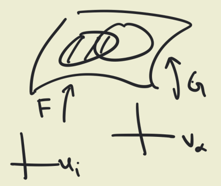
(M,g) $\quad u_1,\dots,u_n \quad (u_i)$
$\quad v_1,\dots,v_n \quad (v_\alpha)$



If $X = X^i\frac{\partial}{\partial u_i} = X^\alpha \frac{\partial}{\partial v_\alpha}$
$Y = Y^j\frac{\partial}{\partial u_j} = Y^\beta\frac{\partial}{\partial v_\beta}$.

$$\Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial u_i}g_{j\ell} + \frac{\partial}{\partial u_j}g_{i\ell} - \frac{\partial}{\partial u_\ell}g_{ij}\right)$$

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2}g^{\gamma\eta}\left(\frac{\partial}{\partial v_\alpha}g_{\beta\eta} + \cdots \right)$$

$$g = g_{ij}\,du^i\otimes du^j = g_{ij}\left(\frac{\partial u_i}{\partial v_\alpha}dv^\alpha\right)\otimes\left(\frac{\partial u_j}{\partial v_\beta}dv^\beta\right) = \underbrace{g_{ij}\frac{\partial u_i}{\partial v_\alpha}\frac{\partial u_j}{\partial v_\beta}}_{g_{\alpha\beta} = g\left(\frac{\partial}{\partial v_\alpha},\frac{\partial}{\partial v_\beta}\right)} dv^\alpha\otimes dv^\beta$$

$$\nabla_X Y = \nabla_{X^i\frac{\partial}{\partial u_i}}\left(Y^j\frac{\partial}{\partial u_j}\right) = X^i\left(\frac{\partial Y^j}{\partial u_i}\frac{\partial}{\partial u_j} + Y^j\Gamma_{ij}^k\frac{\partial}{\partial u_k}\right)$$

Check: $\quad X^\alpha\left(\frac{\partial Y^\beta}{\partial v_\alpha}\frac{\partial}{\partial v_\beta} + Y^\beta\Gamma_{\alpha\beta}^\gamma\frac{\partial}{\partial v_\gamma}\right)$

$$X = X^i\frac{\partial}{\partial u_i} = \underbrace{X^i\frac{\partial v_\alpha}{\partial u_i}}_{X^\alpha}\frac{\partial}{\partial v_\alpha}$$

$\nabla : \Gamma^\infty(TM)\times\Gamma^\infty(TM)\to\Gamma^\infty(TM)$ $\qquad \Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\cdots$

Levi-Civita connection, Riemannian connection, covariant derivative.

① $\Gamma_{ij}^k = \Gamma_{ji}^k$ (torsion-free) $\qquad \Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial u_i}g_{j\ell} + \frac{\partial}{\partial u_j}g_{i\ell} - \frac{\partial}{\partial u_\ell}g_{ij}\right)$

② $\frac{\partial g_{ij}}{\partial u_k} = \Gamma_{ik}^\ell g_{\ell j} + \Gamma_{kj}^\ell g_{i\ell}$ ← (metric compatibility)

$$g_{\ell j}\Gamma_{ik}^\ell = \frac{1}{2}g^{\ell q}\left(\cancel{\frac{\partial}{\partial u_i}g_{kq}} + \frac{\partial}{\partial u_k}g_{iq} - \cancel{\frac{\partial}{\partial u_q}g_{ik}}\right)\cdot g_{\ell j} \qquad \delta_{qj}$$

$$g_{i\ell}\Gamma_{kj}^\ell = \frac{1}{2}g^{\ell q}\left(\frac{\partial}{\partial u_k}g_{jq} + \cancel{\frac{\partial}{\partial u_j}g_{kq}} - \cancel{\frac{\partial}{\partial u_q}g_{kj}}\right)\cdot g_{i\ell} \qquad \delta_{qi}$$

①′ $\nabla_X Y - \nabla_Y X = [X,Y]$

②′ $Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X,\nabla_Z Y)$

{ invariantly. invariant notations } = indep. of local coordinates

Proof ② ⇒ ②′

$X = X^i\frac{\partial}{\partial u_i}$, $Y = Y^j\frac{\partial}{\partial u_j}$, $Z = Z^k\frac{\partial}{\partial u_k}$. $\qquad \frac{\partial g_{ij}}{\partial u_k} = \Gamma_{ik}^\ell g_{\ell j} + \Gamma_{kj}^\ell g_{i\ell}$ ↑ ②

