MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #3 • Tensors and Differential Forms • Due Date: 08/04/2018, 11:59PM

1. Show that the Lie derivative of a 1-form

$$\alpha = \sum_i \alpha_i \, du^i$$

along a vector field $X = \sum_j X^j \frac{\partial}{\partial u_j}$ is locally expressed as:

$$\mathcal{L}_X \alpha = \sum_{i,j} \left(X^i \frac{\partial \alpha_j}{\partial u_i} + \alpha_i \frac{\partial X^i}{\partial u_j} \right) du^j.$$

Solution: Denote Φ_t to be the flow map of X, then for any p on the manifold, we have

$$\begin{aligned}
(\mathcal{L}_X \alpha)_p &:= \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* \alpha_{\Phi_t(p)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \sum_i (\alpha_i \circ \Phi_t) d(u^i \circ \Phi_t) \qquad \text{(Note that } \Phi_* \text{ and } d \text{ commute)} \\
&= \sum_{i,j} \underbrace{\frac{\partial \alpha_i}{\partial u_j} X^j}_{\frac{d}{dt}(\alpha_i \circ \Phi_t)} + \alpha_i \left.\frac{d}{dt}\right|_{t=0} \frac{\partial (u_i \circ \Phi_t)}{\partial u_j} \, du^j \\
&= \sum_{i,j} \underbrace{\frac{\partial \alpha_i}{\partial u_j} X^j}_{\frac{d}{dt}(\alpha_i \circ \Phi_t)} + \alpha_i \frac{\partial}{\partial u_j} \left.\frac{d}{dt}\right|_{t=0} (u_i \circ \Phi_t) \, du^j
\end{aligned}$$

The proof is completed by showing

$$\left.\frac{d}{dt}\right|_{t=0} (u_i \circ \Phi_t) = \sum_k \frac{\partial u_i}{\partial u_k} X^k = \sum_k \delta_{ik} X^K = X^i.$$

2. Consider a smooth manifold M^n and the following symmetric $(2,0)$-tensors on M:

$$g = \sum_{i,j} g_{ij} \, du^i \otimes du^j \qquad\qquad h = \sum_{i,j} h_{ij} \, du^i \otimes du^j$$

where $(u_1, \ldots, u_n)$ are local coordinates of M. Suppose the matrix $[g_{ij}]$ is positive definite (all its eigenvalues are positive).

(a) Let $(v_1, \ldots, v_n)$ be another local coordinates system, and the local expression of g in terms of this system be:

$$g = \sum_{i,j} \widetilde{g}_{ij} \, dv^i \otimes dv^j.$$

Express $\widetilde{g}_{ij}$ in terms of g_{ij}, and then show that the matrix $[\widetilde{g}_{ij}]$ is also symmetric and positive definite.

Solution: By chain rule, we have $du^i = \sum_\alpha \frac{\partial u_i}{\partial v_\alpha} dv^\alpha$, so

$$\begin{aligned} g &= \sum_{i,j} g_{ij}\, du^i \otimes du^j \\ &= \sum_{i,j} g_{ij} \left(\sum_\alpha \frac{\partial u_i}{\partial v_\alpha} dv^\alpha \right) \otimes \left(\sum_\beta \frac{\partial u_j}{\partial v_\beta} dv^\beta \right) \\ &= \sum_{i,j,\alpha,\beta} g_{ij} \frac{\partial u_i}{\partial v_\alpha} \frac{\partial u_j}{\partial v_\beta}\, dv^\alpha \otimes dv^\beta. \end{aligned}$$

Thus, we have

$$\widetilde{g}_{ij} = \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial u_\alpha}{\partial v_i} \frac{\partial u_\beta}{\partial v_j}. \tag{1}$$

Hence $[\widetilde{g}] = A^T[g]A$ where $A = \frac{\partial(u_\beta)}{\partial(v_j)}$. Clearly, $[\widetilde{g}]^T = A^T[g]^T(A^T)^T = A^T[g]A = [\widetilde{g}]$, so $[\widetilde{g}]$ is also symmetric. Also, $[g]$ and $[\widetilde{g}]$ are similar matrices, thus $[g]$ is positive definite if and only if $[\widetilde{g}]$ is positive definite.

