PROBLEM 1

(a) (i) 1-forms : α, β

2-forms : ω, η

(ii) $\omega \wedge \eta = 0$

$\underset{\text{2-form}}{\omega} \wedge \underset{\text{2-form}}{\eta}$, so $\omega \wedge \eta$ is a 4-form in $\mathbb{R}^3$.

$4 > 3$, so $\omega \wedge \eta = 0$.

(iii) $d\alpha = \eta$

(iv) η (since $d\eta = d(d\alpha) = 0$)

(b)

Pick $\boxed{\int_\Sigma \iota_\Sigma^* \eta}$

(I don't think the other one is do-able.)

$$\int_\Sigma \iota_\Sigma^* \eta = \int_\Sigma \iota_\Sigma^*(d\alpha) = \int_\Sigma d(\iota_\Sigma^* \alpha)$$

$$= \int_{\partial\Sigma} \iota_{\partial\Sigma}^* \iota_\Sigma^* \alpha \quad \text{(Stoke's Theorem)}$$

$\iota_{\partial\Sigma} : \partial\Sigma \to \Sigma$ inclusion map.

On $\partial\Sigma = \{(x,y,0) : x^2+y^2=1\}$, we have $z=0$,

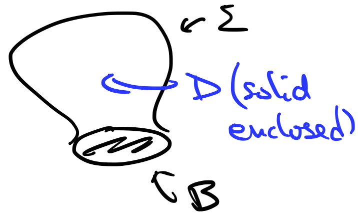
hence $\iota_{\partial\Sigma}^*(\iota_\Sigma^* \alpha) = \iota_\Sigma^*\left(x\,dx + (y+1)\,dy\right) = \iota_\Sigma^* d\left(\frac{x^2+(y+1)^2}{2}\right)$

$= d\left(\iota_\Sigma^*\left(\frac{x^2+(y+1)^2}{2}\right)\right)$, i.e. exact.

$$\therefore \int_{\partial\Sigma} \iota_{\partial\Sigma}^* \iota_\Sigma^* \alpha = \int_{\partial\Sigma} \text{exact 1-form} = \boxed{0}$$

$\partial\Sigma$ has no boundary.

<u>Alternatively</u>, one can glue the solid ball $\{x^2+y^2<1\}$ to Σ to form a closed surface. Denote D to be the solid enclosed, then $\Sigma \cup B = \partial D$ so



$$\int_{\Sigma \cup B} \iota^* \eta = \int_D \underbrace{d\eta}_{0} = 0 \Rightarrow \int_{\Sigma} \iota_{\Sigma}^* \eta = -\int_B \iota_B^* \eta$$

$\iota_B^* \eta = 0$ on B since $z = 0$ on B (so $\iota_B^* dz = d(\iota_B^* z) = d(\underbrace{z \circ \iota_B}_{=0}) = 0$)

$$\iota_B^* \eta = -xy\cos(xy\cdot 0)\, dy \wedge d(0) + 2xe^{0}(0-1)\, d(0) \wedge dx + y\cdot(0)\cos(0)\, dx \wedge dy = 0$$

$$\therefore \int_{\Sigma} \iota_{\Sigma}^* \eta = -\int_B \iota_B^* \eta = \boxed{0}.$$

PROBLEM 2

$$\mathcal{L}_X(du^i) = i_X \cancelto{0}{d(du^i)} + d(i_X du^i)$$

$$= d\left(du^i(X)\right) = d(X^i) = \sum_k \frac{\partial X^i}{\partial u_k} du^k$$

$$\therefore \mathcal{L}_X g = \mathcal{L}_X\Big(\sum_{i,j} g_{ij}\, du^i \otimes du^j\Big)$$

$$= \sum_{i,j}\Big(X(g_{ij})\, du^i \otimes du^j + g_{ij}\, \mathcal{L}_X du^i \otimes du^j + g_{ij}\, du^i \otimes \mathcal{L}_X du^j\Big)$$

$$= \sum_{i,j}\Big(\sum_k X^k \frac{\partial g_{ij}}{\partial u_k} du^i \otimes du^j\Big)$$

$$+ \sum_{i,j}\Big(g_{ij} \sum_k \frac{\partial X^i}{\partial u_k} du^k \otimes du^j\Big)$$

$$+ \sum_{i,j}\Big(g_{ij}\, du^i \otimes \sum_k \frac{\partial X^j}{\partial u_k} du^k\Big)$$

