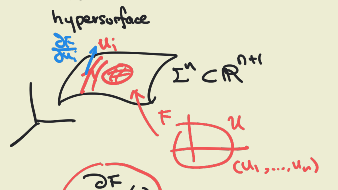
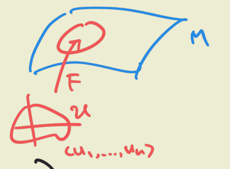
Euclidean hypersurface



Abstract manifold



$\frac{\partial F}{\partial u_i}(p)$ $\longrightarrow$ $\frac{\partial}{\partial u_i}(p) : C^\infty(M) \to \mathbb{R}$

$f \mapsto \frac{\partial f}{\partial u_i}(p) = \frac{\partial}{\partial u_i}(f \circ F).$

$T_p\Sigma = \operatorname{span}\left\{\frac{\partial F}{\partial u_i}(p)\right\}_{i=1}^n$ $\qquad$ $T_pM = \operatorname{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$

$g_{ij} = \left\langle \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle$

$g = g_{ij}\, du^i \otimes du^j$

Definition:

M^n abstract manifold.

g is called a Riemannian metric on M $\leftarrow$ call (M,g) is a Riemannian manifold.

if $g : TM \times TM \to \mathbb{R}$ satisfies. (g: 2-tensor)

① $g(X,Y) = g(Y,X) \quad \forall X,Y \in TM.$

② $g(X,X) \geq 0 \quad \forall X \in TM$ and equality holds iff $X = 0.$ $\longrightarrow |X| = \sqrt{g(X,X)}$

$g_{ij} = g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) \rightsquigarrow g = g_{ij}\, du^i \otimes du^j = g_{ij}\, du^i du^j$

② $\Leftrightarrow$ $[g_{ij}]$ is positive-definite

$\overset{\text{def}}{\Leftrightarrow}$ all eigenvalues of $[g]$ are positive.

e.g. $(\mathbb{R}^n, \delta)$ is a Riem. mfd. ($(x_1,\dots,x_n)$; δ: standard dot product)

$\delta = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$
$= dx^1 dx^1 + \dots + dx^n dx^n$
$= (dx^1)^2 + \dots + (dx^n)^2$

(x,y,z) $\quad \delta = dx^2 + dy^2 + dz^2 = dx\,dx + dy\,dy + dz\,dz$

e.g. Euclidean hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$

$g :=$ the first fund. form $\qquad g : T\Sigma \times T\Sigma \to \mathbb{R}.$

(g: Riem. metric on Σ^n)

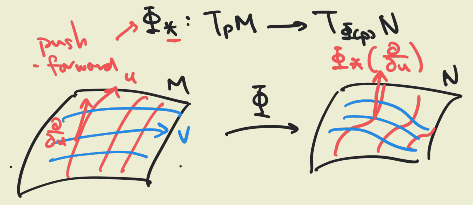
$\delta =$ Euclidean dot prod. of $\mathbb{R}^{n+1}$

$\iota : \Sigma^n \to \mathbb{R}^{n+1}$
$p \mapsto p$

$g = \iota^*\delta$

$\Phi : M \to N$

$\Phi_* : T_pM \to T_{\Phi(p)}N$ (push-forward)



$\omega : TN \to \mathbb{R}$

$(\Phi^*\omega)(X)$ ($X \in TM$)
$= \omega(\underbrace{\Phi_* X}_{\in TN})$

$\Phi^* : T^*N \to T^*M$ ($\omega \in T^*N$; Φ^*: pull-back)

$(\iota^*\delta)(\underbrace{X,Y}_{\in T\Sigma})$
$= \delta(\iota_* X, \iota_* Y)$
$= \delta(X,Y) = X \cdot Y$

$\iota : \Sigma \to \mathbb{R}^{n+1}$

$\iota_* X = X$

e.g. $\Phi : M \to N$, $\qquad (N,g)$ is Riem. mfd.

Suppose Φ is an immersion.

$\overset{\text{def}}{\Leftrightarrow} \Phi_{*p} : T_pM \to T_{\Phi(p)}N$ is injective.

$\tilde{g} : TM \times TM \to \mathbb{R}, \qquad \tilde{g} = \Phi^* g$

① $\tilde{g}(X,Y) = (\Phi^*g)(X,Y) = g(\Phi_* X, \Phi_* Y)$

② $\tilde{g}(X,X) = g(\Phi_* X, \Phi_* X) \geq 0$

equality $\Leftrightarrow \Phi_* X = 0 \longrightarrow X = 0.$

e.g. (M,g), $(N,\tilde{g})$ Riem. mfd.

