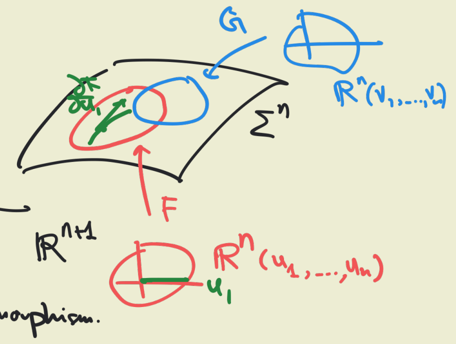
MATH 6250 Week 1

Regular Euclidean hypersurface



$\forall p \in \Sigma$, $\exists\, F(u_1,\dots,u_n) \to \Sigma$

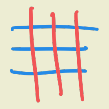
① F is C^∞.

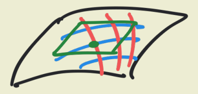
② $F: \mathcal{U} \to F(\mathcal{U})$ is homeomorphism.

③* $\left\{ \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_n} \right\}$ linearly indep.

$T_p\Sigma = \text{span}\left\{ \frac{\partial F}{\partial u_1}(p), \dots, \frac{\partial F}{\partial u_n}(p) \right\}$

has dim $= n$.





Definition: (1st fundamental form)

$\delta = \langle \cdot, \cdot \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ dot product

$X, Y \mapsto X \cdot Y$

$g(\cdot,\cdot) : T_p\Sigma^n \times T_p\Sigma^n \to \mathbb{R} \leftarrow$ 1st fund. form.

$\overset{\shortparallel}{\text{span}\left\{\frac{\partial F}{\partial u_i}\right\}}$ $du^i : T_p\Sigma \to \mathbb{R}$, $du^i\left(\frac{\partial F}{\partial u_j}\right) = \delta_{ij}$

$\delta = dx^1 \otimes dx^1 + \dots + dx^{n+1} \otimes dx^{n+1}$

$(S \otimes T)(X, Y) := \underbrace{S(X)}_{\text{in } \mathbb{R}} T(Y)$ $T, S : T_p\Sigma \to \mathbb{R}$

$dx^i\left(\frac{\partial}{\partial x_j}\right) := \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$\delta(X, Y) = \left(dx^1 \otimes dx^1 + \dots + dx^{n+1} \otimes dx^{n+1}\right)\left(X^1 \frac{\partial}{\partial x_1} + \dots + X^{n+1} \frac{\partial}{\partial x_{n+1}}, Y^1 \frac{\partial}{\partial x_1} + \dots + Y^{n+1} \frac{\partial}{\partial x_{n+1}}\right)$

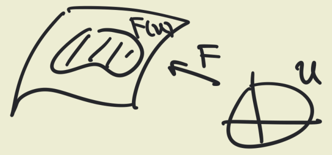
$X = X^i \frac{\partial}{\partial x_i}$, $Y = Y^j \frac{\partial}{\partial x_j}$ $= X^1Y^1 + \dots + X^{n+1}Y^{n+1}$

$g = \sum_{i,j} g_{ij}\, du^i \otimes du^j$

$g\left(\frac{\partial F}{\partial u_k}, \frac{\partial F}{\partial u_l}\right) = g_{ij}\, du^i \otimes du^j \left(\frac{\partial F}{\partial u_k}, \frac{\partial F}{\partial u_l}\right)$

$= g_{ij}\, \delta_{ik}\, \delta_{jl} = g_{kj}\, \delta_{jl} = g_{kl}$

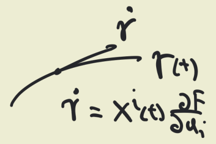
$\Rightarrow g = \underbrace{g\left(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right)}_{g_{ij}} du^i \otimes du^j$. $[g] = [g_{ij}]$



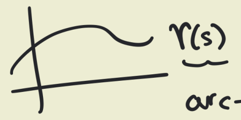
Area of $F(\mathcal{U}) = \int_{\mathcal{U}} \sqrt{\det[g_{ij}]}\, du^1 \cdots du^n$