$$= \sum_{i,j,k} X^k \frac{\partial g_{ij}}{\partial u_k} du^i \otimes du^j$$

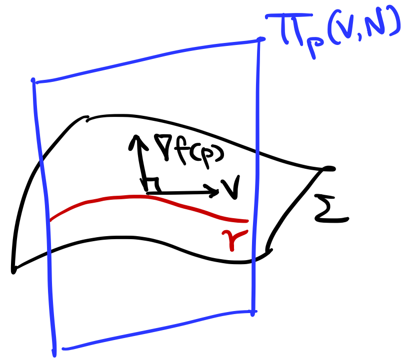
$$+ \sum_{i,j,k} g_{kj} \frac{\partial X^k}{\partial u_i} du^i \otimes du^j \qquad i \leftrightarrow k \quad \text{switch index}$$

$$+ \sum_{i,j,k} g_{ik} \frac{\partial X^k}{\partial u_j} du^i \otimes du^j \qquad j \leftrightarrow k$$

$$= \sum_{i,j,k}\left(X^k \frac{\partial g_{ij}}{\partial u_k} + g_{kj} \frac{\partial X^k}{\partial u_i} + g_{ik} \frac{\partial X^k}{\partial u_j}\right) du^i \otimes du^j$$

PROBLEM 3

(a) From multivariable calculus, we can take $\underbrace{N = \nabla f(p)}_{}$ $\overset{\neq 0 \text{ (given)}}{}$ since Σ is a level set of f.



Normal of $\Pi_p(V,N)$: $\underbrace{V \times \nabla f(p)}$

Equation of $\Pi_p(V,N)$:

$$(\underset{\substack{\| \\ (x,y,z)}}{\vec{x}} - \vec{p}) \cdot (V \times \nabla f(p)) = 0$$

$\therefore\ \gamma = \Pi_p(V,N) \cap \overset{\{f(x,y,z)=0\}}{\Sigma}$ is given by the system:

$$\begin{cases} (\vec{x}-\vec{p}) \cdot (V \times \nabla f(p)) = 0 \\ f(x,y,z) = 0 \end{cases}$$

Let $\Phi(\underbrace{x,y,z}_{\mathbb{R}^3}) = \Big(((x,y,z)-\vec{p}) \cdot (V \times \nabla f(p)),\ f(x,y,z) \Big) \in \mathbb{R}^2$

then $\Phi^{-1}(\vec{0}) = \{(x,y,z) : \Phi(x,y,z) = (0,0)\} = \Pi_p(V,N) \cap \Sigma$

(b) First we show Φ is a $\underbrace{\text{submersion at } p}$ (then by $\overset{\Phi \text{ is } C^\infty}{\text{continuity}}$ it is a submersion in a neighborhood of p)

$$\frac{\partial}{\partial x}\Big(((x,y,z)-\vec{p}) \cdot \underbrace{(V \times \nabla f(p))}_{\leftarrow \text{ constant vector}} \Big) = (V \times \nabla f(p))^x$$

x-component of $V \times \nabla f(p)$

$$[\Phi_*]_p = \begin{bmatrix} (V \times \nabla f(p))^x & (V \times \nabla f(p))^y & (V \times \nabla f(p))^z \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{bmatrix}$$

y, z components of $V \times \nabla f(p)$.

$$= \begin{bmatrix} V \times \nabla f(p) \\ \nabla f(p) \end{bmatrix}$$

Since $\underbrace{V \times \nabla f(p)}_{\neq 0} \perp \underbrace{\nabla f(p)}_{\neq 0 \leftarrow \text{given}}$, $\{V \times \nabla f(p), \nabla f(p)\}$ are linearly independent.

$$|V \times \nabla f(p)| = \underbrace{|V|}_{\neq 0}\,\underbrace{|\nabla f(p)|}_{\neq 0}\,\underbrace{\sin\tfrac{\pi}{2}}_{=1} \neq 0.$$

given

$\Rightarrow$ column rank of $[\Phi_*]_p = 2$

$\Rightarrow$ row rank of $[\Phi_*]_p = 2$

$\therefore\ \Phi_*$ is surjective $\Rightarrow$ Φ is a submersion at p.

By continuity (since Φ is C^∞), $\exists$ open set $\mathcal{U} \subset \mathbb{R}^3$ containing p such that

$$\Phi|_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^2$$

is a submersion at every point on $\mathcal{U}$.

$\Rightarrow \gamma \cap \mathcal{U} = \Phi^{-1}(\vec{0}) \cap \mathcal{U} = \left(\Phi|_{\mathcal{U}}\right)^{-1}(\vec{0})$ is a C^∞ 1-manifold
$\quad \uparrow_{=3-2}$.

