#1 (a) 2-manifold regular surface

(b) $\mathcal{O}_x \cup \mathcal{O}_y \cup \mathcal{O}_z = \Sigma$ since for any $p \in \Sigma$, f is a submersion at $p \Rightarrow [f_*]_p = \left[\frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial y}(p) \quad \frac{\partial f}{\partial z}(p)\right]$ is surjective

At least one of $f_x(p)$, $f_y(p)$ and $f_z(p)$ is non-zero

$\Rightarrow p \in \mathcal{O}_x \cup \mathcal{O}_y \cup \mathcal{O}_z$.

(c)(i) For $p \in \mathcal{O}_x$, we have $\frac{\partial f}{\partial x}(p) \neq 0$ and by implicit function theorem, Σ can be locally expressed as a graph $x = g(y,z)$ such that $f(g(y,z), y, z) = 0$.

Using the graphical parametrization $F(u,v) = (\underbrace{g(u,v)}_{x}, \underset{\uparrow}{u}, \underset{\uparrow}{v})$, ω_x can be locally expressed as:
$ y \quad z$

$$\omega_x = \frac{1}{f_x}\, d(y \circ \iota) \wedge d(z \circ \iota) = \frac{1}{f_x}\, du \wedge dv \neq 0.$$

since $\left\{\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right\}_p$ are linearly indep.

Hence $\omega_x(p) \neq 0 \quad \forall p \in \mathcal{O}_x$.

(ii) On $\mathcal{O}_x \cap \mathcal{O}_y$, one can still parametrize Σ by $F(u,v) = (\underbrace{g(u,v)}_{x}, \underbrace{u}_{y}, \underbrace{v}_{z})$.

then
$$\omega_y = \frac{1}{f_y}\, d(z \circ \iota) \wedge d(x \circ \iota)$$
$$= \frac{1}{f_y}\, dv \wedge \left(\frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv\right)$$
$$= \frac{1}{f_y} \cdot \frac{\partial g}{\partial u}\, dv \wedge du \qquad \text{——— } (*)$$

On the other hand, we have $f(g(u,v), u, v) = 0 \quad \forall (u,v)$

By chain rule, we have:

$$0 = \frac{\partial f}{\partial x}\frac{\partial g}{\partial u} + \frac{\partial f}{\partial y}$$

$$\Rightarrow \frac{1}{f_y} = -\frac{1}{f_x g_u}$$

$$\begin{array}{ccc} & f & \\ x & y & z \\ u\ v & u & v \end{array}$$

Put it into $(*)$, we get:

from (i)

$$\omega_y = \left(-\frac{1}{f_x g_u}\right) \cdot g_u\, dv \wedge du = \frac{1}{f_x}\, du \wedge dv = \omega_x.$$

(d) Similar to (c)(ii), we have

$$\begin{cases} \omega_x = \omega_z & \text{on } \mathcal{O}_x \cap \mathcal{O}_z \\ \omega_y = \omega_z & \text{on } \mathcal{O}_y \cap \mathcal{O}_z . \end{cases}$$

Therefore, the following in a well-defined 2-form on the whole Σ:

$$\Omega := \begin{cases} \omega_x & \text{on } \mathcal{O}_x \\ \omega_y & \text{on } \mathcal{O}_y \\ \omega_z & \text{on } \mathcal{O}_z . \end{cases}$$

From (c)(i), $\Omega(p) \neq 0$ on $\mathcal{O}_x$. But similarly we can also prove $\Omega(q) \neq 0$ on $\mathcal{O}_y$, and $\Omega(r) \neq 0$ on $\mathcal{O}_z$

with the same proof.

Therefore, Ω is a non-vanishing global 2-form of Σ
$\Rightarrow$ Σ is orientable.

#2 (a) $d\left(y^3\,dx - (x^3-3x)\,dy + (w^3-3w)\,dz - z^3\,dw\right)$

$$= 3y^2\,dy\wedge dx - (3x^2-3)\,dx\wedge dy + (3w^2-3)\,dw\wedge dz - 3z^2\,dz\wedge dw$$

$$= -3(x^2+y^2-1)\,dx\wedge dy - 3(z^2+w^2-1)\,dz\wedge dw$$

Since $d\circ\iota^* = \iota^*\circ d$, we get:

$$d\sigma = -3(x^2+y^2-1)\circ\iota \; d(x\circ\iota)\wedge d(y\circ\iota) - 3(z^2+w^2-1)\circ\iota \; d(z\circ\iota)\wedge d(w\circ\iota)$$

On $\mathbb{T}^2$, we have $x^2+y^2=1$ and $z^2+w^2=1$, so

$$(x^2+y^2-1)\circ\iota = (z^2+w^2-1)\circ\iota = 0 \quad \text{on } \mathbb{T}^2$$

$\Rightarrow d\sigma = 0$.