$\int |\dot{r}|\, dt$

$|\dot{r}|^2 = \langle \dot{r}, \dot{r} \rangle = g_{ij} X^i X^j$



$\dot{r} = X^i(t) \frac{\partial F}{\partial u_i}$



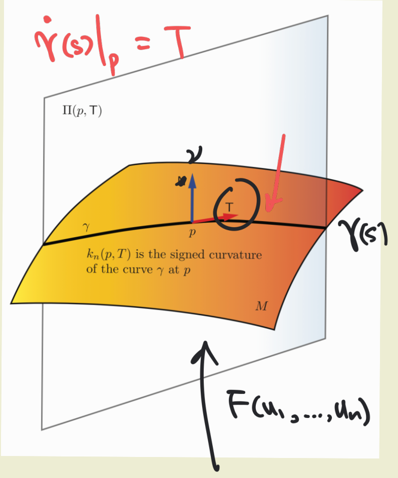
$\gamma(s)$ arc-length parametrized. $K(s) = |\ddot{\gamma}(s)|$

$|\dot{\gamma}(s)| = 1$

$k(s) := \underset{/\!/ N(s)}{\ddot{\gamma}(s)} \cdot N(s)$ $|k(s)| = K(s)$.

circle with radius R, $k(s) = \pm \frac{1}{R}$.

Normal curvature



T given. $T \in T_p\Sigma$

$|T| = 1$.

$k_n(p, T) =$ signed curvature of γ at p

$= \ddot{\gamma}(s) \cdot \nu$

normal.

$\gamma(s) = F(u_1(s), \dots, u_n(s))$

$X^i \frac{\partial F}{\partial u_i} = T = \dot{\gamma}(s) = \frac{\partial F}{\partial u_i} \frac{du_i}{ds}\Big|_p$ tangent

$\ddot{\gamma}(s) = \left(\frac{\partial^2 F}{\partial u_i \partial u_j} \frac{du_j}{ds}\right) \frac{du_i}{ds} + \frac{\partial F}{\partial u_i} \frac{d^2 u_i}{ds^2}$.

$\Rightarrow \ddot{\gamma}(s) \cdot \nu = \left(\frac{\partial^2 F}{\partial u_i \partial u_j} \cdot \nu\right) \underset{X^i}{\frac{du_i}{ds}} \underset{X^j}{\frac{du_j}{ds}}\Big|_p$

$= \underbrace{\left(\frac{\partial^2 F}{\partial u_i \partial u_j} \cdot \nu\right)}_{h_{ij}} X^i X^j$.

$h_{ij} \leftarrow$ second fundament form.

$-\frac{\partial F}{\partial u_i} \cdot \frac{\partial \nu}{\partial u_j}$

$h := h_{ij}\, du^i \otimes du^j$, then $h(T, T) = h_{ij} X^i X^j$

$T = X^i \frac{\partial F}{\partial u_i}$

Theorem:

$\left.\begin{array}{l} \max\{k_n(p,T) : |T| = 1\} \\ \min\{k_n(p,T) : |T| = 1\} \end{array}\right\}$ are eigenvalues of $[g]^{-1}[h]$

$g^{ik} h_{kj}$

$k_n(p,T) = h_{ij}\big|_p X^i X^j$, $T = X^i \frac{\partial F}{\partial u_i}$

$|T| = 1 \Leftrightarrow g_{ij} X^i X^j = 1$.

Proof: $f(x_1, \dots, x_n) = h_{ij} X^i X^j$ — $\frac{\partial}{\partial X_k}(h_{ij} X^i X^j) = h_{ij}(\delta_{ik} X^j + X^i \delta_{jk})$

$= h_{kj} X^j + h_{ik} X^i$

$\begin{cases} \frac{\partial f}{\partial X_k} = \lambda \frac{\partial}{\partial X_k}(g_{ij} X^i X^j) \\ g_{ij} X^i X^j = 1 \end{cases}$

$\Rightarrow h_{kj} X^j + h_{ik} X^i = \lambda (g_{kj} X^j + g_{ik} X^i)$

$\Rightarrow \cancel{2}\, h_{ki} X^i = \cancel{2} \lambda\, g_{ki} X^i \Rightarrow [h]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda [g] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow [g]^{-1}[h] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow g^{ik} h_{kj} X^j = \lambda X^i$

$\Rightarrow h_{kj} X^j = \lambda g_{ik} X^i$

$\Rightarrow h_{kj} X^j X^k = \lambda g_{ik} X^i X^k = \lambda |T|^2 = \lambda$

$\overset{\shortparallel}{h(T,T)} = k_n(p, T)$ □

principal curvatures: eigenvalues of $[g]^{-1}[h]$. $g^{ik} h_{kj}$

$\{\lambda_1, \dots, \lambda_n\}$. $\{\kappa_1, \dots, \kappa_n\}$.

