SELECTED SOLUTION. #1, #2, #3a,b.

#1. • First fundamental form of Σ:

$$\frac{\partial G}{\partial u_i} = \underbrace{\frac{\partial}{\partial u_i}(f\circ F)}_{=: \frac{\partial f}{\partial u_i}} \cdot F + (f\circ F)\frac{\partial F}{\partial u_i}$$

$$\Rightarrow \tilde{g}_{ij} = \left\langle \frac{\partial G}{\partial u_i}, \frac{\partial G}{\partial u_j} \right\rangle = \boxed{\frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j} + (f\circ F)^2 g_{ij}}$$

Note $F \perp \frac{\partial F}{\partial u_i}$ for S^n.

• Second fundamental form of Σ:

First need to find unit normal to Σ.
Use $\left\{ \frac{\partial F}{\partial u_1}(p), \ldots, \frac{\partial F}{\partial u_n}(p), F(p) \right\}$ as basis of $\mathbb{R}^{n+1}$ at $f(p)p$.

Express $\nu = \sum_{i=1}^{n} a^i \frac{\partial F}{\partial u_i} + bF$.

We solve:

$$\left\langle \frac{\partial G}{\partial u_k}, \nu \right\rangle = 0 \qquad \forall k = 1, \ldots, n$$

$$\Leftrightarrow \left\langle \frac{\partial f}{\partial u_k} F + (f\circ F)\frac{\partial F}{\partial u_k}, a^i \frac{\partial F}{\partial u_i} + bF \right\rangle = 0$$

$$\Leftrightarrow a^i (f\circ F) g_{ik} + b\frac{\partial f}{\partial u_k} = 0$$

$$\Leftrightarrow a^i = -\frac{b}{f\circ F} g^{ik}\frac{\partial f}{\partial u_k} = -b g^{ik}\frac{\partial}{\partial u_k}\log f$$

$$a^i = -b\nabla^i \log f \leftarrow$$

∇ = Levi-Civita connection w.r.t. g.

We also need $|\nu| = 1$, so

$$1 = \left\langle a^i \frac{\partial F}{\partial u_i} + bF, a^j \frac{\partial F}{\partial u_j} + bF \right\rangle = a^i a^j g_{ij} + b^2$$

$$= b^2 \left(g_{ij} \nabla^i \log f \cdot \nabla^j \log f + 1 \right) = b^2 \left(1 + |\nabla \log f|^2 \right)$$

$$\Rightarrow b = 1 \Big/ \sqrt{1 + |\nabla \log f|^2} \quad \Rightarrow \quad \nu = \frac{-\sum_{i=1}^{n} (\nabla^i \log f)\frac{\partial F}{\partial u_i} + F}{\sqrt{1 + |\nabla \log f|^2}}$$

for S^n, $h_{ij} = g_{ij}$, unit normal $= F$.

$$\frac{\partial^2 G}{\partial u_i \partial u_j} = \frac{\partial}{\partial u_j}\left(\frac{\partial f}{\partial u_i} F + (f\circ F)\frac{\partial F}{\partial u_i} \right)$$

$$= \frac{\partial^2 f}{\partial u_i \partial u_j} F + \frac{\partial f}{\partial u_i}\frac{\partial F}{\partial u_j} + \frac{\partial f}{\partial u_j}\frac{\partial F}{\partial u_i} + \underbrace{(f\circ F)\frac{\partial^2 F}{\partial u_i \partial u_j}}_{= (f\circ F)\left(\Gamma^l_{ij}\frac{\partial F}{\partial u_l} + g_{ij}F\right)}$$

By $\nu = \dfrac{-\sum_{k=1}^{n} (\nabla^k \log f)\frac{\partial F}{\partial u_k} + F}{\sqrt{1+|\nabla \log f|^2}}$,

one can easily show:

$$\tilde{h}_{ij} = \left\langle \frac{\partial^2 G}{\partial u_i \partial u_j}, \nu \right\rangle$$

$= \frac{\partial}{\partial u_j}\log f = \frac{1}{f}\frac{\partial f}{\partial u_j}$

$$= \frac{1}{\sqrt{1+|\nabla \log f|^2}}\left(-\frac{\partial f}{\partial u_i} g_{jk}\nabla^k \log f - \frac{\partial f}{\partial u_j} g_{ik}\nabla^k \log f - (f\circ F) g_{kl}\Gamma^l_{ij}\nabla^k \log f + \frac{\partial^2 f}{\partial u_i \partial u_j} + (f\circ F) g_{ij} \right)$$