$M \times N \qquad T(M\times N) = TM \oplus TN$

$(g \oplus \tilde{g})\big((X,Y), (Z,W)\big)$ ($X \in TM$, $Y \in TN$, $Z \in TM$, $W \in TN$)

$:= g(X,Z) + \tilde{g}(Y,W).$

$$\begin{array}{ccc} & M\times N & \\ \pi_M \swarrow & & \searrow \pi_N \\ M & & N \end{array}$$

then $g \oplus \tilde{g} = \pi_M^* g + \pi_N^* \tilde{g}$

e.g. $D := \{(x,y) : x^2+y^2 < 1\}.$

$g := \dfrac{4(dx \otimes dx + dy \otimes dy)}{(1-x^2-y^2)^2}$ $\qquad [g] = \begin{bmatrix} \frac{4}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{4}{(1-x^2-y^2)^2} \end{bmatrix}$

hyperbolic metric on D. (Poincaré)

e.g. $\mathbb{CP}^n = \{[z_1 : \dots : z_{n+1}] \mid (z_1,\dots,z_{n+1}) \neq (0,\dots,0)\}.$

e.g. $[1 : i : 0 : \dots : 0] = [\pi : \pi i : 0 : \dots : 0]$
$= [i : -1 : 0 : \dots : 0].$

$\mathbb{CP}^1 \quad \{[0 : z]\} \sqcup \{[w : *]\}$ ($z \neq 0$, $w \neq 0$)
$\quad = \underbrace{\{[0:1]\}}_{\{\infty\}} \sqcup \underbrace{\{[1:*]\}}_{\cong \mathbb{C}}$

$F^{(j)}(z_1^{(j)}, \dots, z_{j-1}^{(j)}, z_{j+1}^{(j)}, \dots, z_{n+1}^{(j)}) = [z_1^{(j)} : \dots : z_{j-1}^{(j)} : 1 : z_{j+1}^{(j)}, \dots, z_{n+1}^{(j)}]$

Fubini–Study metric $\qquad z_j^{(j)} := 1.$

$$g_{FS} = 2\,\mathrm{Re}\left(\frac{\partial}{\partial z_p}\frac{\partial}{\partial \bar{z}_q} \log\left(|z_1|^2 + \dots + |z_{n+1}|^2\right) dz^p \otimes d\bar{z}^q\right)$$

$z_p = x_p + \sqrt{-1}\, y_p \rightsquigarrow \frac{\partial}{\partial z_p} = \frac{1}{2}\left(\frac{\partial}{\partial x_p} - \sqrt{-1}\frac{\partial}{\partial y_p}\right)$

$\frac{\partial}{\partial \bar{z}_p} = \overline{\frac{\partial}{\partial z_p}}$

$dz^p = dx^p + \sqrt{-1}\, dy^p$
$d\bar{z}^p = \overline{dz^p}$ $\longrightarrow$ $dz^p\left(\frac{\partial}{\partial z_q}\right) = \delta_{pq}$

$dz^p\left(\frac{\partial}{\partial \bar{z}_q}\right) = 0.$

$g_{ij} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = 0$

$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$

$$[g] = \begin{array}{c} \begin{array}{cc} i & \bar{i} \end{array} \\ \left[\begin{array}{c|c} 0 & g_{\bar{i}j} \\ \hline g_{i\bar{j}} & 0 \end{array}\right] \end{array} \begin{array}{c} j \\ \bar{j} \end{array}$$

$g_{i\bar{j}} = \overline{g_{\bar{i}j}}$

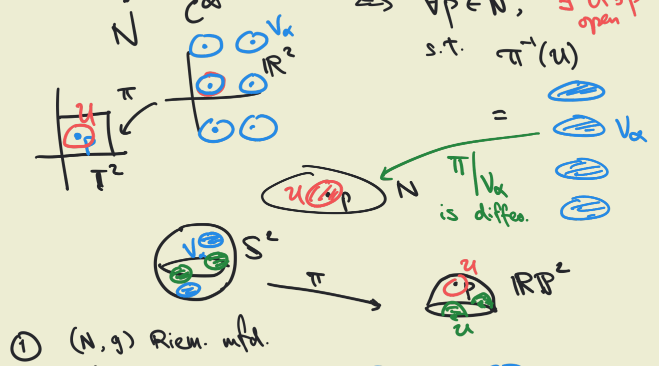
Next: e.g. $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, \qquad \mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2.$

$M \xrightarrow{\pi} N$ surjective C^∞

Def: $\pi : M \to N$ is a covering map

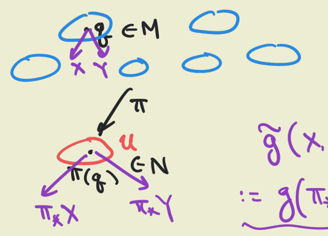
$\Leftrightarrow \forall p \in N$, $\exists\, \mathcal{U} \ni p$ open s.t. $\pi^{-1}(\mathcal{U}) = \sqcup V_\alpha$

$\pi|_{V_\alpha}$ is diffeo.



