MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #4 • Generalized Stokes' Theorem • Due Date: 06/05/2018, 11:59PM

1. Given that M is a smooth m-manifold *without* boundary, and N is a smooth n-manifold *with* boundary. Show that the product manifold $M \times N$ is a smooth $(m+n)$-manifold *with* boundary, and that $\partial(M \times N) = M \times \partial N$.

Solution: Let $\{F_\alpha\}$ be a family of local parameterizations of M, $\{G_\beta\}$ and $\{\overline{G}_\gamma\}$ be families of interior and boundary types of local parameterizations of N such that

$$F_\alpha : U_\alpha \subset \mathbb{R}^m \to M$$
$$G_\beta : V_\beta \subset \mathbb{R}^n \to N$$
$$\overline{G}_\gamma : \overline{V}_\gamma \subset \mathbb{R}^m_+ \to N$$

and that the transition function

$$G_\beta^{-1} \circ G_{\beta'} \qquad G_\beta^{-1} \circ \overline{G}_\gamma \qquad \overline{G}_\gamma^{-1} \circ G_\beta \qquad \overline{G}_\gamma^{-1} \circ \overline{G}_{\gamma'}$$

are smooth on the overlapping domain for any β, β', γ and γ'. Clearly, $\{U_\alpha \times V_\beta, U_\alpha \times \overline{V}_\gamma\}$ is a covering of $M \times N$. Now consider

$$F_\alpha \times G_\beta : U_\alpha \times V\gamma \subset \mathbb{R}^{m+n} \to M \times N$$
$$F_\alpha \times \overline{G}_\beta : U_\alpha \times \overline{V}\gamma \subset \mathbb{R}^{m+n}_+ \to M \times N,$$

check that

$$(F_\alpha \times G_\beta)^{-1} \circ (F'_\alpha \times G_{\beta'}) = (F_\alpha^{-1} \circ F'_\alpha) \times (G_\beta^{-1} \circ G_{\beta'})$$
$$(F_\alpha \times G_\beta)^{-1} \circ (F'_\alpha \times \overline{G}_\gamma) = (F_\alpha^{-1} \circ F'_\alpha) \times (G_\beta^{-1} \circ \overline{G}_\gamma)$$
$$(F_\alpha \times \overline{G}_\gamma)^{-1} \circ (F'_\alpha \times G_\beta) = (F_\alpha^{-1} \circ F'_\alpha) \times (\overline{G}_\gamma^{-1} \circ G_\beta)$$
$$(F_\alpha \times \overline{G}_\gamma)^{-1} \circ (F'_\alpha \times \overline{G}_{\gamma'}) = (F_\alpha^{-1} \circ F'_\alpha) \times (\overline{G}_\gamma^{-1} \circ \overline{G}_{\gamma'})$$

are smooth for all α, α', β, β', γ and γ'. Thus, the product manifold $M \times N$ is a smooth $(m+n)$-manifold *with* boundary. Also, we have

$$\begin{aligned}
&\partial(M \times N) \\
=&\bigcup_{\alpha,\gamma}\{F_\alpha \times G_\gamma(u_1, \cdots, u_m, \overline{v}_1, \cdots, \overline{v}_{n-1}, 0) : (u_1, \cdots, u_m) \in U_\alpha, (\overline{v}_1, \cdots, \overline{v}_{n-1}, 0) \in \overline{V}_\gamma\} \\
=&\bigcup_{\alpha,\gamma}\{F_\alpha(u_1, \cdots, u_m) \times G_\gamma(\overline{v}_1, \cdots, \overline{v}_{n-1}, 0)\} \\
=&\bigcup_{\alpha}\{F_\alpha(u_1, \cdots, u_m)\} \times \bigcup_{\gamma} G_\gamma(\overline{v}_1, \cdots, \overline{v}_{n-1}, 0)\} \\
=&M \times \partial N
\end{aligned}$$

2. Prove the following about orientability:

   (a) Show that the n-dimensional sphere $\mathbb{S}^n$ is orientable.

Solution: We regard n-dimensional sphere $\mathbb{S}^n$ as a hypersurface in $\mathbb{R}^{n+1}$. Clearly, the unit outward normal vector at point x is x itself which is continuous on the whole $\mathbb{S}^n$. Hence, $\mathbb{S}^n$ is orientable.

