香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code:	MATH 4033
Course Title:	Calculus on Manifolds
Semester:	Spring 2016-17
Date and Time:	16 May 2017, 4:30PM-7:30PM

Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- It is an **OPEN-NOTES** exam. Authorized reference materials are the instructor's lecture notes and homework solutions. No other reference materials are allowed.
- Answer **ALL FOUR** problems. Write your solutions in the yellow book.
- You must **SHOW YOUR WORK** to receive credits in every problem.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

Student's Signature: ______________________________

Student's Name: ______________________ **HKUST ID:** __________

1. Let $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose $\Sigma := f^{-1}(0)$ is non-empty, and f is a submersion at every $p \in \Sigma$.
   (a) Which of the following MUST be true about Σ? List the correct term(s) in the answer book. **[5]**

   compact 2-manifold regular surface with boundary connected

   (b) Define the following subsets of Σ: **[3]**

$$\mathcal{O}_x := \{(x,y,z) \in \Sigma : f_x(x,y,z) \neq 0\}$$
$$\mathcal{O}_y := \{(x,y,z) \in \Sigma : f_y(x,y,z) \neq 0\}$$
$$\mathcal{O}_z := \{(x,y,z) \in \Sigma : f_z(x,y,z) \neq 0\}$$

   where f_x, f_y, and f_z are partial derivatives of f by x, y and z respectively.
   What is $\mathcal{O}_x \cup \mathcal{O}_y \cup \mathcal{O}_z$? Explain briefly.

   (c) Denote the inclusion map by $\iota : \Sigma \to \mathbb{R}^3$. Consider the following locally defined 2-forms: **[12]**

$$\omega_x := \iota^*\left(\frac{1}{f_x}\, dy \wedge dz\right) \quad \text{on } \mathcal{O}_x$$
$$\omega_y := \iota^*\left(\frac{1}{f_y}\, dz \wedge dx\right) \quad \text{on } \mathcal{O}_y$$
$$\omega_z := \iota^*\left(\frac{1}{f_z}\, dx \wedge dy\right) \quad \text{on } \mathcal{O}_z$$

   Show that:
   i. $\omega_x(p) \neq 0$ for any $p \in \mathcal{O}_x$.
   ii. $\omega_x = \omega_y$ on $\mathcal{O}_x \cap \mathcal{O}_y$.

   (d) Show that Σ is orientable using results from (c). You may also use the *analogous* results of (c), such as $\omega_y(q) \neq 0$ for any $q \in \mathcal{O}_y$, and $\omega_y = \omega_z$ on $\mathcal{O}_y \cap \mathcal{O}_z$, etc. **[5]**

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2. Consider the following torus $\mathbb{T}^2$ in $\mathbb{R}^4$:

$$\mathbb{T}^2 := \{(x,y,z,w) \in \mathbb{R}^4 : x^2 + y^2 = 1 \quad \text{and} \quad z^2 + w^2 = 1\},$$

   which can be locally parametrized by $\mathbf{F} : (0, 2\pi) \times (0, 2\pi) \to \mathbb{T}^2$:

$$\mathbf{F}(\theta_1, \theta_2) = (\cos\theta_1,\, \sin\theta_1,\, \cos\theta_2,\, \sin\theta_2)$$

   Denote $\iota : \mathbb{T}^2 \to \mathbb{R}^4$ to be the inclusion map. Consider the following 1-form on $\mathbb{T}^2$:

$$\sigma := \iota^*\left(y^3\, dx - (x^3 - 3x)\, dy + (w^3 - 3w)\, dz - z^3\, dw\right).$$

   (a) Show that σ is closed. **[5]**

   (b) Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ parametrized by $\mathbf{G}(t) = (\cos t, \sin t)$. Consider the map $\Phi : \mathbb{S}^1 \to \mathbb{T}^2$ given by: **[10]**

$$\underbrace{(p,q)}_{\text{coordinates in } \mathbb{R}^2} \mapsto \underbrace{(p,\, q,\, (p-q)/\sqrt{2},\, (p+q)/\sqrt{2})}_{\text{coordinates in } \mathbb{R}^4}.$$

   Express $\Phi^*\sigma$ in terms of dt.

   (c) Using (b), show that σ is not exact. You may use the following results without proof: **[10]**

$$\int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \sin^2 t\, dt = \pi, \qquad \int_0^{2\pi} \cos^4 t\, dt = \int_0^{2\pi} \sin^4 t\, dt = \frac{3\pi}{4}.$$

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3. Using Mayer-Vietoris sequences, show that $H^1_{\mathrm{dR}}(\mathbb{RP}^2) = 0$. **[25]**

   [Hint and remark: Consider the standard coordinate cover of $\mathbb{RP}^2$. It may be necessary to use two or more Mayer-Vietoris sequences. You may use the following fact without proof: $H^2_{\mathrm{dR}}(\mathbb{RP}^2) = 0$.]

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4. (a) Explain why any pair of 2-forms ω and η on a smooth manifold M must commute when taking wedge product, i.e. $\omega \wedge \eta = \eta \wedge \omega$. **[3]**

   (b) Let M^{2m} and N^{2n} be smooth manifolds with dimensions $2m$ and $2n$ respectively. Denote the projection maps from $M \times N$ to each of M and N by: **[10]**

$$\begin{aligned} \pi_M : M \times N &\to M \\ (p, q) &\mapsto p \end{aligned} \qquad\qquad \begin{aligned} \pi_N : M \times N &\to N \\ (p, q) &\mapsto q \end{aligned}$$

   Let α be a smooth 2-form on M, and β be a smooth 2-form on N. Show that:

$$(\pi_M^*\alpha + \pi_N^*\beta)^{m+n} = C(\pi_M^*\alpha)^m \wedge (\pi_N^*\beta)^n$$

   for some non-zero constant C depending only on m and n.

   (c) Suppose further that both M^{2m} and N^{2n} are compact orientable manifolds without boundary. Show that the following map is well-defined: **[12]**

$$\begin{aligned} H^2_{\mathrm{dR}}(M) \times H^2_{\mathrm{dR}}(N) &\to \mathbb{R} \\ ([\alpha], [\beta]) &\mapsto \int_{M \times N} (\pi_M^*\alpha + \pi_N^*\beta)^{m+n}. \end{aligned}$$

End of Exam