Mean curvature : $H = \lambda_1 + \dots + \lambda_n = \text{Tr}\,[g]^{-1}[h] = g^{ik} h_{ki} = g^{ij} h_{ij}$

Gauss curvature : $K = \lambda_1 \cdots \lambda_n = \det([g]^{-1}[h]) = \frac{\det[h]}{\det[g]}$

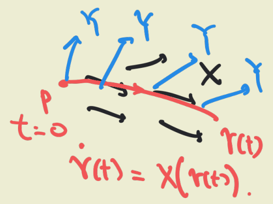
Gauss's Theorema Egregium (2D only)

K depends only on g_{ij}.

X, Y vector fields on $\Sigma^n \subset \mathbb{R}^{n+1}$

$(D_X Y)_p =$ directional derivative of Y along X

$= \frac{d}{dt} Y(\gamma(t)) \Big|_{t=0}$



C^∞ vector fields on Σ

$\nabla : \Gamma^\infty(T\Sigma) \times \Gamma^\infty(T\Sigma) \to \Gamma^\infty(T\Sigma)$ covariant derivative

$\nabla_X Y := (D_X Y)^{T\Sigma} =$ projection of $D_X Y$ onto $T\Sigma$.

$\nabla_{\frac{\partial F}{\partial u_i}} \frac{\partial F}{\partial u_j} = ?$ $\nabla_{fX} Y = f \nabla_X Y$

$\nabla_X (fY) = X(f)\, Y + f \nabla_X Y$.

$\nabla_{\frac{\partial F}{\partial u_i}} \frac{\partial F}{\partial u_j} =: \Gamma_{ij}^k \frac{\partial F}{\partial u_k}$. ?

$\underbrace{D_{\frac{\partial F}{\partial u_i}} \frac{\partial F}{\partial u_j}}_{\shortparallel} = \underbrace{\nabla_{\frac{\partial F}{\partial u_i}} \frac{\partial F}{\partial u_j}}_{T\Sigma} + \underbrace{\overset{h_{ij}}{\cancel{c_{ij}}}\, \nu}_{N\Sigma}$

$\underbrace{\left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \nu \right\rangle}_{h_{ij}} = 0 + c_{ij}$ –

Claim: $\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l}\right)$

Proof: $+\frac{\partial}{\partial u_i} g_{jl} = \frac{\partial}{\partial u_i}\left\langle \frac{\partial F}{\partial u_j}, \frac{\partial F}{\partial u_l} \right\rangle = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \frac{\partial F}{\partial u_l} \right\rangle$

$+ \left\langle \frac{\partial F}{\partial u_j}, \frac{\partial^2 F}{\partial u_i \partial u_l} \right\rangle$

$= \left\langle \Gamma_{ij}^k \frac{\partial F}{\partial u_k}, \frac{\partial F}{\partial u_l} \right\rangle + \dots$

$= g_{kl} \Gamma_{ij}^k + \cancel{g_{kj} \Gamma_{il}^k}$

$+\frac{\partial}{\partial u_j} g_{il} = g_{kl} \Gamma_{ji}^k + \cancel{g_{ki} \Gamma_{jl}^k}$ $\frac{\partial^2 F}{\partial u_j \partial u_l}$

$-\frac{\partial}{\partial u_l} g_{ij} = \cancel{-g_{kj} \Gamma_{li}^k} - \cancel{g_{ki} \Gamma_{lj}^k}$ $= \frac{\partial^2 F}{\partial u_l \partial u_j}$

$\Rightarrow (+\ +\ -) = 2 g_{kl} \Gamma_{ij}^k \Rightarrow g^{lp}(+\ +\ -) = 2\delta_{kp} \Gamma_{ij}^k$

$= 2\Gamma_{ij}^p$

$\Rightarrow \Gamma_{ij}^p = \cdot\ -\ -$ ■

Christoffel symbols.