Alternatively, one can argue by observing that the stereographic atlas of $\mathbb{S}^n$ consists of only two parametrizations F_+ and F_- whose overlap is connected. Hence, one can always rearrange the variables of say F_- to guarantee that $\det D(\mathsf{F}_+^{-1} \circ \mathsf{F}_-) > 0$ on the overlap.

(b) Show that the Klein bottle defined in HW2 is not orientable.

Solution: Consider the two parametrizations of $\mathbb{R}^2/\sim$:

$$\begin{aligned} \mathsf{G}_\alpha : (0,1) \times (0,1) &\to \mathbb{R}^2/\sim \\ (x_\alpha, y_\alpha) &\mapsto [(x_\alpha, y_\alpha)] \end{aligned} \qquad \begin{aligned} \mathsf{G}_\beta : (0,1) \times (0.5,1.5) &\to \mathbb{R}^2/\sim \\ (x_\beta, y_\beta) &\mapsto [(x_\beta, y_\beta)] \end{aligned}$$

where the equivalence relation $\sim$ defined on $\mathbb{R}^2$:

$$(x,y) \sim (x',y') \iff (x',y') = \big((-1)^n x + m,\ y+n\big) \text{ for some integers } m \text{ and } n.$$

From HW 2, we have

$$\mathsf{G}_\beta^{-1} \circ \mathsf{G}_\alpha(x_\alpha, y_\alpha) = \begin{cases} (1 - x_\alpha, y_\alpha + 1) & \text{if } (x_\alpha, y_\alpha) \in (0,1) \times (0,0.5) \\ (x_\alpha, y_\alpha) & \text{if } (x_\alpha, y_\alpha) \in (0,1) \times (0.5,1) \end{cases}.$$

which implies

$$D(\mathsf{G}_\beta^{-1} \circ \mathsf{G}_\alpha) = \begin{cases} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } (x_\alpha, y_\alpha) \in (0,1) \times (0,0.5) \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } (x_\alpha, y_\alpha) \in (0,1) \times (0.5,1) \end{cases}$$

such that

$$\det\big(D(\mathsf{G}_\beta^{-1} \circ \mathsf{G}_\alpha)\big) = \begin{cases} -1 & \text{if } (x_\alpha, y_\alpha) \in (0,1) \times (0,0.5) \\ 1 & \text{if } (x_\alpha, y_\alpha) \in (0,1) \times (0.5,1) \end{cases}.$$

To complete the proof that the Klein bottle is non-orientable, we assume on the contrary that there exist a non-vanishing smooth 2-form ω globally defined on $\mathbb{R}^2/\sim$. In terms of local coordinates, we have

$$\begin{cases} \omega = \varphi_\alpha \, dx_\alpha \wedge dx_\alpha \\ \omega = \varphi_\beta \, dx_\beta \wedge dy_\beta \end{cases}$$

As the domain of G_α is connected, $\omega(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha})$ is either positive on the whole domain $(0,1) \times (0,1)$, or negative on the whole domain. Similar for $\omega(\frac{\partial}{\partial x_\beta}, \frac{\partial}{\partial y_\beta})$.

WLOG assume $\omega(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}) > 0$ and $\omega(\frac{\partial}{\partial x_\beta}, \frac{\partial}{\partial y_\beta}) < 0$ (the other cases are similar). Then, on the overlap of the two coordinate systems, we first have:

$$0 < \omega\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}\right) = \varphi_\alpha$$
$$0 > \omega\left(\frac{\partial}{\partial x_\beta}, \frac{\partial}{\partial y_\beta}\right) = \varphi_\beta$$

Moreover, by change-of-coordinates, we also have:

$$\varphi_\alpha = \varphi_\beta \det \frac{\partial(x_\beta, y_\beta)}{\partial(x_\alpha, y_\alpha)} \qquad \Longrightarrow \qquad \frac{\varphi_\alpha}{\varphi_\beta} = \det D(\mathsf{G}_\beta^{-1} \circ \mathsf{G}_\alpha),$$

showing that $\det D(\mathsf{G}_\beta^{-1} \circ \mathsf{G}_\alpha) < 0$ on **everywhere** of the overlap, which contradicts to our computations. Hence, the Klein bottle is not orientable.