$g_{kl}\Gamma^l_{ij}\nabla^k f = \Gamma^l_{ij}\frac{\partial f}{\partial u_l}$

$$= \frac{1}{\sqrt{1+|\nabla \log f|^2}}\left(\nabla_i\nabla_j f + f g_{ij} - \frac{2}{f}\frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j} \right)$$

$\nabla_i\nabla_j f = \frac{\partial^2 f}{\partial u_i \partial u_j} - \Gamma^l_{ij}\frac{\partial f}{\partial u_l}$

abb. $f := f\circ F$

Recall $\nabla^k = g^{kq}\nabla_q$.

$$= \frac{1}{\sqrt{1+|\nabla \log f|^2}} \cdot f g_{ik}\left(\delta_{kj} + \frac{1}{f}\nabla^k\nabla_j f - \frac{2}{f^2}\nabla^k f \frac{\partial f}{\partial u_j} \right)$$

$$= \frac{1}{\sqrt{1+|\nabla \log f|^2}} f g_{ik}\left(\delta_{kj} + f^2 \nabla^k\left(\frac{1}{f^2}\nabla_j f\right) \right)$$

from experience.

- mean curvature:

By Sherman-Morrison's formula: $(A+uv^T)^{-1} = A^{-1} - \dfrac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}.$

$$\tilde g^{ij} = \underbrace{\frac{1}{f^2}g^{ij}}_{A^{-1}} - \frac{\frac{1}{f^2}g^{ik}\frac{\partial f}{\partial u_k}\frac{\partial f}{\partial u_l}\frac{1}{f^2}g^{lj}}{1+\frac{\partial f}{\partial u_i}\cdot\frac{1}{f^2}g^{ij}\cdot\frac{\partial f}{\partial u_j}}$$

$$= \frac{1}{f^2}g^{ij} - \frac{\frac{1}{f^2}\nabla^i\log f\cdot\nabla^j\log f}{1+|\nabla\log f|^2}$$

$$= \frac{1}{f^2}g^{ik}\left(\delta_{kj} - \underbrace{\frac{\frac{\partial}{\partial u_k}\log f\cdot\nabla^j\log f}{1+|\nabla\log f|^2}}_{\text{rank 1 matrix}}\right)$$

has eigenvalues: $\left\{1-\dfrac{|\nabla\log f|^2}{1+|\nabla\log f|^2},\ 1,\dots,1\right\}$

$= \left\{\dfrac{1}{1+|\nabla\log f|^2},\ 1,\dots,1\right\}$

rank 1 matrix with eigenvector $\frac{\partial}{\partial u_j}\log f$

since $\left(\frac{\partial}{\partial u_k}\log f\cdot\nabla^j\log f\right)\frac{\partial}{\partial u_j}\log f$

$= \underbrace{|\nabla\log f|^2}_{\text{eigenvalue}}\frac{\partial}{\partial u_k}\log f$

$$\therefore\ \tilde H = \tilde g^{ij}\tilde h_{ij}$$

$$= \frac{1}{f^2}g^{ik}\left(\delta_{kj} - \frac{\nabla_k\log f\cdot\nabla^j\log f}{1+|\nabla\log f|^2}\right)$$

$$\frac{1}{\sqrt{1+|\nabla\log f|^2}}\, f\, g_{il}\left(\delta_{lj} + f^2\nabla^l\left(\frac{1}{f^2}\nabla_j f\right)\right)$$

$$= \frac{1}{f\sqrt{1+|\nabla\log f|^2}}\left(\delta_{kj} - \frac{\nabla_k\log f\cdot\nabla^j\log f}{1+|\nabla\log f|^2}\right)\left(\delta_{kj} + f^2\nabla^k\left(\frac{1}{f^2}\nabla_j f\right)\right)$$

$\left[f^2 \nabla^k \left(\frac{1}{f^2} \nabla_j f \right) \right]$ is a symmetric matrix

$\Rightarrow$ eigenvalues $\in \mathbb{R}$.