(b) Show that the quantity $\dfrac{\det[h_{ij}]}{\det[g_{ij}]}$ is independent of local coordinates, i.e.

$$\frac{\det[h_{ij}]}{\det[g_{ij}]} = \frac{\det[\widetilde{h}_{ij}]}{\det[\widetilde{g}_{ij}]}.$$

Solution: From (a), we have $[\widetilde{g}] = A^T[g]A$ where $A = \frac{\partial(u_\beta)}{\partial(v_j)}$. Similarly, we can show $[\widetilde{h}] = A^T[h]A$. Note that A depends only on transition maps but not the tensors (i.e. the same A for both g and h). Hence,

$$\frac{\det[\widetilde{h}]}{\det[\widetilde{g}]} = \frac{\det(A^T)\det[h]\det(A)}{\det(A^T)\det[g]\det(A)} = \frac{\det[h]}{\det[g]}.$$

3. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

(a) Express ω in spherical coordinates (ρ, θ, φ) defined by:

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}$$

Solution: By direct computation,

$$\begin{aligned}
dx &= \sin\varphi\cos\theta\,d\rho + \rho\cos\varphi\cos\theta\,d\varphi - \rho\sin\varphi\sin\theta\,d\theta \\
dy &= \sin\varphi\sin\theta\,d\rho + \rho\cos\varphi\sin\theta\,d\varphi + \rho\sin\varphi\cos\theta\,d\theta \\
dz &= \cos\varphi\,d\rho - \rho\sin\varphi\,d\varphi \\
dy\wedge dz &= -\rho\sin\theta\,d\rho\wedge d\varphi + \rho^2\sin^2\varphi\cos\theta\,d\varphi\wedge d\theta + \rho\sin\varphi\cos\varphi\cos\theta\,d\theta\wedge d\rho \\
dz\wedge dx &= \rho\cos\theta\,d\rho\wedge d\varphi + \rho^2\sin^2\varphi\sin\theta\,d\varphi\wedge d\theta + \rho\sin\varphi\cos\varphi\sin\theta\,d\theta\wedge d\rho \\
dx\wedge dy &= \rho^2\sin\varphi\cos\varphi\,d\varphi\wedge d\theta - \rho\sin^2\varphi\,d\theta\wedge d\rho.
\end{aligned}$$

Hence, we have

$$\omega = -\rho^3\sin\varphi\,d\theta\wedge d\varphi.$$

(b) Compute $d\omega$ in both rectangular and spherical coordinates.

Solution: In rectangular coordinates,

$$d\omega = dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy = 3\,dx\wedge dy\wedge dz.$$

In spherical coordinates,

$$d\omega = -\frac{\partial(\rho^3\sin\varphi)}{\partial\rho}\,d\rho\wedge d\theta\wedge d\varphi == -3\rho^2\sin\varphi\,d\rho\wedge d\theta\wedge d\varphi.$$

(c) Let $\mathbb{S}^2$ be the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$. Consider the inclusion map $\iota:\mathbb{S}^2\to\mathbb{R}^3$. Compute the pull-back $\iota^*\omega$, and express it in terms of spherical coordinates.

Solution: We have $\rho=1$, $\iota^*(d\theta)=d\theta$ and $\iota^*(d\varphi)=d\varphi$. Hence,

$$\begin{aligned}
\iota^*\omega &= -\rho^3\sin\varphi\left(\iota^*(d\theta)\wedge\iota^*(d\varphi)\right) \\
&= -\sin\varphi\,d\theta\wedge d\varphi.
\end{aligned}$$

(d) Let $\delta = dx\otimes dx + dy\otimes dy + dz\otimes dz$ be a symmetric $(2,0)$-tensor on $\mathbb{R}^3$. Compute $\iota^*\delta$, and express it in terms of spherical coordinates.

Solution: Since $\rho=1$, we have

$$\begin{aligned}
\iota^*(dx) &= d(\iota^*x) = d(x\circ\iota) = dx = \cos\varphi\cos\theta\,d\varphi - \sin\varphi\sin\theta\,d\theta \\
\iota^*(dy) &= d(\iota^*y) = d(y\circ\iota) = dy = \cos\varphi\sin\theta\,d\varphi + \sin\varphi\cos\theta\,d\theta \\
\iota^*(dz) &= d(\iota^*z) = d(z\circ\iota) = dz = -\sin\varphi\,d\varphi.
\end{aligned}$$

Hence,

$$\begin{aligned}
\iota^*\delta &= \iota^*(dx)\otimes\iota^*(dx)+\iota^*(dy)\otimes\iota^*(dy)+\iota^*(dz)\otimes\iota^*(dz)\\
&= \big(\cos\varphi\cos\theta\big)^2\,d\varphi\otimes d\varphi+\big(\sin\varphi\sin\theta\big)^2\,d\theta\otimes d\theta\\
&\quad-\big(\cos\varphi\cos\theta\big)\big(\sin\varphi\sin\theta\big)\,d\theta\otimes d\varphi\\
&\quad+\big(\cos\varphi\sin\theta\big)^2\,d\varphi\otimes d\varphi+\big(\sin\varphi\cos\theta\big)^2\,d\theta\otimes d\theta\\
&\quad+\big(\cos\varphi\sin\theta\big)\big(\sin\varphi\cos\theta\big)\,d\theta\otimes d\varphi\\
&\quad+\sin^2\varphi\,d\varphi\otimes d\varphi\\
&= \sin^2\varphi\,d\theta\otimes d\theta+d\varphi\otimes d\varphi.
\end{aligned}$$

