**MATH 4033 • Spring 2018 • Calculus on Manifolds**
**Problem Set #4 • Generalized Stokes' Theorem • Due Date: 06/05/2018, 11:59PM**

1. Given that $M$ is a smooth $m$-manifold *without* boundary, and $N$ is a smooth $n$-manifold *with* boundary. Show that the product manifold $M \times N$ is a smooth $(m+n)$-manifold *with* boundary, and that $\partial(M \times N) = M \times \partial N$.

2. Prove the following about orientability:
   (a) Show that the $n$-dimensional sphere $\mathbb{S}^n$ is orientable.
   (b) Show that the Klein bottle defined in HW2 is not orientable.
   (c) Show that the tangent bundle $TM$ of any smooth manifold $M$ must be orientable (no matter whether $M$ is orientable or not).
   (d) A complex manifold $M^{2n}$ is a smooth manifold equipped with an atlas $\{\mathsf{F}_\alpha : \mathcal{U}_\alpha \subset \mathbb{C}^n \to M^{2n}\}$ such that the transition functions are holomorphic, i.e. by writing $(u_1 + iv_1, \ldots, u_n + iv_n) = \mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha(x_1 + iy_1, \ldots, x_n + iy_n)$, each $u_k$ and $v_k$ are (real) differentiable functions of $(x_1, y_1, \ldots, x_n, y_n)$, and the Cauchy-Riemann equations are satisfied:
   $$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j} \qquad\qquad \frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j}$$
   for any $k, j \in \{1, \ldots, n\}$. Show that any complex manifold must be orientable.

3. Let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ be a 2-form on $\mathbb{R}^3$.
   (a) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin. Compute directly the integral:
   $$\int_{\mathbb{S}^2} \iota^*\omega$$
   where $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map.
   (b) Let $\Sigma$ be a compact, orientable, simply-connected regular surface in $\mathbb{R}^3$ without boundary, and $\iota : \Sigma \to \mathbb{R}^3$ be the inclusion map. Using generalized Stokes' Theorem, show that:
   $$\frac{1}{3}\int_{\Sigma} \iota^*\omega$$
   is equal to the volume of the solid $D$ enclosed by $\Sigma$.
   [Remark: You may assume without proof that such $\Sigma$ must enclose a solid $D$, and that $\Sigma = \partial D$.]

4. Let $\omega$ be the $n$-form on $\mathbb{R}^{n+1}\backslash\{0\}$ defined by:
   $$\omega = \frac{1}{|\mathsf{x}|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1}\, x_i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$
   where $\mathsf{x} = (x_1, \ldots, x_{n+1})$ and $|\mathsf{x}| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.
   (a) Let $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the inclusion of the unit $n$-sphere $\mathbb{S}^n$. Show that $\displaystyle\int_{\mathbb{S}^n} \iota^*\omega \neq 0$.
   (b) Hence, show that $\omega$ is closed but is not exact on $\mathbb{R}^{n+1}\backslash\{0\}$.

5. On a smooth manifold $M$, a smooth positive-definite symmetric $(2,0)$-tensor $g$ is called a *Riemannian metric* on $M$. Using partitions of unity, show that every smooth manifold has at least one Riemannian metric.