$$\begin{aligned} Z(g(X,Y)) &= Z(g_{ij}X^iY^j) \\ &= Z^k\frac{\partial}{\partial u_k}(g_{ij}X^iY^j) \\ &= Z^k\left(\frac{\partial g_{ij}}{\partial u_k}X^iY^j + g_{ij}\frac{\partial X^i}{\partial u_k}Y^j + g_{ij}X^i\frac{\partial Y^j}{\partial u_k}\right) \\ &= Z^k\Big(X^iY^j\Gamma_{ik}^\ell g_{\ell j} + \Gamma_{kj}^\ell g_{i\ell}X^iY^j \\ &\qquad + g_{ij}\frac{\partial X^i}{\partial u_k}Y^j + g_{ij}X^i\frac{\partial Y^j}{\partial u_k}\Big) \end{aligned}$$

$$\begin{aligned} g(\nabla_Z X, Y) &= g\left(\nabla_{Z^k\frac{\partial}{\partial u_k}}\left(X^i\frac{\partial}{\partial u_i}\right), Y^j\frac{\partial}{\partial u_j}\right) \\ &= Z^k g\left(\frac{\partial X^i}{\partial u_k}\frac{\partial}{\partial u_i} + X^i\Gamma_{ki}^\ell\frac{\partial}{\partial u_\ell}, Y^j\frac{\partial}{\partial u_j}\right) \\ &= Z^k\left(Y^j\frac{\partial X^i}{\partial u_k}g_{ij} + Y^jX^i\Gamma_{ki}^\ell g_{\ell j}\right) \end{aligned}$$

If $D : \Gamma^\infty(TM)\times\Gamma^\infty(TM)\to\Gamma^\infty(TM)$ satisfies.

① $D_{X_1+X_2}(Y_1+Y_2) = D_{X_1}Y_1 + D_{X_1}Y_2 + D_{X_2}Y_1 + D_{X_2}Y_2$.

② $D_{fX}Y = fD_XY$

③ $D_X(fY) = X(f)Y + fD_XY$.

then D is called a connection on M.

If further D satisfies:

④ $D_XY - D_YX = [X,Y]$ ⟺ $\gamma_{ij}^k = \gamma_{ji}^k$

⑤ $Z(g(X,Y)) = g(D_ZX,Y) + g(X,D_ZY)$.

$\frac{\partial}{\partial u_k}g_{ij} = \gamma_{kj}^\ell g_{i\ell} + \gamma_{ki}^\ell g_{j\ell}$ ←

then: D = Levi-Civita connection (wrt. g).

Proof: $D_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j} = \gamma_{ij}^k\frac{\partial}{\partial u_k}$ $\qquad$ (WANT: $\gamma_{ij}^k = \Gamma_{ij}^k$)

$$- \frac{\partial}{\partial u_k}g_{ij} = -\cancel{\gamma_{kj}^\ell g_{i\ell}} - \cancel{\gamma_{ki}^\ell g_{j\ell}}$$
$$+ \frac{\partial}{\partial u_i}g_{kj} = +\gamma_{ij}^\ell g_{k\ell} + \cancel{\gamma_{ik}^\ell g_{j\ell}} \qquad \} \; i\leftrightarrow k$$
$$+ \frac{\partial}{\partial u_j}g_{ki} = +\gamma_{ji}^\ell g_{k\ell} + \cancel{\gamma_{jk}^\ell g_{i\ell}} \qquad \} \; i\leftrightarrow j$$

$$\Rightarrow -\partial_k g_{ij} + \partial_i g_{kj} + \partial_j g_{ik} = 2g_{k\ell}\gamma_{ij}^\ell$$
$$\Rightarrow \gamma_{ij}^\ell = \frac{1}{2}g^{k\ell}(\quad + \quad - \quad) = \Gamma_{ij}^k.$$

Tensorial

$T(X,Y) : \Gamma^\infty(TM)\times\Gamma^\infty(TM)\to\Gamma^\infty(TM)$.

T is tensorial at $\underbrace{X}_{\text{the first slot.}}$ $\overset{\text{def}}{\Longleftrightarrow}$ $T(fX,Y) = fT(X,Y)$ $\forall f\in C^\infty(M)$.

$\nabla_XY = \nabla(X,Y)$. $\qquad \nabla_{fX}Y = f\nabla_XY$

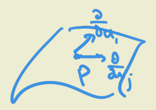
∴ Tensorial at X.

$\nabla_X(fY) \neq f\nabla_XY$

Not tensorial at the Y-slot.

If $T(X,Y)$ is tensorial at both X, Y.