① (N,g) Riem. mfd.

$M \xrightarrow{\pi} (N,g)$ covering

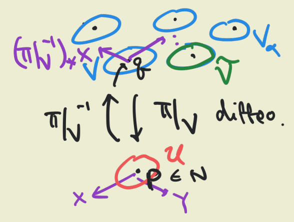


$\tilde{g}(X,Y) := g(\pi_* X, \pi_* Y)$

$\tilde{g} = \pi^* g.$

② $(M,\tilde{g})$ Riem. mfd.

$(M,\tilde{g}) \xrightarrow{\pi} N$ covering

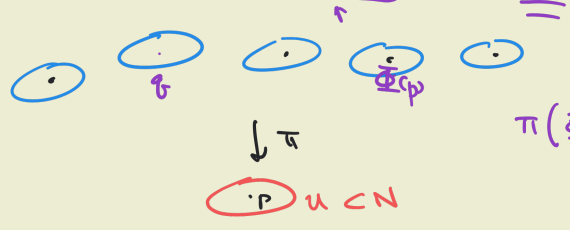


$g(X,Y) := \tilde{g}\big((\pi|_V^{-1})_* X, (\pi|_V^{-1})_* Y\big)$

Deck transformation

$M \xrightarrow{\pi} N \qquad \Phi : M \to M$ diffeo. said to be a deck transformation of $\pi : M \to N$

$\overset{\text{def}}{\Leftrightarrow} \pi \circ \Phi = \pi$



$\Phi(q)$

$\pi(\Phi(q)) = \pi(q) = p$

e.g. $\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}^2/\mathbb{Z}^2 \qquad$ if $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a deck trans.

$\Leftrightarrow \pi \circ \Phi(x_1,x_2) = \pi(x_1,x_2)$

$\Leftrightarrow [\Phi(x_1,x_2)] = [(x_1,x_2)]$

$\Leftrightarrow \Phi(x_1,x_2) - (x_1,x_2) \in \mathbb{Z}^2.$

$\Rightarrow \Phi(x_1,x_2) - (x_1,x_2) = \underbrace{(m_1,m_2)}_{\text{indep. of } x_1,x_2} \in \mathbb{Z}^2$

$\Rightarrow \Phi(x_1,x_2) = (x_1,x_2) + (m_1,m_2).$

e.g. $\mathbb{S}^2 \xrightarrow{\pi} \mathbb{RP}^2. \qquad \Phi : \mathbb{S}^2 \to \mathbb{S}^2.$

$\pi \circ \underbrace{\Phi(x_1,x_2)}_{\mathbb{S}^2} = \pi(x_1,x_2)$

$[\Phi(x_1,x_2)] = [(x_1,x_2)]$

$\Rightarrow \Phi(x_1,x_2) = \pm(x_1,x_2)$

$\Phi(x_1,x_2)\cdot(x_1,x_2) = \pm\underbrace{(x_1,x_2)\cdot(x_1,x_2)}_{|(x_1,x_2)|^2} = \pm 1$

$\Rightarrow \Phi = \mathrm{id}$ or $\Phi = -\mathrm{id}.$

$(M,\tilde{g}) \leftarrow$ Riem.

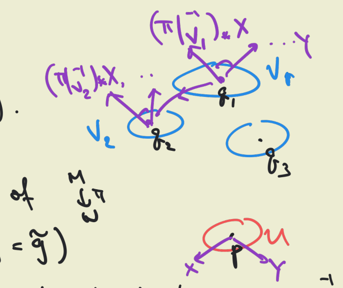
$\downarrow \pi$

N

$\Phi : M \to M.$

$\pi \circ \Phi = \pi$
$\pi_* \circ \Phi_* = \pi_*$

$\to g(X,Y) = \tilde{g}\big((\pi|_V^{-1})_* X, (\pi|_V^{-1})_* Y\big).$



if: ① any deck transformation of $M \xrightarrow{\pi} N$ is an isometry $(\Phi^*\tilde{g} = \tilde{g})$

② deck transformation group acts transitively on any $\pi^{-1}(p)$, $p \in N$.