let eigenvalues be

$\{\lambda_1, \ldots, \lambda_n\}$

$$[\tilde{h}_{ij}] = \frac{f}{\sqrt{1+|\nabla \log f|^2}} [\tilde{g}_{ik}] \underbrace{\left[\delta_{kj} + f^2 \nabla^k \left(\frac{1}{f^2} \nabla_j f \right) \right]}_{\text{eigenvalues: } \{1+\lambda_i\}}$$

then

$$K = \frac{\det[\tilde{h}_{ij}]}{\det[\tilde{g}_{ij}]} = \det[\tilde{h}_{ij}] \det[\tilde{g}^{ij}]$$

$$= \frac{f^n}{(1+|\nabla \log f|^2)^{\frac{n}{2}}} \cancel{\det[g_{ik}]} \, (1+\lambda_1)\cdots(1+\lambda_n) \cdot \frac{1}{f^{2n}} \cancel{\det[g^{ik}]} \cdot \frac{1}{1+|\nabla \log f|^2}$$

$$= \frac{1}{f^n} \cdot \frac{(1+\lambda_1)\cdots(1+\lambda_n)}{(1+|\nabla \log f|^2)^{\frac{n}{2}+1}}$$

$\blacksquare$

#2

$$g_{FS} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\left(1+|z_1|^2+\cdots+|z_n|^2\right) dz^i \otimes d\bar{z}^j$$

$$+ \frac{\partial^2}{\partial \bar{z}_i \partial z_j} \log\left(1+|z_1|^2+\cdots+|z_n|^2\right) d\bar{z}^i \otimes dz^j$$

conjugate of each other.

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\left(1+|z_1|^2+\cdots+|z_n|^2\right)$$

$$= \frac{\partial}{\partial z_i}\left(\frac{1}{1+Z^2} \frac{\partial}{\partial \bar{z}_j}\left(z_1\bar{z}_1+\cdots+z_n\bar{z}_n\right)\right)$$

$$Z := \sqrt{|z_1|^2+\cdots+|z_n|^2}$$

$$= \frac{\partial}{\partial z_i}\left(\frac{z_j}{1+Z^2}\right)$$

$$= \frac{(1+Z^2)\delta_{ij} - z_j \bar{z}_i}{(1+Z^2)^2} = \frac{1}{1+Z^2}\left(\delta_{ij} - \frac{z_j}{\sqrt{1+Z^2}} \frac{\bar{z}_i}{\sqrt{1+Z^2}}\right)$$

$$[z_j \bar{z}_i] = \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix} [z_1 \cdots z_n]$$ has rank 1 ☆

with eigenvector $(\bar{z}_1, \ldots, \bar{z}_n)^T$:

$$[z_j \bar{z}_i] \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix} = \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix} \underbrace{[z_1 \cdots z_n] \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix}}_{= \text{scalar } Z^2} = Z^2 \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix}$$

$\therefore \left[\delta_{ij} - \frac{z_j}{\sqrt{1+Z^2}} \frac{\bar{z}_i}{\sqrt{1+Z^2}}\right]$ has eigenvalues $\underbrace{1 - \frac{Z^2}{1+Z^2}}_{= \frac{1}{1+Z^2} > 0}, 1, \ldots, 1$

$\therefore [g_{i\bar{j}}]$ is positive-definite.

Similarly for $[g_{\bar{i}j}]$.

Finally, given any $X \in T_pM$, we can write

$$X = X^i \frac{\partial}{\partial z_i} + X^{\bar i} \frac{\partial}{\partial \bar z_i} \quad \text{where} \quad X^{\bar i} = \overline{X^i} \quad \text{since } X \text{ is real.}$$

$$g(X,X) = g\left(X^i \partial_i + \overline{X^i}\, \partial_{\bar i} \,,\; X^j \partial_j + \overline{X^j}\, \partial_{\bar j}\right)$$

$$= g_{i\bar j}\, X^i \overline{X^j} + g_{\bar i j}\, \overline{X^i}\, X^j \qquad \left(\text{since } g_{ij} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = 0\right)$$

and $g_{\bar i \bar j} = 0$ too.

$$\geq \underbrace{\frac{1}{1+Z^2}}_{\substack{\text{lowest eigenvalues} \\ \text{of } [g_{i\bar j}]}} \sum_{i=1}^{n} |X^i|^2 + \frac{1}{1+Z^2} \sum_{i=1}^{n} |X^i|^2$$

$$\geq 0$$

If $g(X,X) = 0$, then $\sum_{i=1}^{n} |X^i|^2 = 0 \Rightarrow X = 0$.

