$\Lambda^0T^*M \xrightarrow{d} \Lambda^1T^*M \xrightarrow{d} \Lambda^2T^*M \longrightarrow \cdots$

$f \longmapsto df = \sum_i \frac{\partial f}{\partial u_i} du^i$

If $d\omega = 0$, then $\omega$ is closed.
If $\omega = d\eta$, then $\omega$ is exact.

$d^2 = 0 \rightsquigarrow$ exact $\Rightarrow$ closed.

But closed $\not\Rightarrow$ exact

$\mathbb{R}^2\setminus\{0\}$. $\quad \omega = -\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$ closed. $\leftarrow$ computation

$\int_{\bigcirc} \iota^*\omega = 2\pi \neq 0. \rightsquigarrow$ not exact (Stokes')

$d(\omega + df) = \underbrace{d\omega}_{0} + \underbrace{d^2 f}_{0} = 0. \Rightarrow \omega + df$ is closed.

If $\omega + df = d\phi \Rightarrow \omega = d(\phi - f)$ contradiction!
$\Rightarrow \omega + df$ is not exact

WANT: Regard $\omega$, $\omega + df$ to be the "same" when counting.

Define: $\omega \sim \tilde\omega \overset{def}{\iff} \omega - \tilde\omega = d\eta \quad \exists \eta$.

$[\omega] := \{\tilde\omega : \omega \sim \tilde\omega\} = \{\omega + d\eta : \eta \in \Lambda^{k-1}\}$. ($\uparrow \Lambda^k$)

Fact: If $\omega \sim \tilde\omega$ then $[\omega] = [\tilde\omega]$.

$[\omega]$ cohomology class. represented by $\omega$.

$\{[\omega] : \omega$ k-form, $d\omega = 0\}$.

$= \{\omega \in \Lambda^k : d\omega = 0\} \big/ \{d\eta : \eta \in \Lambda^{k-1}\} = H^k_{dR}(M)$

$d^2 = 0 \Rightarrow \Lambda^{k-1} \xrightarrow{d_{k-1}} \Lambda^k \xrightarrow{d_k} \Lambda^{k+1}$

$d_k \circ d_{k-1} = 0.$

$\omega = d_{k-1}\eta \Rightarrow d\omega = 0$

{exact k-form} $\subset$ {closed k-form}.

$H^k_{dR}(M) = $ {closed k-form} / {exact k-form}

$= \{[\omega] : \omega \in \Lambda^k, d\omega = 0\}$.

topological invariant.

$H^k_{dR}(M) = $ k-th deRham cohomology group ($k \geq 1$)

If $M \xrightarrow{\Phi} N$ diffeomorphic, then $H^k_{dR}(M) \overset{iso}{\cong} H^k_{dR}(N)$ for $k \geq 1$.

$\Phi^* : \Lambda^k N \to \Lambda^k M.$
$\omega \longmapsto \Phi^*\omega$

$f : \mathbb{Q} \to \cancel{\mathbb{Z}}$, $f(\frac{p}{q}) = p$. $\quad f(\frac{1}{2}) = 1$, $f(\frac{2}{4}) = 2$

Define: $\Phi^* : H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$
$[\omega] \longmapsto [\Phi^*\omega]$

$\Phi^*[\omega] := [\Phi^*\omega] \longleftarrow$ To justify this "def":

If $[\omega] = [\hat\omega]$, check $[\Phi^*\omega] = [\Phi^*\hat\omega]$

If $[\omega] = [\hat\omega]$ ($\Rightarrow \omega \sim \hat\omega$), then $\omega - \hat\omega = d\eta$.

$\Phi^*\omega - \Phi^*\hat\omega = \Phi^*(\omega - \hat\omega) = \Phi^* d\eta$
$= d(\Phi^*\eta)$

$\Rightarrow \Phi^*\omega \sim \Phi^*\hat\omega \Rightarrow [\Phi^*\omega] = [\Phi^*\hat\omega]$

Claim: $\Phi^* : H^k_{dR}(N) \to H^k_{dR}(M)$ is an isomorphism.
($\Phi : M \to N$ is diffeomorphism).

Proof $\Phi \circ \Phi^{-1} = id$. $\quad \Phi^{-1} \circ \Phi = id$.

$\Phi : M \to N$, $\quad \Phi^* : H^k_{dR}(N) \to H^k_{dR}(M)$
$\Phi^{-1} : N \to M$ $\quad (\Phi^{-1})^* : H^k_{dR}(M) \to H^k_{dR}(N)$

$(\Phi \circ \Phi^{-1})^* = id^*$ $\qquad \Phi^{-1} \circ \Phi = id \Rightarrow (\Phi^{-1} \circ \Phi)^* = id^* = id.$

$(\Phi^{-1})^* \circ \Phi^* = id$ $\longleftrightarrow$ $\Phi^* \circ (\Phi^{-1})^*$

$\Phi^*$, $(\Phi^{-1})^*$ are invertible.

