**MATH 6250I • Fall 2018 • Riemannian Geometry**
**Problem Set #1 • Due Date: 30/09/2018**

1. Let $\mathbb{S}^n$ be the unit sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$, and $f : \mathbb{S}^n \to (0, \infty)$ be a smooth (i.e. $C^\infty$) scalar function. Write $X = (x_1, \cdots, x_{n+1})$. Consider the set $\Sigma$ defined by:

$$\Sigma := \{f(X)X : X \in \mathbb{S}^n\}.$$

Let $F(u_1, \cdots, u_n)$ be a smooth local parametrization of $\mathbb{S}^n$ and denote the first fundamental form under this local coordinate system by $g_{ij} := g\big(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\big)$. Now consider the induced local parametrization $G(u_1, \cdots, u_n)$ of $\Sigma$, defined as

$$G(u_1, \cdots, u_n) = f\big(F(u_1, \cdots, u_n)\big)F(u_1, \cdots, u_n).$$

Calculate the first and second fundamental forms $\widetilde{g}_{ij}$ and $\widetilde{h}_{ij}$, the mean curvature, and the Gauss curvature of $\Sigma$.

2. Verify that the Fubini-Study metric defined in Example 8.8 is indeed a Riemannian metric.

3. Consider $\mathbb{R}^{n+1}$, with coordinates denoted by $(x_0, x_1, \cdots, x_n)$, and the Minkowski metric:

$$\eta = -dx^0 \otimes dx^0 + \sum_{j=1}^{n} dx^j \otimes dx^j.$$

Consider the subset

$$\mathbb{H}^n := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : -x_0^2 + x_1^2 + \cdots + x_n^2 = -1 \text{ and } x_0 > 0\}.$$

(a) Show that $g_{\mathbb{H}} := \iota^*\eta$, where $\iota : \mathbb{H}^n \to \mathbb{R}^{n+1}$ is the inclusion map, is a Riemannian metric on $\mathbb{H}^n$.

(b) Let $\mathbb{B}^n$ be the open ball of radius $1$ in $\mathbb{R}^n$ with coordinates $(y_1, \cdots, y_n)$. Consider the Riemannian metric

$$g_{\mathbb{B}} := \frac{4(dy^1 \otimes dy^1 + \cdots + dy^n \otimes dy^n)}{(1 - y_1^2 - \cdots - y_n^2)^2}.$$

Show that $(\mathbb{H}^n, g_{\mathbb{H}})$ and $(\mathbb{B}^n, g_{\mathbb{B}})$ are isometric via the map

$$(x_0, x_1, \cdots, x_n) \mapsto \frac{1}{1 + x_0}(x_1, \cdots, x_n).$$

(c) Let $\mathbb{R}^n_+$ be the upper half-space of $\mathbb{R}^n$ with coordinates $(u_1, \cdots, u_n)$ and $u_1 > 0$ for $\mathbb{R}^n_+$. Consider the Riemannian metric

$$g_+ := \frac{du^1 \otimes du^1 + \cdots + du^n \otimes du^n}{u_1^2}.$$

Show that $(\mathbb{B}^n, g_{\mathbb{B}})$ and $(\mathbb{R}^n_+, g_+)$ are isometric via the map

$$\underline{x} := (x_1, \cdots, x_n) \mapsto \frac{\big(1 - |\underline{x}|^2, 2x_2, \cdots, 2x_n\big)}{|\underline{x} - (1, 0, \cdots, 0)|^2}.$$

4. Let $(M, g)$ be a Riemannian manifold, and $\Sigma$ be a submanifold of $M$. Denote $\iota : \Sigma \to M$ the inclusion map. Then, $\overline{g} := \iota^* g$ is a Riemannian metric on $\Sigma$. Show that the Levi-Civita connection of $(\Sigma, \overline{g})$, denoted by $\overline{\nabla}$, is given by:

$$\overline{\nabla}_X Y = (\nabla_X Y)^T := \text{ projection of } \nabla_X Y \text{ onto } T\Sigma$$

for any $X, Y \in \Gamma^\infty(T\Sigma)$.

5. Exercises 8.8, 8.9, 8.10 and 8.13 in the instructor's lecture notes.