4. The purpose of this exercise is to show that any closed 1-form ω on $\mathbb{R}^3$ must be exact. Let

$$\omega = P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz$$

be a closed 1-form on $\mathbb{R}^3$. Define $f:\mathbb{R}^3\to\mathbb{R}$ by:

$$f(x,y,z)=\int_{t=0}^{t=1}\left(xP(tx,ty,tz)+yQ(tx,ty,tz)+zR(tx,ty,tz)\right)\,dt.$$

(a) Show that $df=\omega$.

Solution: Taking the exterior derivative of ω, we have

$$d\omega=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dy\wedge dz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dz\wedge dx.$$

Since ω is closed, we have $d\omega=0$ which implies

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y},\quad \frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}$$

Then by using chain rule, we compute that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \int_{t=0}^{t=1}\left(P+x\frac{\partial P}{\partial x}t+y\frac{\partial Q}{\partial x}t+z\frac{\partial R}{\partial x}t\right)dt\\
&= \int_{t=0}^{t=1}\left(P+x\frac{\partial P}{\partial x}t+y\frac{\partial P}{\partial y}t+z\frac{\partial P}{\partial z}t\right)dt\\
&= \int_{t=0}^{t=1}\left(\frac{d}{dt}tP(tx,ty,tz)\right)dt\\
&= tP(tx,ty,tz)\Big|_{t=0}^{t=1}\\
&= P(x,y,z)
\end{aligned}$$

Similarly, we have

$$\frac{\partial f}{\partial y}=Q(x,y,z),\quad \frac{\partial f}{\partial z}=R(x,y,z)$$

Hence,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=Pdx+Qdy+Rdz=\omega$$

(b) Point out exactly where you have used the fact that $d\omega = 0$ in your solution to (a).

Solution: In (a), we have used the fact that $d\omega = 0$ **everywhere** on $\mathbb{R}^3$ to derive the equalities

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad ,$$

which are then used to convert the integral:

$$\int_{t=0}^{t=1} \left(P + x\frac{\partial P}{\partial x}t + y\frac{\partial Q}{\partial x}t + z\frac{\partial R}{\partial x}t \right) dt = \int_{t=0}^{t=1} \left(P + x\frac{\partial P}{\partial x}t + y\frac{\partial P}{\partial y}t + z\frac{\partial P}{\partial z}t \right) dt$$

(c) Explain why your solution to (a) would fail if the domain of ω is $\mathbb{R}^3 \backslash \{p\}$ (where p is a fixed point in $\mathbb{R}^3$).

Solution: If the domain of ω is $\mathbb{R} \setminus \{p\}$, then either P, Q, R is undefined at p. Hence it does not make sense to talk about

$$\left.\frac{\partial Q}{\partial x}\right|_p = \left.\frac{\partial P}{\partial y}\right|_p, \quad \left.\frac{\partial R}{\partial y}\right|_p = \left.\frac{\partial Q}{\partial z}\right|_p, \quad \left.\frac{\partial P}{\partial z}\right|_p = \left.\frac{\partial R}{\partial x}\right|_p.$$

If the straight line joining $(0,0,0)$ to (x,y,z) passes through this point p, then the integrand in

$$\int_{t=0}^{t=1} \left(P + x\frac{\partial P}{\partial x}t + y\frac{\partial Q}{\partial x}t + z\frac{\partial R}{\partial x}t \right) dt$$

is not continuous. It may not be legitimate to switch $\dfrac{\partial}{\partial x}$ and the integral sign.

5. Consider $\mathbb{R}^4$ with coordinates (t,x,y,z), which is also denoted as (x_0, x_1, x_2, x_3) in this problem. Denote $*$ to be the Minkowski Hodge-star operator on $\mathbb{R}^4$ (see P.104).