$\therefore \Phi^*$ is an isomorphism between $H^k_{dR}(N)$ and $H^k_{dR}(M)$.

{exact forms} = {closed forms}

Poincaré lemma:
If $\mathcal{U} \subset \mathbb{R}^n$ (star-shaped), then $H^k_{dR}(\mathcal{U}) = \{[0]\}$ $\forall k \geq 1$.


star-shaped.

Proof: ($k = 1$).
Given $\omega = \sum_i \omega_i du^i$, $d\omega = 0$ on $\mathcal{U}$.

Want: find $f : \mathcal{U} \to \mathbb{R}$ s.t. $\omega = df$. $\quad \uparrow -$P.E. $= +$K.E.

$H^k_{dR}(\text{disk}) = 0 \quad \forall k \geq 1.$

$f(\vec x) := \int_{L_{\vec x}} \omega = \int_{L_x} \sum_i \omega_i du^i$

$\vec p \xrightarrow{L_{\vec x}} \vec x \in \mathcal{U} \subset \mathbb{R}^n$. $\quad \vec r(t) = (1-t)\vec p + t\vec x$. $\quad \boxed{u_i = (1-t)p_i + tx_i}$

$= \int_0^1 \sum_j \omega_j((1-t)\vec p + t\vec x)(x_j - p_j)\,dt$

$\frac{\partial f}{\partial x_i} = \int_0^1 \frac{\partial}{\partial x_i}\left(\sum_j \omega_j((1-t)\vec p + t\vec x)\cdot(x_j - p_j)\right)dt.$

$= \int_0^1 \left(\sum_{j,k} \frac{\partial \omega_j}{\partial u_k}\frac{\partial u_k}{\partial x_i}(x_j - p_j) + \sum_j \omega_j((1-t)\vec p + t\vec x)\delta_{ij}\right)dt$

$= \int_0^1 \sum_{j,k} \frac{\partial \omega_j}{\partial u_k}\cdot t\,\delta_{ik}(x_j - p_j) + \omega_i((1-t)\vec p + t\vec x)\,dt$

$= \int_0^1 \sum_j \frac{\partial \omega_j}{\partial u_i} t(x_j - p_j) + \omega_i((1-t)\vec p + t\vec x)\,dt$

$= \int_0^1 \left(\sum_j \frac{\partial \omega_i}{\partial u_j}\cdot t(x_j - p_j)\right) + \omega_i((1-t)\vec p + t\vec x)\,dt$

$\omega = \sum_i \omega_i du^i$
$0 = d\omega = \sum_{i<j}\left(\frac{\partial \omega_i}{\partial u_j} - \frac{\partial \omega_j}{\partial u_i}\right)du^i \wedge du^j$

$\iff \frac{\partial \omega_i}{\partial u_j} = \frac{\partial \omega_j}{\partial u_i} \quad \forall i,j$

$t\frac{d}{dt}\omega_i((1-t)\vec p + t\vec x)$

$\omega_i - u_j - x_k,\ t$

$u_j = (1-t)p_j + tx_j$
$\frac{\partial u_j}{\partial t} = (x_j - p_j)$

$= \int_0^1 \left(t\frac{d}{dt}\omega_i((1-t)\vec p + t\vec x) + \omega_i((1-t)\vec p + t\vec x)\right)dt$

$= \left[t\,\omega_i((1-t)\vec p + t\vec x)\right]_0^1 - \cancel{\int_0^1 \omega_i((1-t)\vec p + t\vec x)\,dt} + \cancel{\int_0^1 \omega_i((1-t)\vec p + t\vec x)}$

$= \omega_i(\vec x)$

$\Rightarrow \frac{\partial f}{\partial x_i} = \omega_i \iff \omega = df.$

$[a:b:c] = [e:f:g] \overset{def}{\iff} \exists \lambda \neq 0$ s.t. $(a,b,c) = \lambda(e,f,g)$.

$[\omega] = [\hat\omega] \overset{def}{\iff} \omega - \hat\omega = d\eta \quad \exists \eta$.

$d\omega = 0.$ $\qquad [\omega] = [0] \iff \omega - 0 = d\eta \quad \exists \eta$
$\iff \omega$ is exact.

