep.
$$K(X,Y) := \frac{Rm^{(4,0)}(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$
sectional curvature along $\{X,Y\}$

= Gauss curvature of $\Sigma_p(X,Y)$ at p.



$\tilde g := \iota^*g$
$\tilde g(U,V) := g(U,V)$



$\Pi \subset T_pM$

$$\frac{Rm(X_1,Y_1,Y_1,X_1)}{|X_1|^2|Y_1|^2 - g(X_1,Y_1)^2} = \frac{Rm(X_2,Y_2,Y_2,X_2)}{|X_2|^2|Y_2|^2 - g(X_2,Y_2)^2}$$

$X_1 = aX_2 + bY_2$ $\quad Rm(\underbrace{aX_2+bY_2}_{X_1}, \dots$
$Y_1 = cX_2 + cY_2$
$\Pi \subset T_pM$
$K_p(\Pi) := K_p(X,Y)$. where $\{X,Y\}$ is any basis for Π.