(c) Show that the tangent bundle TM of any smooth manifold M must be orientable (no matter whether M is orientable or not).

Solution: Let $F(u_1, \cdots, u_n) : U \to O$ be a local parametrization of M, the induced local parametrization $\widetilde{F} : U \times \mathbb{R}^n \to TM$ of the tangent bundle TM is

$$\widetilde{F}((u_1, \cdots, u_n, a^1, \cdots, a^n) = \left(F(u_1, \cdots, u_n), a^1\frac{\partial}{\partial u_1} + \cdots + a^n\frac{\partial}{\partial u_n}\right) \in TM.$$

Let $G(u_1, \cdots, u_n) : U \to O$ be another local parametrization of M, the induced local parametrization $\widetilde{G} : U \times \mathbb{R}^n \to TM$ of the tangent bundle TM is

$$\widetilde{G}((v_1, \cdots, v_n, a^1, \cdots, a^n) = \left(G(v_1, \cdots, v_n), a^1\frac{\partial}{\partial v_1} + \cdots + a^n\frac{\partial}{\partial v_n}\right) \in TM.$$

Then,

$$\widetilde{G}^{-1} \circ \widetilde{F} = \left(G^{-1} \circ F(u_1, \cdots, u_n), a^j\frac{\partial v_1}{\partial u_j}, \cdots, a^j\frac{\partial v_n}{\partial u_j}\right)$$

implies

$$\det D(\widetilde{G}^{-1} \circ \widetilde{F}) = \det \begin{bmatrix} D(G^{-1} \circ F) & 0 \\ * & D(G^{-1} \circ F) \end{bmatrix}.$$

Hence, $\det D(\widetilde{G}^{-1} \circ \widetilde{F}) = \big(D(G^{-1} \circ F)\big)^2 > 0$ since $G^{-1} \circ F$ is invertible. Therefore, TM is orientable.

(d) A complex manifold M^{2n} is a smooth manifold equipped with an atlas $\{\mathsf{F}_\alpha : \mathcal{U}_\alpha \subset \mathbb{C}^n \to M^{2n}\}$ such that the transition functions are holomorphic, i.e. by writing $(u_1 + iv_1, \ldots, u_n + iv_n) = \mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha(x_1 + iy_1, \ldots, x_n + iy_n)$, each u_k and v_k are (real) differentiable functions of $(x_1, y_1, \ldots, x_n, y_n)$, and the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j} \qquad\qquad \frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j}$$

for any $k, j \in \{1, \ldots, n\}$. Show that any complex manifold must be orientable.

Solution: Let

$$A = \frac{\partial(u_1, \cdots, u_n)}{\partial(x_1, \cdots, x_n)} \text{ and } B = \frac{\partial(v_1, \cdots, v_n)}{\partial(x_1, \cdots, x_n)}.$$

By the Cauchy-Riemann equations, we have

$$D(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

Now we consider a matrix

$$P = \begin{bmatrix} I_n & iI_n \\ I_n & -iI_n \end{bmatrix} \text{ with } P^{-1} = \frac{1}{2}\begin{bmatrix} I_n & I_n \\ -iI_n & iI_n \end{bmatrix},$$

then

$$PD(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha)P^{-1} = \frac{1}{2}\begin{bmatrix} I_n & iI_n \\ I_n & -iI_n \end{bmatrix}\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} I_n & I_n \\ -iI_n & iI_n \end{bmatrix} = \begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix}.$$

Hence,

$$\begin{aligned} \det D(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) &= \det\left(PD(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha)P^{-1}\right) \\ &= \det \begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix} \\ &= \det(A+iB)\det(A-iB) \\ &= |\det(A+iB)|^2 \end{aligned}$$

On the other hand, by chain rule, we have

$$\begin{aligned} \det(A+iB) &= \det\left(\frac{\partial(u_1+iv_1, \cdots, u_n+iv_n)}{\partial(x_1, \cdots, x_n)}\right) \\ &= \det\left(\frac{\partial(u_1+iv_1, \cdots, u_n+iv_n)}{\partial(x_1+iy_1, \cdots, x_n+iy_n)}\right) \\ &= \det\left(D(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha)\right) \\ &\neq 0, \end{aligned}$$

since $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ is invertible. Thus, we obtain

$$\det D(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) = |\det(A+iB)|^2 > 0$$

which implies any complex manifold must be orientable.