Poincaré lemma
$\mathcal{U} \subset \mathbb{R}^n$ star-shaped, then any closed k-form $\omega$ on $\mathcal{U}$ must be exact on $\mathcal{U}$ ($\omega = d\eta$) ($\eta$: k-1 form on $\mathcal{U}$)

$[\omega] = [d\eta] = [0] \rightarrow H^k_{dR}(\mathcal{U}) = \{[0]\}$. $k \geq 1$.


$d\omega = 0$ on M.
$\Rightarrow d\omega = 0$ on $F_\alpha(\mathcal{U}_\alpha)$
$\Rightarrow \omega = d\eta_\alpha$ on $F_\alpha(\mathcal{U}_\alpha)$ $\rightarrow \sum \frac{\partial \eta_\alpha}{\partial u_i} du^i = \omega$

$\eta = \sum_\alpha \rho_\alpha \eta_\alpha$

$d\eta = \sum_\alpha \frac{\partial}{\partial u_i}(\rho_\alpha \eta_\alpha)du^i = \sum_\alpha \left(\underbrace{\frac{\partial \rho_\alpha}{\partial u_i}\eta_\alpha}_{?} + \rho_\alpha \underbrace{\frac{\partial \eta_\alpha}{\partial u_i}du^i}_{\omega}\right)$

$=$

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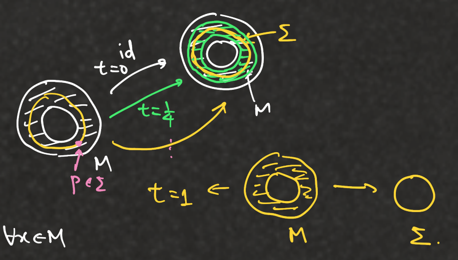
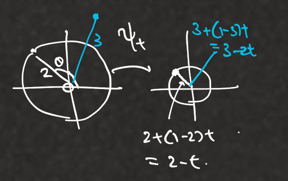
$\Sigma \subset M$

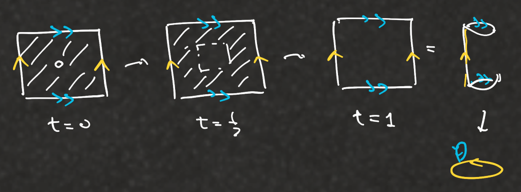



if $t \in [0,1]$
$\exists\, \Psi_t : M \to M$
① $\Psi_0 = id_M$
② $\Psi_1(x) \in \Sigma \quad \forall x \in M$
③ $\Psi_t(p) = p \quad \forall t \in [0,1], \forall p \in \Sigma$
$\overset{then}{\Longrightarrow}$ say $\Sigma$ is a deformation retract of M.

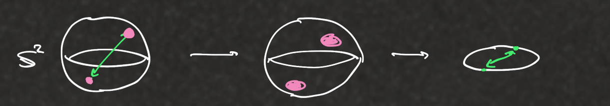
e.g. $M = \mathbb{R}^2\setminus\{0\}$. $\Sigma =$ unit circle

$\Psi_t : \mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$
$\Psi_t(re^{i\theta}) := (r + t(1-r))e^{i\theta}$

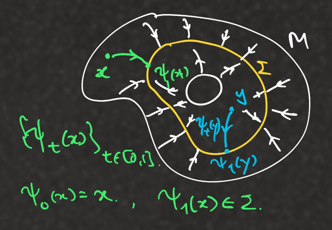


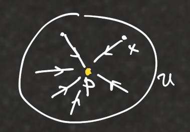

$\mathbb{RP}^2\setminus\{p\} \longrightarrow \mathbb{RP}^1 \cong S^1$.




Theorem: If $\Sigma$ is a deformation retract of M, then $H^k_{dR}(M) \cong H^k_{dR}(\Sigma) \quad \forall k \geq 1$.




$\Psi_t : M \to M$.
① $\Psi_0 = id_M$
② $\Psi_1 : M \to \Sigma$
③ $\Psi_t(p) = p \quad \forall p \in \Sigma,\ \forall t \in [0,1]$.



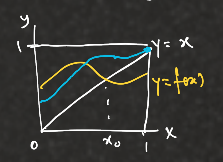

$\Psi_t(x) = (1-t)x + tp.$

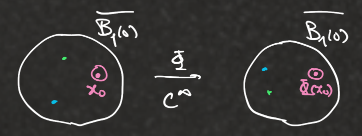
$f(x) := \int_{\{\Psi_t(x)\}} \omega$ (curve) $\qquad d\omega = 0$, $df = ?$

Key: $\Psi_1^* \circ \iota^*\omega - \omega = \pm df$ (closed 1-form) $\quad f(x) = \int_{\Psi_t(x)} \omega$.