$T\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right) = T_{ij}$ (✓ Given) at p



$\underbrace{T(X,Y)}_{\text{in } T_pM} = T\left(X^i\frac{\partial}{\partial u_i}, Y^j\frac{\partial}{\partial u_j}\right) = X^iY^jT_{ij}$ at p.

$\nabla(X,Y) = \nabla_XY$ $\qquad \nabla\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right) = \Gamma_{ij}^k\frac{\partial}{\partial u_k}$ at p.

$$\nabla_{X^i\frac{\partial}{\partial u_i}}\left(Y^j\frac{\partial}{\partial u_j}\right) = X^i\left(\frac{\partial Y^j}{\partial u_i}\frac{\partial}{\partial u_j} + \Gamma_{ij}^kY^j\frac{\partial}{\partial u_k}\right).$$

? $\qquad$ at p

Remark: D_1, D_2 connections (not tensorial)

$\Rightarrow D_1 - D_2$ is tensorial.

$$(D_1-D_2)_X(fY) = \begin{matrix}(D_1)_X(fY) \\ -(D_2)_X(fY)\end{matrix} = \begin{matrix}\cancel{X(f)Y} + f(D_1)_XY \\ -\cancel{X(f)Y} - f(D_2)_XY\end{matrix}$$

Check: $T(X,Y) := D_XY - D_YX$ not tensorial

$S(X,Y) := D_XY - D_YX - [X,Y]$ is tensorial.

Given $X, Y\in\Gamma^\infty(TM)$, $Z\in\Gamma^\infty(TM)$

$\nabla_Z(X\otimes Y) := \nabla_ZX\otimes Y + X\otimes\nabla_ZY$

$\omega\in\Gamma^\infty(T^*M)$, $\quad \omega : TM\to\mathbb{R}$

$\nabla_X\omega = ?$ ← tangent

WANT: $X(\underbrace{\omega(Y)}_{C^\infty(M,\mathbb{R})}) = (\nabla_X\omega)(Y) + \omega(\nabla_XY)$

$\nabla_X\omega\in\Gamma^\infty(T^*M)$: $TM\to\mathbb{R}$

$(\nabla_X\omega)(Y) := X(\omega(Y)) - \omega(\nabla_XY)$

$$\begin{aligned}\left(\nabla_{\frac{\partial}{\partial u_i}}du^k\right)\left(\frac{\partial}{\partial u_j}\right) &= \cancel{\frac{\partial}{\partial u_i}\left(\overset{\delta_{kj}}{du^k\left(\frac{\partial}{\partial u_j}\right)}\right)} - du^k\left(\nabla_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j}\right) \\ &= -du^k\left(\Gamma_{ij}^p\frac{\partial}{\partial u_p}\right) \\ &= -\Gamma_{ij}^p\delta_{kp} = -\Gamma_{ij}^k\end{aligned}$$

$$\nabla_{\frac{\partial}{\partial u_i}}du^k = \underbrace{-\Gamma_{ij}^kdu^j}$$

$$\underbrace{\nabla_{\frac{\partial}{\partial u_i}}}_{\nabla_i}(\alpha_kdu^k) = \frac{\partial\alpha_k}{\partial u_i}du^k - \Gamma_{ij}^k\alpha_kdu^j$$

$$\nabla_i(\alpha_kdu^k) = \left(\frac{\partial\alpha_j}{\partial u_i} - \Gamma_{ij}^k\alpha_k\right)du^j \qquad j\leftrightarrow k$$

// symbolical

$$\underbrace{(\nabla_i\alpha_k)}_{?}du^k = \left(\frac{\partial\alpha_k}{\partial u_i} - \Gamma_{ik}^j\alpha_j\right)du^k$$

$$\therefore \quad \nabla_i\alpha_k := \frac{\partial\alpha_k}{\partial u_i} - \Gamma_{ik}^j\alpha_j$$

(1,1)-tensor: $\quad T = T_i^j\,du^i\otimes\frac{\partial}{\partial u_j}$ ←

$$T\left(\frac{\partial}{\partial u_k}\right) = T_i^j\left(du^i\otimes\frac{\partial}{\partial u_j}\right)\left(\frac{\partial}{\partial u_k}\right) = T_i^j\,du^i\left(\frac{\partial}{\partial u_k}\right)\frac{\partial}{\partial u_j} = T_k^j\frac{\partial}{\partial u_j}.$$

$$\begin{aligned}\nabla_qT &= \nabla_q\left(T_i^j\,du^i\otimes\frac{\partial}{\partial u_j}\right) \\ &= \frac{\partial T_i^j}{\partial u_q}du^i\otimes\frac{\partial}{\partial u_j} - T_i^j\Gamma_{qk}^i\,du^k\otimes\frac{\partial}{\partial u_j} + T_i^j\,du^i\otimes\Gamma_{qj}^k\frac{\partial}{\partial u_k}. \\ &= \underbrace{\left(\frac{\partial T_i^j}{\partial u_q} - T_k^j\Gamma_{qi}^k + T_i^k\Gamma_{qk}^j\right)}_{\nabla_qT_i^j}du^i\otimes\frac{\partial}{\partial u_j}\end{aligned}$$