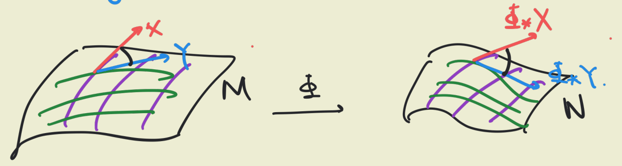
(if M is simply-connected, then satisfies ②)

then g is well-defined.

Def: $\Phi : (M,g) \to (N,\tilde{g})$ diffeo. is said to an isometry

$\overset{\text{def}}{\Leftrightarrow} \Phi^*\tilde{g} = g$

$\tilde{g}(\Phi_* X, \Phi_* Y) = g(X,Y)$



$\Phi(x,y) = (u,v) \qquad M \xrightarrow{\Phi} N \qquad u \circ \Phi : M \to \mathbb{R}.$

$\Phi^*(du) = d(\Phi^* u) = d(u \circ \Phi)$

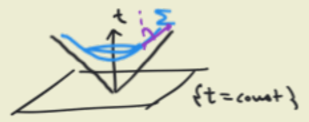
e.g. let $\Phi(x,y) = (\underbrace{x^2+2y}_{u}, \underbrace{y^3 + \sin x}_{v})$

$\Phi^* du = d(\Phi^* u) = d(x^2+2y) = 2x\,dx + 2dy.$

$\mathbb{R}^{n,1} \qquad (x_0, \underbrace{x_1, \dots, x_n}_{\text{space}})$ ($x_0 = t$)

$\eta = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n.$

(η: Minkowski metric)

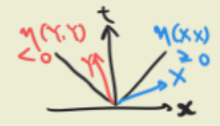


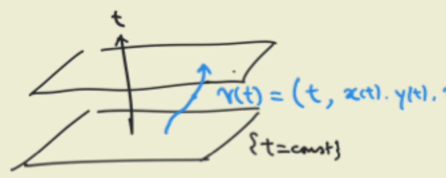
$\iota : \Sigma \to \mathbb{R}^{n,1}$

HW:
Show $\iota^*\eta$ is Riem. metric on Σ.

$X \in T\Sigma.$

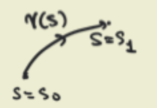
i.e. $(\iota^*\eta)(X,X) = \eta(X,X) \geq 0.$





$\eta(\dot{\gamma}, \dot{\gamma}) < 0.$

(M,g) Riem.



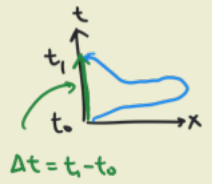
$L(\gamma) = \int_{s_0}^{s_1} \underbrace{|\dot{\gamma}(s)|}_{\sqrt{g(\dot{\gamma},\dot{\gamma})}} ds$

$(\mathbb{R}^{3,1}, \eta)$ Minkowski space. $\qquad \eta = -dt^2 + dx^2 + dy^2 + dz^2$

$\int_{t_0}^{t_1} \sqrt{-\eta(\dot{\gamma},\dot{\gamma})}\, dt \qquad$ proper time.

$= \int_{t_0}^{t_1} \sqrt{(dt^2 - dx^2 - dy^2 - dz^2)(\dot{\gamma},\dot{\gamma})}\, dt$

$\dot{\gamma} = \frac{\partial}{\partial t} + x'(t)\frac{\partial}{\partial x} + \dots$



$= \int_{t_0}^{t_1} \sqrt{1 - x'(t)^2 - y'(t)^2 - z'(t)^2}\, dt$

$\leq t_1 - t_0.$

Given (M,g) Riem. mfd.

$g = g_{ij}\, du^i \otimes du^j$

$\Gamma_{ij}^k := \frac{1}{2} g^{kl}\left(\frac{\partial}{\partial u_i} g_{jl} + \frac{\partial}{\partial u_j} g_{il} - \frac{\partial}{\partial u_l} g_{ij}\right)$

$\nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial u_i} := \Gamma_{ij}^k \frac{\partial}{\partial u_k}, \qquad \nabla_{X^j \frac{\partial}{\partial u_j}}\left(Y^i \frac{\partial}{\partial u_i}\right)$

$\nabla_{fX} Y = f\nabla_X Y. \qquad \nabla_X(fY) = f\nabla_X Y + X(f)\, Y$

$\nabla : TM \times TM \to TM$