(b) First find the local coordinate expression of Φ:

$$\begin{aligned}
& F^{-1}\circ\Phi\circ G(t) \\
&= F^{-1}\circ\Phi(\cos t, \sin t) \\
&= F^{-1}\left(\cos t, \sin t, \tfrac{1}{\sqrt{2}}(\cos t-\sin t), \tfrac{1}{\sqrt{2}}(\cos t+\sin t)\right) \\
&= F^{-1}\left(\cos t, \sin t, \cos(t+\tfrac{\pi}{4}), \sin(t+\tfrac{\pi}{4})\right) \\
&= (t, t+\tfrac{\pi}{4})
\end{aligned}$$

To find $\Phi^*\sigma$, we need to compute $(\Phi^*\sigma)(\frac{\partial}{\partial t}) = \sigma(\Phi_*\frac{\partial}{\partial t})$:

Since $D(F^{-1}\circ\Phi\circ G) = [1 \quad 1]$, we have $\Phi_*\frac{\partial}{\partial t} = \frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}$
eval. at $(\theta_1,\theta_2)=(t, t+\frac{\pi}{4})$

$\therefore\ \sigma(\Phi_*\frac{\partial}{\partial t}) = \sigma(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2})$ To find this, we first express σ in terms of $d\theta_i$'s.

$$\iota^*(y^3dx) = \sin^3\theta_1\, d(\cos\theta_1) = -\sin^4\theta_1\, d\theta_1$$

$$\begin{aligned}
\iota^*(-(x^3-3x)dy) &= -(\cos^3\theta_1-3\cos\theta_1)\, d(\sin\theta_1) = -(\cos^3\theta_1-3\cos\theta_1)\cos\theta_1\, d\theta_1 \\
&= (-\cos^4\theta_1+3\cos^2\theta_1)\, d\theta_1
\end{aligned}$$

Similarly, we have:

$$\iota^*((w^3-3w)dz) = (-\sin^4\theta_2+3\sin^2\theta_2)\, d\theta_2$$

$$\iota^*(-z^3dw) = -\cos^4\theta_2\, d\theta_2$$

$$\Rightarrow \sigma = -\sin^2\theta_1\, d\theta_1 + (-\cos^4\theta_1+3\cos^2\theta_1)\, d\theta_1 + (-\sin^4\theta_2+3\sin^2\theta_2)\, d\theta_2 - \cos^4\theta_2\, d\theta_2$$

$$\Rightarrow \sigma(\Phi_*\tfrac{\partial}{\partial t})\Big|_t = \sigma(\tfrac{\partial}{\partial\theta_1}+\tfrac{\partial}{\partial\theta_2})\Big|_{(t,t+\frac{\pi}{4})}$$

$$= \underbrace{(-\sin^4 t-\cos^4 t+3\cos^2 t)}_{\sigma(\frac{\partial}{\partial\theta_1})|_t}\ \underbrace{-\sin^4(t+\tfrac{\pi}{4})+3\sin^2(t+\tfrac{\pi}{4})-\cos^4(t+\tfrac{\pi}{4})}_{\sigma(\frac{\partial}{\partial\theta_2})|_{t+\frac{\pi}{4}}}$$

$$\therefore\ \Phi^*\sigma = \left(-\sin^4 t-\cos^4 t+3\cos^2 t-\sin^4(t+\tfrac{\pi}{4})+3\sin^2(t+\tfrac{\pi}{4})-\cos^4(t+\tfrac{\pi}{4})\right)dt$$

(c) By contradiction: assume σ is exact and $\exists$ 1-form η on T^2 s.t. $\sigma = d\eta$
then $\Phi^*\sigma = \Phi^*d\eta = d\Phi^*\eta \Rightarrow \int_{S^1}\Phi^*\sigma = \int_{S^1} d(\Phi^*\eta) = 0$ by Stokes' Theorem $(\partial S=\phi)$.

However

$$\int_{S^1}\Phi^*\sigma = \int_0^{2\pi}\left(-\sin^4 t-\cos^4 t+3\cos^2 t-\sin^4(t+\tfrac{\pi}{4})+3\sin^2(t+\tfrac{\pi}{4})-\cos^4(t+\tfrac{\pi}{4})\right)dt$$

$= -\frac{3\pi}{4}-\frac{3\pi}{4}+3\pi-\frac{3\pi}{4}+3\pi-\frac{3\pi}{4} = 3\pi \neq 0.$ Contradiction! $\Rightarrow$ σ is NOT exact.

#3. Let $U_i = \{ [x_1 : x_2 : x_3] \in \mathbb{RP}^2 \; : \; x_i \neq 0 \}$ $(i=1,2,3)$.