$i\leftrightarrow k$ $\qquad$ $k\leftrightarrow j$

$$\nabla_qT = \nabla_q\left(T_i^j\,du^i\otimes\frac{\partial}{\partial u_j}\right) =: (\nabla_qT_i^j)\,du^i\otimes\frac{\partial}{\partial u_j}$$

(2,0)-tensor

$g = g_{ij}\,du^i\otimes du^j$ $\qquad \nabla_kg_{ij} = 0$.
$h = h_{ij}\,du^i\otimes du^j$ $\qquad \nabla_kh_{ij} = \nabla_ih_{kj}$ $\quad$ | HW.

$\nabla_j\left(\nabla_i\frac{\partial}{\partial u_k}\right) = \nabla_j\left(\Gamma_{ik}^\ell\frac{\partial}{\partial u_\ell}\right) = \dots$ ← tangent

$\nabla_j\nabla_iX^k = ?$ ← component.

$\nabla_iX = \underline{X^j\Gamma_{ij}^k\frac{\partial}{\partial u_k}}$ ← tangent $\qquad X = X^i\frac{\partial}{\partial u_j}$

$\nabla_i\left(X^j\frac{\partial}{\partial u_j}\right) = \underline{\nabla_iX^i}\frac{\partial}{\partial u_j}$

∇X ← ☆ (1,1)-tensor.

$(\nabla X)(Y) := \nabla_YX$.

$\nabla X = \nabla_iX\,du^i = X^j\Gamma_{ij}^k\,du^i\otimes\frac{\partial}{\partial u_k}$.

$$\nabla_q(\nabla X) = \nabla_q\left(X^j\Gamma_{ij}^k\,du^i\otimes\frac{\partial}{\partial u_k}\right) = \nabla_q(\nabla_iX^k\,du^i)$$
$$= (\cdot \;-\; -\cdot)\,du^i\otimes\frac{\partial}{\partial u_k}$$

$\underline{\nabla_q\nabla_iX^k} = ?$

$\nabla_j := \nabla_{\frac{\partial}{\partial u_j}}$, $\qquad \nabla^i := g^{ij}\nabla_j$ ✓

$$\underbrace{\nabla^if}_{C^\infty} = g^{ij}\nabla_jf = g^{ij}\frac{\partial f}{\partial u_j}$$

$\nabla f := \nabla^if\frac{\partial}{\partial u_i} = g^{ij}\frac{\partial f}{\partial u_j}\frac{\partial}{\partial u_i}$. $\qquad df = \nabla_if\,du^i = \frac{\partial f}{\partial u_i}du^i$

$\Sigma = \{f = c\}$. $\qquad \nabla f\perp T\Sigma\ni Y$.
$\qquad \Leftrightarrow g(\nabla f, Y) = 0$.

$X\in TM$, $\quad \nabla X \quad$ (1,1)-tensor. $\quad \nabla_iX^j$
$\qquad\quad \nabla X \quad$ (0,2)-tensor $\quad \nabla^iX^j = g^{ik}\nabla_kX^j$

$\nabla_q\left(S_{ij}h_k^j\,du^i\otimes du^k\right)$ $\qquad \frac{\partial}{\partial u_j}(du^\ell) = \delta_{j\ell}$

$=: \underbrace{\nabla_q(S_{ij}h_k^j)}_{\text{stuff.}}du^i\otimes du^k$ $\qquad h = h_k^j\,du^k\otimes\frac{\partial}{\partial u_j}$

then: $\nabla_q(S_{ij}h_k^j) = (\nabla_qS_{ij})h_k^j + S_{ij}\nabla_qh_k^j$

$(S*h)(X) := \sum_j S\left(X,\frac{\partial}{\partial u_j}\right)\underbrace{h(du^j)}_{h_k^j\frac{\partial}{\partial u_k}}$

$(S*h)\left(\frac{\partial}{\partial u_i}\right) = S_{ij}h_k^j$