(c) From (b) we have shown $\gamma \cap \mathcal{U}$ is a submanifold of $\mathcal{U} \subset \mathbb{R}^3$

$\Rightarrow \underset{\swarrow \text{inclusion map}}{\iota_\gamma} : \gamma \cap \mathcal{U} \to \mathbb{R}^3$ is an immersion

We need $j : \gamma \cap \mathcal{U} \to \Sigma$ to be an immersion. $\nabla f \neq 0$.
$\quad$ ↖ inclusion map.

We also know $\iota_\Sigma : \Sigma \to \mathbb{R}^3$ is an immersion since $\overset{f^{-1}(0)}{\Sigma}$ is a regular surface.

Consider

$$\begin{tikzcd} \gamma \cap \mathcal{U} \arrow[r, "j"] \arrow[rr, bend right, "\iota_\gamma"'] & \Sigma \arrow[r, "\iota_\Sigma"] & \mathbb{R}^3 \end{tikzcd}$$

$$\iota_\gamma = \iota_\Sigma \circ j \Rightarrow (\iota_\gamma)_* = (\iota_\Sigma)_* \circ j_*$$

To show $\mathrm{Ker}(j_*) = \{0\}$, we let T s.t. $j_*(T) = 0$

then $(\iota_\gamma)_*(T) = ((\iota_\Sigma)_* \circ j_*(T)) = 0$

$\Rightarrow T = 0$ as $(\iota_\gamma)_*$ is injective.

$\therefore \mathrm{Ker}(j_*) = \{0\}$ too $\Rightarrow j_*$ is injective.

Hence $\gamma \cap \mathcal{U}$ is a submanifold of Σ.

PROBLEM 4

(a) We calculate $\det D(F_j^{-1} \circ F_i)$ for $i<j$ (the case $j<i$ is similar)

$$
\begin{aligned}
& F_j^{-1} \circ F_i(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{2n}) \\
&= F_j^{-1}\left([u_1 : \cdots : u_{i-1} : (-1)^i : u_{i+1} : \cdots : u_j : \cdots u_{2n}]\right) \\
&= F_j^{-1}\left(\left[\frac{u_1}{u_j} : \cdots : \frac{u_{i-1}}{u_j} : \frac{(-1)^i}{u_j} : \frac{u_{i+1}}{u_j} : \cdots : \frac{u_{j-1}}{u_j} : 1 : \frac{u_{j+1}}{u_j} : \cdots : \frac{u_{2n}}{u_j}\right]\right) \\
&= (-1)^j \Big(\frac{u_1}{u_j}, \dots, \frac{u_{i-1}}{u_j}, \underbrace{\frac{(-1)^i}{u_j}}_{i\text{-th}}, \frac{u_{i+1}}{u_j}, \dots, \frac{u_{j-1}}{u_j}, \frac{u_{j+1}}{u_j}, \dots, \frac{u_{2n}}{u_j}\Big)
\end{aligned}
$$

$\Rightarrow \det D(F_j^{-1} \circ F_i)$

$$
= \left((-1)^j\right)^{2n-1}
\begin{array}{c}
\begin{array}{ccccccccc}
\frac{\partial}{\partial u_1} & \cdots & \frac{\partial}{\partial u_{i-1}} & \frac{\partial}{\partial u_{i+1}} & \cdots & \frac{\partial}{\partial u_{j-1}} & \frac{\partial}{\partial u_j} & \frac{\partial}{\partial u_{j+1}} & \cdots \ \frac{\partial}{\partial u_{2n}}
\end{array} \\
\left|\begin{array}{ccc|ccc|c|ccc}
\frac{1}{u_j} & & 0 & & & & -\frac{u_1}{u_j^2} & & & \\
 & \ddots & & & 0 & & \vdots & & 0 & \\
0 & & \frac{1}{u_j} & & & & -\frac{u_{i-1}}{u_j^2} & & & \\
\hline
 & 0 & & & 0 & & -\frac{(-1)^i}{u_j^2} & & 0 & \\
\hline
 & & & \frac{1}{u_j} & & 0 & -\frac{u_{i+1}}{u_j^2} & & & \\
 & 0 & & & \ddots & & \vdots & & 0 & \\
 & & & 0 & & \frac{1}{u_j} & -\frac{u_{j-1}}{u_j^2} & & & \\
\hline
 & & & & & & -\frac{u_{j+1}}{u_j^2} & \frac{1}{u_j} & & \\
 & 0 & & & 0 & & \vdots & & \ddots & \\
 & & & & & & -\frac{u_{2n}}{u_j^2} & & & \frac{1}{u_j}
\end{array}\right|
\end{array}
\begin{array}{l}
\left.\right\} i-1 \\
\leftarrow i\text{-th row} \\
\left.\right\} j-i-1 \\
\left.\right\} 2n-j
\end{array}
$$