For each i, U_i is diffeo to $\mathbb{R}^2$

For each $i \neq j$ (say $i=1, j=2$) :

$$U_1 \cap U_2 = \{ [x_1 : x_2 : x_3] : x_1 \neq 0 \text{ AND } x_2 \neq 0 \}$$
$$= \{ [1 : y_2 : y_3] : y_2 \neq 0, \; y_3 \text{ any real} \}$$
$$\cong \mathbb{R}^2 \setminus \text{infinite} = \underbrace{W_1 \sqcup W_2}_{\text{half-planes}}$$

First compute $H^1(U_i \cup U_j)$

$$U_1 \cup U_2 = \{ [x_1 : x_2 : x_3] : x_1 \neq 0 \text{ OR } x_2 \neq 0 \}$$

is path-connected since: $\forall [x_1 : x_2 : x_3]$ and $[y_1 : y_2 : y_3]$ in $U_1 \cup U_2$, we have $(x_1, x_2) \neq 0$ and $(y_1, y_2) \neq 0$. Since $\mathbb{R}^2 \setminus (0,0)$ is path-connected, $\exists$ a path $\vec{r}(t)$ in $\mathbb{R}^2 \setminus (0,0)$ connecting (x_1, x_2) and (y_1, y_2)

$(r_1(t), r_2(t))$

then $[r_1(t) : r_2(t) : (1-t)x_3 + t y_3]$, $0 \leq t \leq 1$

is a path connecting $[x_1 : x_2 : x_3]$ and $[y_1 : y_2 : y_3]$.

$$0 \to \overset{\mathbb{R}}{H^0(U_1 \cup U_2)} \to \overset{\mathbb{R}}{H^0(U_1)} \overset{\oplus}{\oplus} \overset{\mathbb{R}}{H^0(U_2)} \to \overset{\mathbb{R}^2}{H^0(U_1 \cap U_2)}$$
$$\to \overset{x}{H^1(U_1 \cup U_2)} \to \overset{0}{H^1(U_1)} \overset{\oplus}{\oplus} \overset{0}{H^1(U_2)}$$

Using alternating sum: $1 - (1+1) + 2 - x = 0$

$\Rightarrow \quad x = 1$

$\therefore \dim H^1(U_1 \cup U_2) = 1$.

Next consider $V_1 = \mathcal{U}_1 \cup \mathcal{U}_2$

$V_2 = \mathcal{U}_3$

then $V_1 \cup V_2 = \mathbb{RP}^2$, $\quad V_1 \cap V_2 = \{[x_1 : x_2 : x_3] : (x_1 \neq 0 \text{ OR } x_2 \neq 0) \text{ AND } x_3 \neq 0\}$

$$= \{[y_1 : y_2 : 1] : y_1 \neq 0 \text{ OR } y_2 \neq 0\}$$

$$\cong \mathbb{R}^2 \setminus \{(0,0)\}.$$

Consider:

$$0 \to \overset{\overset{\text{connected}}{1}}{H^0(\mathbb{RP}^2)} \to \overset{(1\ +}{H^0(V_1)} \oplus \overset{1)}{H^0(V_2)} \to \overset{1}{H^0(V_1 \cap V_2)}$$

$$\to \overset{y}{H^1(\mathbb{RP}^2)} \to \underset{\substack{\text{previous}\\ \text{result}}}{\overset{(1\ +}{H^1(V_1)}} \oplus \underset{\cong \mathbb{R}^2}{\overset{0)}{H^1(V_2)}} \to \overset{1}{H^1(V_1 \cap V_2)} \leftarrow \begin{array}{l}\mathbb{R}^2 \setminus \{(0,0)\} \\ \text{def. retr.} \\ \text{onto } S^1.\end{array}$$

$$\to \underset{\text{given}}{\overset{0}{H^2(\mathbb{RP}^2)}}$$

Consider alternating sum:

$$1 - 2 + 1 - y + 1 - 1 = 0$$

$$\Rightarrow y = 0.$$

$\therefore \ \dim H^1_{dR}(\mathbb{RP}^2) = 0$, i.e. $H^1_{dR}(\mathbb{RP}^2) = 0$.

#4 (a) $(dx^i \wedge dx^j) \wedge (dx^k \wedge dx^l) = dx^i \wedge \underbrace{(dx^k \wedge dx^l)}_{\text{swap twice}} \wedge dx^j$

swap twice

$$= (dx^k \wedge dx^l) \wedge (dx^i \wedge dx^j)$$

(b) Key observation:

$\underbrace{\alpha^k}_{2k\text{-form}} = 0 \quad \forall k > m \quad$ since $\dim M = 2m$.