$\Rightarrow \Psi_1^*\iota^*[\omega] = [\omega]$

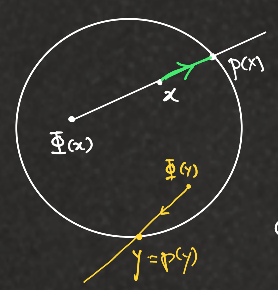
$f : [0,1] \to [0,1]$ continuous.
then $\exists x_0 \in [0,1]$ s.t. $f(x_0) = x_0$.




Brouwer's fixed point theorem
$\Rightarrow \exists x_0 \in \overline{B_1(0)}$ s.t. $\Phi(x_0) = x_0$

Proof: Assume $\Phi(x) \neq x \quad \forall x \in \overline{B_1(0)}$.




$\Psi_t(x) := (1-t)x + t\,p(x)$
① $\Psi_0(x) = x \quad \forall x \in \overline{B_1(0)}$
② $\Psi_1(x) = p(x) \quad \forall x \in \overline{B_1(0)}$, $\in \partial B_1(0)$
③ $\Psi_t(y) = y \quad \forall t \in [0,1]$, $\forall y \in \partial B_1(0)$

$\therefore \Psi_t$ is a deform. retraction from $\overline{B_1(0)}$ to $\partial B_1(0)$.

$\{[0]\} = H^1_{dR}(\overline{B_1(0)}) \underset{\text{deform. retract}}{=} H^1_{dR}(\bigcirc) = H^1_{dR}(\mathbb{R}^2\setminus\{0\}) \neq \{[0]\}$.

(disk) is star-shaped, Poincaré lemma

$\left[\frac{-y\,dx + x\,dy}{x^2+y^2}\right] \neq [0]$. closed, not exact

$\cdots \xrightarrow{T_{k-1}} G_{k-1} \xrightarrow{T_k} G_k \xrightarrow{T_{k+1}} G_{k+1} \xrightarrow{T_{k+2}} \cdots$

$Im(T_k) \subset G_k \qquad Ker(T_{k+1}) \subset G_k$
$= \{T_k(x) : x \in G_{k-1}\} \qquad = \{x \in G_k : T_{k+1}(x) = 0\}$

long exact sequence $\overset{def}{\iff} Im(T_k) = Ker(T_{k+1}) \quad \forall k$.

$0 \longmapsto 0$ $\qquad x \longmapsto 0 \quad \forall x \in G_3$.

$0 \xrightarrow{T_0} G_1 \xrightarrow{T_1} G_2 \xrightarrow{T_2} G_3 \xrightarrow{T_3} 0$

trivial group $\qquad$ trivial group

short exact sequence $\iff Im(T_k) = Ker(T_{k+1}) \quad \forall k$.

$Im(T_0) = Ker(T_1) \rightarrow \{0\} = Ker\,T_1 \iff T_1$ injective.

$Im(T_2) = Ker(T_3) \rightarrow Im(T_2) = G_3 \iff T_2$ surjective.

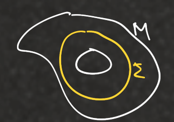
e.g.

$z \longmapsto e^{2\pi i z}$

$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{f} \mathbb{C}\setminus\{0\} \longrightarrow 0$

$n \longmapsto n + 0i$

$Im(\iota) = \mathbb{Z}$.
$Ker(f) = \{z \in \mathbb{C} : f(z) = 1\}$
$= \{z \in \mathbb{C} : e^{2\pi i z} = 1\}$
$= \mathbb{Z}$.




$H^k_{dR}(\Sigma) \underset{\iota^*}{\overset{\Psi_1^*}{\rightleftarrows}} H^k_{dR}(M)$

$\iota : \Sigma \to M$
$\iota^* : \Lambda^k T^*M \to \Lambda^k T^*\Sigma.$
$\iota^* : H^k_{dR}(M) \longrightarrow H^k_{dR}(\Sigma) \qquad \iota^*[\omega] = [\iota^*\omega].$

$\Psi_1 : M \to \Sigma$
$\Psi_1^* : H^k_{dR}(\Sigma) \to H^k_{dR}(M)$

$\Psi_1^* \circ \iota^* = (\iota \circ \Psi_1)^*$
$\iota^* \circ \Psi_1^* = (\Psi_1 \circ \iota)^* = id$

$\Psi_1 \circ \iota : \Sigma \to \Sigma$
$\Psi_1(\iota(p)) = p \Rightarrow \Psi_1 \circ \iota = id_\Sigma$