MATH 6250I • Fall 2018 • Riemannian Geometry
Problem Set #2 • Due Date: 04/11/2018

1. (Exercise 9.3) Find the first variation formula of the energy functional $E(\gamma)$ and show that if γ_0 minimizes $E(\gamma_s)$, then one has $\nabla_{\gamma_0'(t)}\gamma_0'(t) = 0$.

2. Consider the hyperbolic space $\mathbb{H}^n$ with the upper-half space model:
$$\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 > 0\}, \quad g := \frac{\sum_{i=1}^n dx^i \otimes dx^i}{x_1^2}.$$
   (a) Show that $(\mathbb{H}^n, g)$ has constant negative sectional curvature.
   (b) In the case $n = 2$, show that straight-lines normal to the x_2-axis, and semi-circles:
$$x_1^2 + (x_2 - a)^2 = r^2, \quad x_1 > 0$$
   meeting orthogonally to the x_1-axis are geodesics of $(\mathbb{H}^2, g)$.

3. (Exercise 10.2) Prove the second Bianchi identity using geodesic normal coordinates.

4. (Exercise 10.3) Give a complete proof of Proposition 10.4.

5. (Kähler Manifolds) A complex manifold M^{2n} is a smooth manifold that admits an atlas $\{F(z_1, \cdots, z_n) : U \subset \mathbb{C}^n \to M\}$ such that the transition maps are holomorphic. Write $z_i = x_i + y_i\sqrt{-1}$, $dz^i = dx^i + \sqrt{-1}dy^i$, and $d\bar{z}^i = \overline{dz^i}$. Let $\frac{\partial}{\partial z_i}$ be the dual vector of dz^i, and $\frac{\partial}{\partial \bar{z}_i}$ be the dual vector of $d\bar{z}^i$. The complex structure J is a $(1,1)$-tensor that acts on basis vectors by
$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$
   (a) Show that $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{z}_i}$ are eigenvectors of J. What are their eigenvalues?
   (b) A Riemannian metric g on a complex manifold M^{2n} is said to be a *Hermitian metric* (with respect to J) if $g(JX, JY) = g(X, Y)$ for any tangent vectors X and Y. Show that if g is a Hermitian metric on M, then
$$g_{ij} := g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = 0 \quad \text{and} \quad g_{\bar{i}\bar{j}} := g\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right) = 0.$$
   (c) Define a 2-tensor ω by:
$$\omega(X, Y) := g(JX, Y).$$
   Show that if g is Hermitian, then ω is a two-form. Express ω using local coordinates (in terms of dz^i and $d\bar{z}^i$'s).
   (d) Let g be a Hermitian metric on M^{2n}. Show that the following are equivalent:
   - $\nabla J = 0$
   - $d\omega = 0$ (d means exterior derivative)
   - $\nabla\omega = 0$

   If any (hence all) condition is fulfilled, we then call g a *Kähler metric* (with respect to J) on M, and the triple (M, J, g) is a called a *Kähler manifold*.
   (e) Show that the Fubini-Study metric on $\mathbb{CP}^n$ is a Kähler metric.

Choose ONE from below:

朱古力味的屎

Poo that smells like chocolate.

Consider a Kähler manifold (M, J, g) with local holomorphic coordinates $(z_1, \cdots, z_n)$. Write in short $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}_i}$, and denote the metric components and Christoffel symbols by:

$$g_{i\bar{j}} = g\left(\partial_i, \partial_{\bar{j}}\right), \quad \Gamma_{ij}^k = dz^k\left(\nabla_{\partial_i}\partial_j\right), \quad \Gamma_{ij}^{\bar{k}} = d\bar{z}^k\left(\nabla_{\partial_i}\partial_j\right)$$

and similarly for others. Show that

$$\Gamma_{ij}^k = g^{k\bar{l}}\frac{\partial}{\partial z_i}g_{j\bar{l}}, \quad \Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k},$$

and all other Christoffel symbols are zero. Using these, deduce that

$$R_{i\bar{j}k}^l := dz^l\left(\mathrm{Rm}\left(\partial_i, \partial_{\bar{j}}\right)\partial_k\right) = -\frac{\partial}{\partial \bar{z}_j}\Gamma_{ik}^l \quad \text{and} \quad R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\det[g_{i\bar{j}}].$$

Finally, show that the Fubini-Study metric on $\mathbb{CP}^n$ is an Einstein metric.
[Remark: Since g_{FS} is both a Kähler metric and an Einstein metric, we usually call it a Kähler-Einstein metric.]

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Chocolate that smells like poo.

Let M be a compact manifold without boundary. Consider functional

$$\mathcal{F}(g, f) := \int_M (R_g + |\nabla f|^2)e^{-f}\, d\mu_g,$$

where g is any Riemannian metric on M, and f is any smooth scalar function on M. Here R_g is the scalar curvature of g, and the norm $|\nabla f|^2 = g^{ij}\partial_i f \partial_j f$ is with respect to g.

Suppose $g(t)$ and $f(t)$ is a smooth family of Riemannian metrics and scalar functions on M such that

$$\frac{\partial g(t)}{\partial t} = v(t) \quad \text{and} \quad \frac{\partial f(t)}{\partial t} = h(t).$$

Show that

$$\begin{aligned}&\frac{\partial}{\partial t}\mathcal{F}(g(t), f(t)) \\ &= -\int_M g\left(v, \mathrm{Ric} + \nabla\nabla f\right)e^{-f}\, d\mu_g + \int_M \left(\frac{\mathrm{Tr}_g v}{2} - h\right)(2\Delta f - |\nabla f|^2 + R)e^{-f}\, d\mu_g.\end{aligned}$$

Here $\nabla\nabla f$ is the 2-tensor with local components $\nabla_i\nabla_j f$, and $\Delta f = g^{ij}\nabla_i\nabla_j f$. All geometric quantities in the integrand are with respect to $g(t)$.