MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #5 • de Rham Cohomology • Due Date: 20/05/2017, 11:59PM

1. The purpose of this exercise is to prove that $H^2(\mathbb{R}^3) = 0$, i.e. every closed 2-form on $\mathbb{R}^3$ must be exact. Consider a closed form:

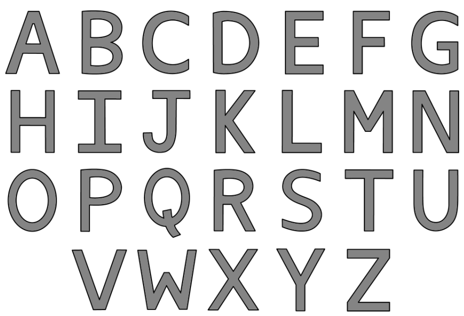
$$\omega = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$$

where A, B and C are smooth scalar functions of (x, y, z). Define the following 1-form:

$$\begin{aligned}\alpha := &\left(\int_0^1 A(tx, ty, tz)t\,dt\right)(y\,dz - z\,dy) \\ &+ \left(\int_0^1 B(tx, ty, tz)t\,dt\right)(z\,dx - x\,dz) \\ &+ \left(\int_0^1 C(tx, ty, tz)t\,dt\right)(x\,dy - y\,dx)\end{aligned}$$

First, compute $d\alpha$; then use the result to show that ω is exact.

2. Consider the following alphabet. Each letter is regarded as a solid region.



Answer the following without justification:

(a) Which letter(s) is/are contractible?

(b) Which letter(s) is/are star-shaped?

(c) Which letter(s) has/have non-zero 1st Betti number b_1?

3. Prove the following statements about deformation retracts by explicitly constructing Ψ_t.

(a) Show that the Möbius strip Σ defined in Example 4.11 deformation retracts onto a circle. [Hence, $H^1_{\text{dR}}(\Sigma) = H^1_{\text{dR}}(\mathbb{S}^1) = \mathbb{R}$.]

(b) The zero section Σ_0 of the tangent bundle TM of a smooth manifold M is defined to be:

$$\Sigma_0 := \{(p, 0_p) \in p \times T_pM : p \in M\}$$

where 0_p is the zero vector in T_pM. Show that Σ_0 is a deformation retract of TM. [Hence, $H^*_{\text{dR}}(TM) = H^*_{\text{dR}}(\Sigma_0) = H^*_{\text{dR}}(M)$.]

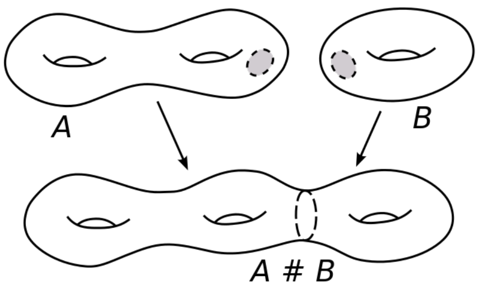
In the following problems, you may assume the Poincaré's Lemma and Deformation Retract Invariance hold on any H^k. Also, we may use the following fact without proof:

On a compact, connected orientable n-manifold M without boundary, then:

- $\dim H^n(M) = 1$
- $H^n(M \backslash \{p\}) = 0$ for any $p \in M$.

4. Let $\mathbb{T}^2$ be the 2-dimensional torus. Show that $b_1(\mathbb{T}^2) = 2$.

5. Given two compact smooth 2-manifolds A and B without boundary, its connected sum $A\#B$ is a 2-manifold obtained by removing an open ball in each of A and B, and then gluing them along the two boundary circles:



   (a) Show that $A\#B$ is orientable if both A and B are so. [Hint: use partitions of unity to construct a global non-vanishing 2-form.]
   (b) Using Mayer-Vietoris sequence, show that $b_1(A\#B) = b_1(A) + b_1(B)$.

6. (∞ points (bonus)) Prove or disprove: "Every Hodge cohomology class of a non-singular complex projective manifold $X \subset \mathbb{CP}^N$ is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X."

End of all MATH 4033 homework.
"The chain will be broken and all men will have their reward."
(from *Les Misérables*)