cofactor expansion along i-th row

$$
= (-1)^{(2n-1)j} \cdot \underbrace{(-1)^{i+(j-1)}}_{\text{cofactor sign}} \left(-\frac{(-1)^i}{u_j^2}\right) \frac{1}{u_j^{2n-2}} = (-1)^{2nj-j+i+j-1+i}(-1) \cdot \frac{1}{u_j^{2n}}
$$

cofactor of the $-\frac{(-1)^i}{u_j^2}$ entry is $\frac{1}{u_j} Id$

$$
= (-1)^{\overset{\text{even}}{2nj}} \frac{1}{u_j^{\overset{\text{even}}{2n}}} > 0
$$

$\therefore$ The atlas $\mathcal{A}$ is oriented (hence $\mathbb{R}P^{2n-1}$ is orientable)

(b) M^n is orientable $\iff$ $\exists$ a non-vanishing n-form Ω_0.

We claim $H^n_{dR}(M^n) = \mathrm{span}\{[\Omega_0]\}$, then it follows $b_n(M^n) = \dim H^n_{dR}(M^n) = 1$.

Let Ω be any (closed) n-form on M^n.
(closed) ← must be.

let $$c = \frac{\int_M \Omega}{\int_M \Omega_0}$$ $\int_M \Omega_0 \neq 0$ since Ω_0 is non-vanishing.

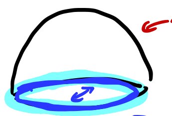
then $$\int_M \Omega - c\Omega_0 = \int_M \Omega - c\int_M \Omega_0 = 0$$

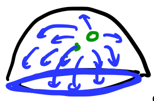
$\therefore$ By given fact, $\Omega - c\Omega_0$ is exact.

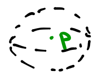
$$\Rightarrow [\Omega] = [c\Omega_0] \text{ in } H^n_{dR}(M^n) = c[\Omega_0].$$

$\boxtimes$

We sketch a "2d" picture for simplicity.

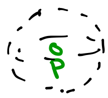
(c) $\mathbb{RP}^3$ =  ← upper S^3, ↖ $\mathbb{RP}^2$

=  $= U$ $(\mathbb{RP}^3 \setminus \{p\})$ ← a point on upper S^3

$\cup$  $= V$ (3d-ball containing p) ← ☆ Not 2d.

deformation retract onto  $\mathbb{RP}^2$

Hence $H^k_{dR}(U) = H^k_{dR}(\mathbb{RP}^2)$

then $U \cap V$ =  $= B^3 \setminus \{p\}$ deform. retract S^2

Consider Mayer-Vietoris sequence:

$$0 \to \underbrace{H^0(\mathbb{RP}^3)}_{1 \text{ (connected)}} \to \underbrace{H^0(U) \oplus H^0(V)}_{1+1 \text{ (both connected)}} \to \underbrace{H^0(U\cap V)}_{1 \text{ (connected)}}$$

$$\to \underbrace{H^1(\mathbb{RP}^3)}_{x} \to \underbrace{H^1(U)}_{b_1(\mathbb{RP}^2)} \oplus \underbrace{H^1(V)}_{=0 \text{ (diffeo to star-shaped ball)}} \to H^1(U\cap V) \leftarrow 0$$

proved in lecture → ($U \cap V$ retracts onto S^2)

$$\to \underbrace{H^2(\mathbb{RP}^3)}_{y} \to \underbrace{H^2(U)}_{b_2(\mathbb{RP}^2)} \oplus \underbrace{H^2(V)}_{=0 \text{ (diffeo. to star-shaped ball)}} \to H^2(U\cap V) \leftarrow 1$$

$1 \leftarrow$ (retracts onto S^2) orientable; $b_2(S^2) = 1$ from (b)

$$\to \underbrace{H^3(\mathbb{RP}^3)}_{1 \text{ (orientable by (a)) use (b).}} \to \underbrace{H^3(U)}_{b_3(\mathbb{RP}^2) = 0} \oplus \underbrace{H^3(V)}_{\leftarrow 0 \text{ (diffeo. to star-shaped ball.)}} \to \cdots$$

$b_3(\mathbb{RP}^2) = 0$, $\underbrace{}_{\dim = 2 < 3}$

Consider alternating sum:

$$\begin{cases} 1 - (1+1) + 1 - x + b_1(\mathbb{RP}^2) = 0 \\ y - b_2(\mathbb{RP}^2) + 1 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = b_1(\mathbb{RP}^2) \\ y = b_2(\mathbb{RP}^2) \end{cases}$$ as desired.