Proof of Gauss's Theorema Egregium:

$\frac{\partial^3 F}{\partial u_i \partial u_j \partial u_k} = \frac{\partial^3 F}{\partial u_j \partial u_i \partial u_k}$

$\frac{\partial^2 F}{\partial u_j \partial u_k} = \nabla_{\frac{\partial F}{\partial u_j}} \frac{\partial F}{\partial u_k} + h_{jk} \nu = \Gamma_{jk}^l \frac{\partial F}{\partial u_l} + h_{jk} \nu$

$\frac{\partial^3 F}{\partial u_i \partial u_j \partial u_k} = \frac{\partial \Gamma_{jk}^l}{\partial u_i} \frac{\partial F}{\partial u_l} + \Gamma_{jk}^l \frac{\partial^2 F}{\partial u_i \partial u_l} + \frac{\partial}{\partial u_i}(h_{jk} \nu)$.

$= \frac{\partial \Gamma_{jk}^l}{\partial u_i} \frac{\partial F}{\partial u_l} + \Gamma_{jk}^l \left(\Gamma_{il}^p \frac{\partial F}{\partial u_p} + h_{il} \nu\right) + \frac{\partial h_{jk}}{\partial u_i} \nu$

$+ h_{jk} \frac{\partial \nu}{\partial u_i}$

$|\nu| = 1$

$\frac{\partial \nu}{\partial u_i} = A_i^q \frac{\partial F}{\partial u_q}$

$\left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial F}{\partial u_r} \right\rangle = A_i^q g_{rq}$

$\overset{\shortparallel}{-h_{ir}}$

$A_i^q = -g^{rq} h_{ir}$

$= \frac{\partial \Gamma_{jk}^p}{\partial u_i} \frac{\partial F}{\partial u_p} + \Gamma_{jk}^l \left(\Gamma_{il}^p \frac{\partial F}{\partial u_p} + h_{il} \nu\right) + \frac{\partial h_{jk}}{\partial u_i} \nu$

$- h_{jk} g^{rp} h_{ir} \frac{\partial F}{\partial u_p}$

$\frac{\partial^3 F}{\partial u_j \partial u_i \partial u_k} = \left(\frac{\partial \Gamma_{ik}^p}{\partial u_j} + \Gamma_{ik}^l \Gamma_{jl}^p - g^{rp} h_{ik} h_{jr}\right) \frac{\partial F}{\partial u_p}$

$+$ normal part.

Compare tangent part:

$\frac{\partial \Gamma_{jk}^p}{\partial u_i} + \Gamma_{jk}^l \Gamma_{il}^p - g^{rp} h_{jk} h_{ir} = \frac{\partial \Gamma_{ik}^p}{\partial u_j} + \Gamma_{ik}^l \Gamma_{jl}^p - g^{rp} h_{ik} h_{jr}$

$\frac{\partial \Gamma_{jk}^p}{\partial u_i} - \frac{\partial \Gamma_{ik}^p}{\partial u_j} + \Gamma_{jk}^l \Gamma_{il}^p - \Gamma_{ik}^l \Gamma_{jl}^p = g^{rp}\left(\underline{h_{jk} h_{ir} - h_{ik} h_{jr}}\right)$

Gauss equation.

$h_{jk} h_{ir} - h_{ik} h_{jr}$

$= g_{pr}\left(\frac{\partial \Gamma_{jk}^p}{\partial u_i} - \frac{\partial \Gamma_{ik}^p}{\partial u_j} + \Gamma_{jk}^l \Gamma_{il}^p - \Gamma_{ik}^l \Gamma_{jl}^p\right) \leftarrow R_{ijkr}$

In 2D : $(i, j, k, r) = (1, 2, 2, 1)$

$\Rightarrow \underbrace{h_{22} h_{11} - h_{12} h_{21}}_{\det[h]} = R_{1221}$