3. Let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ be a 2-form on $\mathbb{R}^3$.

(a) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin. Compute directly the integral:

$$\int_{\mathbb{S}^2} \iota^*\omega$$

where $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map.

Solution: Let $F(\rho, \theta) = (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)$ be a parametrization of $\mathbb{S}^2$. Then we have

$$\begin{aligned}\iota^*\omega &= \iota^*\big(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy\big)\\ &= \sin\rho\,d\rho \wedge d\theta.\end{aligned}$$

Hence, we have

$$\int_{\mathbb{S}^2} \iota^*\omega = \int_0^{2\pi}\int_0^{\pi} \sin\rho\,d\rho d\theta = 4\pi.$$

(b) Let Σ be a compact, orientable, simply-connected regular surface in $\mathbb{R}^3$ without boundary, and $\iota : \Sigma \to \mathbb{R}^3$ be the inclusion map. Using generalized Stokes' Theorem, show that:

$$\frac{1}{3}\int_{\Sigma} \iota^*\omega$$

is equal to the volume of the solid D enclosed by Σ.

[Remark: You may assume without proof that such Σ must enclose a solid D, and that $\Sigma = \partial D$.]

Solution: We first compute

$$\begin{aligned}\omega &= x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy\\ d\omega &= dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy\\ &= 3\,dx \wedge dy \wedge dz\end{aligned}$$

By Stokes' Theorem, we obtain

$$\begin{aligned}\frac{1}{3}\int_{\Sigma=\partial D} \iota^*\omega &= \frac{1}{3}\int_D d\omega\\ &= \int_D dx \wedge dy \wedge dz\\ &= \text{Volume of D.}\end{aligned}$$

4. Let ω be the n-form on $\mathbb{R}^{n+1}\backslash\{0\}$ defined by:

$$\omega = \frac{1}{|\mathsf{x}|^{n+1}}\sum_{i=1}^{n+1}(-1)^{i-1}\,x_i\,dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$

where $\mathsf{x} = (x_1, \ldots, x_{n+1})$ and $|\mathsf{x}| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.

(a) Let $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the inclusion of the unit n-sphere $\mathbb{S}^n$. Show that $\displaystyle\int_{\mathbb{S}^n} \iota^*\omega \neq 0$.

Solution: First note that we cannot apply Stokes' Theorem on directly on $\iota^*\omega$ with $M = \{\mathsf{x} : |\mathsf{x}| \leq 1\}$ and $\partial M = \mathbb{S}^n$, since ω is not smooth at the origin. We proceed by direct computations using higher-dimensional spherical coordinates. Note that we only have to compute the integral of one term only, say:

$$\frac{1}{|\mathsf{x}|^{n+1}} x_{n+1}\, dx^1 \wedge \cdots \wedge dx^n$$

using the coordinates:

$$\begin{aligned}
x_1 &= \cos\phi_1 \\
x_2 &= \sin\phi_1 \cos\phi_2 \\
x_3 &= \sin\phi_2 \cos\phi_3 \\
&\;\vdots \\
x_n &= \sin\phi_1 \cdots \sin\phi_{n-1} \cos\phi_n \\
x_{n+1} &= \sin\phi_1 \cdots \sin\phi_{n-1} \sin\phi_n
\end{aligned}$$

where $\phi_1, \ldots, \phi_{n-1} \in (0, \pi)$ and $\phi_n \in (0, 2\pi)$. By direct computations, one gets:

$$\begin{aligned}
&\iota^*\left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n+1}\, dx^1 \wedge \cdots \wedge dx^n\right) \\
&= \underbrace{\sin\phi_1 \cdots \sin\phi_{n-1} \sin\phi_n}_{x_{n+1}} (-\sin\phi_1\, d\phi_1) \wedge (-\sin\phi_1 \sin\phi_2\, d\phi_2) \wedge \cdots \\
&\qquad \cdots \wedge (-\sin\phi_1 \cdots \sin\phi_{n-1} \sin\phi_n) \\
&= (-1)^n \sin^?\phi_1 \sin^?\phi_2 \cdots \sin^?\phi_{n-1} \sin^2\phi_n.
\end{aligned}$$