$\beta^r = 0 \quad \forall r > n \quad$ since $\dim N = 2n$

Since 2-forms commute, one can apply Binomial Thm:

$$\left(\pi_M^*\alpha + \pi_N^*\beta\right)^{m+n} = \sum_{k=0}^{m+n} C_k^{m+n} (\pi_M^*\alpha)^k \wedge (\pi_N^*\beta)^{m+n-k}$$

$$= \sum_{k=0}^{m+n} C_k^{m+n} \pi_M^*(\alpha^k) \wedge \pi_N^*(\beta^{m+n-k}) \quad \text{———} (*)$$

Since $\begin{cases} \alpha^k = 0 & \text{when } k > m \\ \beta^{m+n-k} = 0 & \text{when } m+n-k > n \\ & \Updownarrow \\ & k < m \end{cases}$,

the only (possibly) non-vanishing term in $(*)$ is the one with $k = m$

$$\therefore \left(\pi_M^*\alpha + \pi_N^*\beta\right)^{m+n} = C_m^{m+n} (\pi_M^*\alpha)^m \wedge (\pi_N^*\beta)^{\overbrace{m+n-m}^{=n}}$$

(c) Suppose $\begin{cases} [\alpha] = [\alpha'] & \text{in } H^2(M) \\ [\beta] = [\beta'] & \text{in } H^2(N) \end{cases}$, i.e. $\exists$ 1-form η on M s.t. $\alpha - \alpha' = d\eta$

$\exists$ 1-form θ on N s.t. $\beta - \beta' = d\theta$.

Need to check:

$$\int_{M\times N} (\pi_M^*\alpha + \pi_N^*\beta)^{m+n} = \int_{M\times N} (\pi_M^*\alpha' + \pi_N^*\beta')^{m+n},$$

or equivalently (by (b)):

$$\int_{M\times N} (\pi_M^*\alpha)^m \wedge (\pi_N^*\beta)^n = \int_{M\times N} (\pi_M^*\alpha')^m \wedge (\pi_N^*\beta')^n$$

Consider

$$\int_{M\times N} (\pi_M^* \alpha)^m \wedge (\pi_N^* \beta)^n = \int_{M\times N} \left(\pi_M^*(\alpha' + d\eta)\right)^m \wedge (\pi_N^* \beta)^n$$

$$= \int_{M\times N} \pi_M^* \Big((\alpha')^m + C_1^m (\alpha')^{m-1} \wedge d\eta + C_2^m (\alpha')^{m-2} \wedge (d\eta)^2 + \cdots + (d\eta)^m \Big) \wedge (\pi_N^* \beta)^n$$

$$= \int_{M\times N} (\pi_M^* \alpha')^m \wedge (\pi_N^* \beta)^n$$

(**)

$$+ \int_{M\times N} \pi_M^* \left(C_1^m (\alpha')^{m-1} \wedge d\eta + C_2^m (\alpha')^{m-2} \wedge (d\eta)^2 + \cdots (d\eta)^m \right) \wedge (\pi_N^* \beta)^n$$

Note that α' is closed (as it represents $H^2(M)$)

By product rule one has:

$$C_1^m (\alpha')^{m-1} \wedge d\eta + C_2^m (\alpha')^{m-2} \wedge (d\eta)^2 + \cdots + (d\eta)^m$$

$$= d\Big(\pm C_1^m (\alpha')^{m-1} \wedge \eta \pm C_2^m (\alpha')^{m-2} \wedge \eta \wedge d\eta \pm \cdots \pm \eta \wedge \underbrace{d\eta \wedge \cdots \wedge d\eta}_{m-1} \Big)$$

where + or − is not essential.

$\therefore$ [highlighted expression] is exact. (say [highlighted expression] $= d\xi$)

Continue on our calculation:

the second integral in (**)

$$= \int_{M\times N} \pi_M^* (d\xi) \wedge (\pi_N^* \beta)^n = \int_{M\times N} d(\pi_M^* \xi) \wedge (\pi_N^* \beta)^n$$

$$= \int_{M\times N} d\left(\pi_M^* \xi \wedge (\pi_N^* \beta)^n \right) = 0 \leftarrow \text{by Stokes' Theorem since } M\times N \text{ has no boundary.}$$

$\uparrow$ since β is closed (and so is $\pi_N^* \beta$)

From (**), we have proved:

$$\int_{M\times N} (\pi_M^* \alpha)^m \times (\pi_N^* \beta)^n = \int_{M\times N} (\pi_M^* \alpha')^m \wedge (\pi_N^* \beta)^n .$$

Proceed similarly on $\beta = \beta' + d\theta$, one can also prove:

$$\int_{M\times N} (\pi_M^* \alpha')^m \times (\pi_N^* \beta)^n = \int_{M\times N} (\pi_M^* \alpha')^m \wedge (\pi_N^* \beta')^n$$

as desired.