#3

(a) Parametrize $\mathbb{H}^n$ by $F: \mathbb{R}^n \to \mathbb{H}^n$

$$F(u_1, \dots, u_n) = \left(\sqrt{1+u_1^2+\dots+u_n^2}, u_1, \dots, u_n\right)$$

Write $\eta = -dx^0 \otimes dx^0 + \underbrace{\delta_{ij}}\, dx^i \otimes dx^j$

$\sum$ over $i, j = 1, 2, \dots, n.$

then $\iota^* dx^0$

$$= d\left(\sqrt{1+u_1^2+\dots+u_n^2}\right)$$

$x_0 = \sqrt{1+u_1^2+\dots+u_n^2}$

$x_i = u_i \quad (i \geq 1)$

$$= \frac{1}{2x_0} \sum_{i=1}^{n} 2u_i\, du^i = \frac{\sum_1^n u_i\, du^i}{x_0}$$

$$\iota^* dx^i = du^i \qquad \forall i = 1, 2, \dots, n.$$

$$\therefore \quad \iota^* \eta = -\frac{\sum_1^n u_i\, du^i \otimes \sum_1^n u_j\, du^j}{x_0^2} + \delta_{ij}\, du^i \otimes du^j$$

$$= \left(\delta_{ij} - \frac{u_i}{x_0}\frac{u_j}{x_0}\right) du^i \otimes du^j$$

Clearly it is symmetric.

$$\left[\delta_{ij} - \frac{u_i}{x_0}\frac{u_j}{x_0}\right] = I - \vec{u}\vec{u}^T \qquad \text{where} \qquad \vec{u} = \frac{1}{x_0}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$(I - \underbrace{\vec{u}\vec{u}^T}_{\substack{\text{rank 1} \\ \text{matrix}}})\vec{u} = \underbrace{(1 - \vec{u}^T\vec{u})}\vec{u}$$

one of the eigenvalues of $I - \vec{u}\vec{u}^T$

$$= 1 - \frac{u_1^2+\dots+u_n^2}{1+u_1^2+\dots+u_n^2} = \frac{1}{x_0^2} > 0.$$

x_0^2

Hence eigenvalues of $I - \vec{u}\vec{u}^T$ are:

$$1, \dots, 1, \frac{1}{x_0^2}$$

$\therefore \iota^*\eta$ is positive definite. $\therefore \iota^*\eta$ is Riemannian.

(b) let $\Phi(x_0, \dots, x_n) = \frac{1}{1+x_0}(x_1, \dots, x_n) : \mathbb{H}^n \to \mathbb{B}^n$.

then when restricted on $\mathbb{H}^n$, we have:

$$\Phi(F(u_1, \dots, u_n)) = \frac{1}{1+x_0}(u_1, \dots, u_n)$$

$$\therefore \quad y_i = \pi_i \circ \Phi \circ F = \frac{u_i}{1+x_0}, \quad x_0 = \sqrt{1+u_1^2+\dots+u_n^2}$$

Easy to see $\quad 1 - y_1^2 - \dots - y_n^2 = \frac{2}{1+x_0}$.

$$\therefore \quad \Phi^* g_B = \Phi^*\left(\frac{4\sum_i dy^i \otimes dy^i}{(1-y_1^2-\dots-y_n^2)^2}\right)$$

$$= \frac{\cancel{4}\sum_i (\Phi^* dy^i) \otimes (\Phi^* dy^i)}{\frac{\cancel{4}}{(1+x_0)^2}}$$

$$\Phi^* dy^i = d(y^i \circ \Phi) = d\left(\frac{u_i}{1+x_0}\right)$$

$$= -\frac{u_i}{(1+x_0)^2} \cdot \frac{1}{2x_0} \cdot 2u_k du^k$$

$$+ \frac{1}{1+x_0} du^i$$

$$= \frac{1}{(1+x_0)^2}\left((1+x_0)du^i - \frac{u_i u_k}{x_0} du^k\right)$$

$\Rightarrow \Phi^* g_B$

$$= (1+x_0)^2 \sum_i (\Phi^* dy^i) \otimes (\Phi^* dy^i)$$

$$= \frac{1}{(1+x_0)^2}\sum_i \Big((1+x_0)^2 du^i \otimes du^i - \frac{1+x_0}{x_0} u_i u_k du^i \otimes du^k - \frac{1+x_0}{x_0} u_i u_k du^k \otimes du^i$$

$$+ \frac{u_i^2 u_k u_l}{x_0^2} du^k \otimes du^l \Big)$$

change of indices.

$$= \sum_i du^i \otimes du^i + \sum_{i,j} \underbrace{\frac{1}{(1+x_0)^2}\left(-\frac{2(1+x_0)}{x_0} u_i u_j + \frac{(x_0^2-1)}{x_0^2} u_i u_j\right)}_{-\frac{u_i u_j}{x_0^2}} du^i \otimes du^j$$

$$= \sum_{i,j}\left(\delta_{ij} - \frac{u_i u_j}{x_0^2}\right) du^i \otimes du^j = g_{\mathbb{H}^n}$$

∎