Note that many terms were gone after taking wedge products, and the above is the only term survived. We denote ? above as a positive integer which we do not care its exact value. Integrating over $\mathbb{S}^n$, we get:

$$\begin{aligned}
&\int_{\mathbb{S}^n} \iota^*\left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n+1}\, dx^1 \wedge \cdots \wedge \cdots \wedge dx^n\right) \\
&= \int_0^{2\pi}\int_0^{\pi} \cdots \int_0^{\pi} (-1)^n \sin^?\phi_1 \sin^?\phi_2 \cdots \sin^?\phi_{n-1} \sin^2\phi_n\, d\phi_1\, d\phi_2 \cdots\, d\phi_n
\end{aligned}$$

Since $\sin\phi_1, \ldots, \sin\phi_{n-1} > 0$ on $(0, \pi)$, and $\displaystyle\int_0^{2\pi} \sin^2\phi_n\, d\phi_n > 0$, we conclude that the above integral is non-zero.

Note that the unit sphere is reflectional symmetric, hence is invariant under the transformation $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ taking $x_{n+1} \mapsto x_n \mapsto x_{n-1} \to \cdots \mapsto x_1 \mapsto x_{n+1}$.

Note that $\det[\Phi_*] = (-1)^n$, hence

$$\begin{aligned}
&\int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n+1}\, dx^1 \wedge \cdots \wedge dx^n \right) \\
&= (-1)^n \int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_n\, dx^{n+1} \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \right) \\
&= \underbrace{(-1)^n(-1)^{1(n-1)}}_{=-1} \int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_n\, dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dx^{n+1} \right)
\end{aligned}$$

Applying the transformation Φ inductively, one can conclude that

$$\begin{aligned}
&\int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n+1}\, dx^1 \wedge \cdots \wedge dx^n \right) \\
&= (-1)^1 \int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_n\, dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dx^{n+1} \right) \\
&= (-1)^2 \int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n-1}\, dx^1 \wedge \cdots \wedge dx^{n-2} \wedge dx^n \wedge dx^{n+1} \right) \\
&= (-1)^3 \int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n-2}\, dx^1 \wedge \cdots \wedge dx^{n-3} \wedge dx^{n-1} \wedge \cdots dx^{n=1} \right) \\
&= \cdots
\end{aligned}$$

As a result, we have:

$$\int_{\mathbb{S}^n} \iota^*\omega = (n+1) \int_{\mathbb{S}^n} \iota^* \left(\frac{1}{|\mathsf{x}|^{n+1}} x_{n+1}\, dx^1 \wedge \cdots \wedge dx^n \right) \neq 0.$$

A more elegant approach is the following (legal but a bit cheating):
Define the following n-form on $\mathbb{R}^{n+1}$:

$$\eta = \sum_{i=1}^{n+1} (-1)^{i-1}\, x_i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}.$$

Although $\eta \neq \omega$, but $\iota^*\eta = \iota^*\omega$ since $|\mathsf{x}| = 1$ on $\mathbb{S}^n$.
Note that η is smooth everywhere on $\mathbb{R}^{n+1}$. Applying Stokes' Theorem on η over the ball $\{|\mathsf{x}| \leq 1\}$, we get:

$$\int_{\mathbb{S}^n} \iota^*\omega = \int_{\mathbb{S}^n} \iota^*\eta = \int_{\{|\mathsf{x}| \leq 1\}} d\eta.$$

Note that $d\eta = (n+1)\, dx^1 \cdots dx^{n+1}$, we get:

$$\int_{\{|\mathsf{x}| \leq 1\}} d\eta = \pm(n+1) \times \text{volume of the unit ball in } \mathbb{R}^{n+1} \neq 0.$$

(b) Hence, show that ω is closed but is not exact on $\mathbb{R}^{n+1}\backslash\{0\}$.

Solution: By direct computations, we have:

$$\begin{aligned} d\omega &= \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}\left(\frac{(-1)^{i-1}x_i}{|\mathsf{x}|^{n+1}}\right) dx^i \wedge dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1} \\ &= \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|\mathsf{x}|^{n+1}}\right) dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^i \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}. \end{aligned}$$

To show $d\omega = 0$, it suffices to show:

$$\sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|\mathsf{x}|^{n+1}}\right) = 0,$$

which is straight-forward (hence omitted). Hence ω is closed.