(a) Compute each of the following:

$$\begin{array}{ccc} *(dt \wedge dx) & *(dt \wedge dy) & *(dt \wedge dz) \\ *(dx \wedge dy) & *(dy \wedge dz) & *(dz \wedge dx) \end{array}$$

Solution: By direct computation, we have

$$\begin{array}{ccc} *(dt \wedge dx) = -dy \wedge dz & *(dt \wedge dy) = -dz \wedge dx & *(dt \wedge dz) = -dx \wedge dy \\ *(dx \wedge dy) = dt \wedge dz & *(dy \wedge dz) = dt \wedge dx & *(dz \wedge dx) = dt \wedge dy \end{array}$$

(b) The four Maxwell's equations are a set of partial differential equations that form the foundation of electromagnetism. Denote the components of the electric field E, magnetic field B, and current density J by

$$\begin{aligned} \mathsf{E} &= E_x\mathsf{i} + E_y\mathsf{j} + E_z\mathsf{k} \\ \mathsf{B} &= B_x\mathsf{i} + B_y\mathsf{j} + B_z\mathsf{k} \\ \mathsf{J} &= j_x\mathsf{i} + j_y\mathsf{j} + j_z\mathsf{k} \end{aligned}$$

All components of E, B and J are considered to be time-dependent. Denote ρ to be the charge density. The four Maxwell's equations assert that:

$$\nabla \cdot \mathsf{E} = \rho \qquad \nabla \cdot \mathsf{B} = 0$$
$$\nabla \times \mathsf{E} = -\frac{\partial \mathsf{B}}{\partial t} \qquad \nabla \times \mathsf{B} = \mathsf{J} + \frac{\partial \mathsf{E}}{\partial t}$$

We are going to convert the Maxwell's equations using the language of differential forms. We define the following analogue of E, B, J and ρ using differential forms:

$$\begin{aligned}
E &= E_x\,dx + E_y\,dy + E_z\,dz \\
B &= B_x\,dy \wedge dz + B_y\,dz \wedge dx + B_z\,dx \wedge dy \\
J &= (j_x\,dy \wedge dz + j_y\,dz \wedge dx + j_z\,dx \wedge dy) \wedge dt + \rho\,dx \wedge dy \wedge dz
\end{aligned}$$

Define the 2-form F by $F := B + E \wedge dt$. Show that the four Maxwell's equations can be rewritten in an elegant way as:

$$dF = 0 \qquad d(*F) = J.$$

Solution: The four Maxwell's equations can be written as

$$\nabla \cdot E = \rho \iff \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho \quad , \tag{2}$$

$$\nabla \cdot B = 0 \iff \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad , \tag{3}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \iff \begin{cases} \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \end{cases} , \tag{4}$$

$$\nabla \times B = J + \frac{\partial B}{\partial t} \iff \begin{cases} \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = j_z + \frac{\partial E_z}{\partial t} \\ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = j_x + \frac{\partial E_x}{\partial t} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = j_y + \frac{\partial E_y}{\partial t} \end{cases} . \tag{5}$$

We now compute

$$\begin{aligned}
dF &= dB + dE \wedge dt \\
&= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dx \wedge dy \wedge dz \\
&\quad + \frac{\partial B_x}{\partial t} dt \wedge dy \wedge dz + \frac{\partial B_y}{\partial t} dt \wedge dz \wedge dx + \frac{\partial B_z}{\partial t} dt \wedge dx \wedge dy \\
&\quad + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dx \wedge dy \wedge dt + \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) dy \wedge dz \wedge dt \\
&\quad + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) dz \wedge dx \wedge dt \\
&= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dx \wedge dy \wedge dz + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t} \right) dx \wedge dy \wedge dt \\
&\quad + \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} \right) dy \wedge dz \wedge dt + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t} \right) dz \wedge dx \wedge dt \\
&= 0\,,
\end{aligned}$$

where in the last step we have applied (2) and (3). Next, we compute that

$$\begin{aligned} *F &= *B + *(E \wedge dt) \\ &= B_x dt \wedge dx + B_y dt \wedge dy + B_z dt \wedge dz + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \end{aligned}$$

Hence,

$$\begin{aligned} d(*F) &= \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) dt \wedge dx \wedge dy + \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right) dt \wedge dy \wedge dz \\ &\quad + \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) dt \wedge dz \wedge dx \\ &\quad + \frac{\partial E_x}{\partial t} dt \wedge dy \wedge dz + \frac{\partial E_y}{\partial t} dt \wedge dz \wedge dx + \frac{\partial E_z}{\partial t} dt \wedge dx \wedge dy \\ &\quad + \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx \wedge dy \wedge dz \\ &= \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} + \frac{\partial E_z}{\partial t}\right) dt \wedge dx \wedge dy + \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} + \frac{\partial E_x}{\partial t}\right) dt \wedge dy \wedge dz \\ &\quad + \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} + \frac{\partial E_y}{\partial t}\right) dt \wedge dz \wedge dx + \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx \wedge dy \wedge dz \\ &= -j_z dt \wedge dx \wedge dy - j_x dt \wedge dy \wedge dz - j_y dt \wedge dz \wedge dx + \rho\, dx \wedge dy \wedge dz \\ &= J \end{aligned}$$

where we have used (1) and (4) in the last step.