However, it cannot be exact. Suppose otherwise $\omega = d\alpha$ for some $(n-1)$-form on $\mathbb{R}^{n+1}\backslash\{0\}$. Applying Stokes' Theorem on $\iota^*\omega$ over $\mathbb{S}^n$ (which is without boundary), one would get:

$$\int_{\mathbb{S}^n} \iota^*\omega = \int_{\mathbb{S}^n} \iota^* d\alpha = \int_{\mathbb{S}^n} d\iota^*\alpha = 0.$$

It contradicts to the result from (a).

5. On a smooth manifold M, a smooth positive-definite symmetric $(2,0)$-tensor g is called a *Riemannian metric* on M. Using partitions of unity, show that every smooth manifold has at least one Riemannian metric.

Solution: Let $\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha\}$ be an atlas of M, and let $\{\rho_\alpha : \mathcal{U}_\alpha \to [0,1]\}$ be a partitions of unity subordinate to $\mathcal{A}$.

Since each $\mathcal{U}_\alpha$ is an open set of $\mathbb{R}^n$, locally there is a dot product δ_α defined as:

$$\delta_\alpha = \sum_{j=1}^{n} du_\alpha^i \otimes du_\alpha^i$$

where $(u_\alpha^1, \ldots, u_\alpha^n)$ are local coordinates of the chart $\mathsf{F}_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha$. Then, the following is a Riemannian metric on M:

$$g := \sum_\alpha \rho_\alpha \delta_\alpha.$$

It is clearly symmetric and smooth. To show that it is positive-definite, we consider an arbitrary vector $X = \sum_i X^i_\beta \frac{\partial}{\partial u^i_\beta}$ expressed in terms of local coordinates of F_β, then:

$$\begin{aligned}
g(X,X) &= \sum_\alpha \sum_{j=1}^n \rho_\alpha \, du^j_\alpha \otimes du^j_\alpha \left(\sum_{i=1}^n X^i_\beta \frac{\partial}{\partial u^i_\beta}, \sum_{k=1}^n X^k_\beta \frac{\partial}{\partial u^k_\beta} \right) \\
&= \sum_\alpha \sum_{j,p,q} \rho_\alpha \frac{\partial u^j_\alpha}{\partial u^p_\beta} \frac{\partial u^j_\alpha}{\partial u^q_\beta} \, du^p_\beta \otimes du^q_\beta \left(\sum_{i=1}^n X^i_\beta \frac{\partial}{\partial u^i_\beta}, \sum_{k=1}^n X^k_\beta \frac{\partial}{\partial u^k_\beta} \right) \\
&= \sum_\alpha \sum_{i,j,k,p,q} \rho_\alpha \frac{\partial u^j_\alpha}{\partial u^p_\beta} \frac{\partial u^j_\alpha}{\partial u^q_\beta} X^i_\beta X^k_\beta \delta_{ip} \delta_{qk} \\
&= \sum_\alpha \sum_{i,j,k} \rho_\alpha \frac{\partial u^j_\alpha}{\partial u^i_\beta} \frac{\partial u^j_\alpha}{\partial u^k_\beta} X^i_\beta X^k_\beta \\
&= \sum_\alpha \rho_\alpha \sum_j \left(\sum_i X^i_\beta \frac{\partial u^j_\alpha}{\partial u^i_\beta} \right)^2 \geq 0
\end{aligned}$$

Moreover, if $X \neq 0$ at a point $p \in M$, then for any α such that $p \in \mathcal{O}_\alpha \cap \mathcal{O}_\beta$, we have:

$$\sum_j \left(\sum_i X^i_\beta \frac{\partial u^j_\alpha}{\partial u^i_\beta} \right)^2 \neq 0$$

at p since the Jacobian $D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta)$ is invertible. Moreover, there must exists at least one α such that $\rho_\alpha(p) \neq 0$, so $g(X,X) > 0$ at p from the above result. Since p and $X \in T_pM$ is arbitrary, we conclude that g is positive-definite.