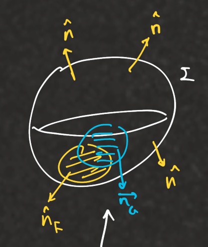


$\exists\ \hat{n} : \Sigma \to \mathbb{S}^2(1)$ smooth map s.t. $\hat{n}(p)$ is a unit normal to Σ at p.

$\overset{\text{def}}{\Longleftrightarrow}$ orientable surface.

$F(u_1, u_2) = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$

$$\hat{n}_F = \frac{\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}}{\left|\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right|}$$

$$\det\frac{\partial(y,z)}{\partial(u_1,u_2)}\hat{i} + \det\frac{\partial(z,x)}{\partial(u_1,u_2)}\hat{j} + \det\frac{\partial(x,y)}{\partial(u_1,u_2)}\hat{k}$$

$$\frac{\partial F}{\partial u_1} \to \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_{u_1} & y_{u_1} & z_{u_1} \\ x_{u_2} & y_{u_2} & z_{u_2} \end{vmatrix}$$

$G(v_1, v_2) = (x, y, z)$

$$\hat{n}_G = \frac{\det\frac{\partial(y,z)}{\partial(v_1,v_2)}\hat{i} + \cdots}{|\qquad|}$$

WANT: $\hat{n}_F = \hat{n}_G$

$$\frac{\partial(y,z)}{\partial(v_1,v_2)} = \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\frac{\partial(y,z)}{\partial(u_1,u_2)} \leftarrow \text{chain rule}$$

$$\Rightarrow \det\frac{\partial(y,z)}{\partial(v_1,v_2)} = \det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\det\frac{\partial(y,z)}{\partial(u_1,u_2)}$$

$$\hat{n}_G = \frac{\det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\left(\det\frac{\partial(y,z)}{\partial(u_1,u_2)}\hat{i} + \cdots\right)}{\left|\det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\left(\det\frac{\partial(y,z)}{\partial(u_1,u_2)}\hat{i} + \cdots\right)\right|} = $$

$$= \frac{\det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}}{\left|\det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\right|}\hat{n}_F$$

$\hat{n}_G = \hat{n}_F$

$\Leftrightarrow \det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)} > 0.$

$\Leftrightarrow \det D(F^{-1}\circ G) > 0.$

$G^{-1}\circ F(u_1,u_2) = (v_1, v_2)$

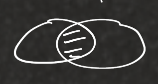
$F^{-1}\circ G(v_1, v_2) = (u_1, u_2)$

Definition

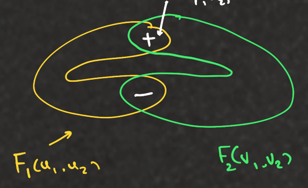
M is an orientable manifold

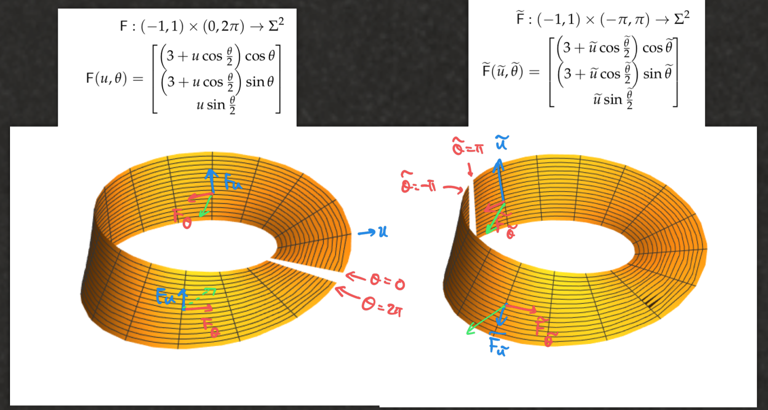
$\overset{\text{def}}{\Longleftrightarrow} \exists$ atlas $\{F_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha\}$ of M such that

$\det D(F_\alpha^{-1}\circ F_\beta) > 0 \quad \forall \alpha,\beta$ (overlapping).



$\det\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)} > 0$





$$\mathsf{F} : (-1,1)\times(0,2\pi) \to \Sigma^2$$

$$\mathsf{F}(u,\theta) = \begin{bmatrix} \left(3 + u\cos\frac{\theta}{2}\right)\cos\theta \\ \left(3 + u\cos\frac{\theta}{2}\right)\sin\theta \\ u\sin\frac{\theta}{2} \end{bmatrix}$$

$$\widetilde{\mathsf{F}} : (-1,1)\times(-\pi,\pi) \to \Sigma^2$$

$$\widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta}) = \begin{bmatrix} \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \\ \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \\ \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \end{bmatrix}$$

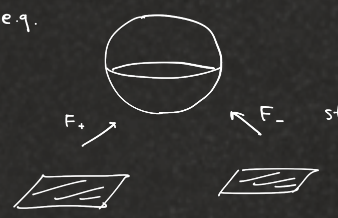
$$\tilde{F}^{-1}\circ F(u,\theta) = \begin{cases} (u, \theta) & \text{if } \theta \in (0,\pi) \\ (-u, \theta - 2\pi) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

$D(\tilde{F}^{-1}\circ F)$

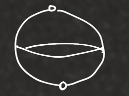
$$= \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } \theta \in (0,\pi) \\ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

$$\det D(\tilde{F}^{-1}\circ F) = \begin{cases} + & \text{if } \theta \in (0,\pi) \\ - & \text{if } \theta \in (\pi, 2\pi). \end{cases}$$

e.g.



overlap:



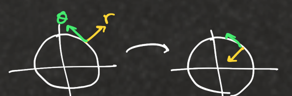
$(F_+^{-1}\circ F_-)(z) = \frac{1}{\bar{z}}$

$re^{i\theta} \mapsto \frac{1}{r}e^{i\theta}$

$\det D(F_+^{-1}\circ F_-) < 0$

$\tilde{F}_-(x,y) := \tilde{F}_-(y,x)$

$\det D(F_+^{-1}\circ \tilde{F}_-) > 0.$



$F(\theta, z) = (\cos\theta, \sin\theta, z) : (0,2\pi)\times\mathbb{R} \to$ Cylinder

$\tilde{F}(\tilde\theta, \tilde{z}) = (\cos\tilde\theta, \sin\tilde\theta, \tilde{z}) : (-\pi,\pi)\times\mathbb{R} \to$ Cylinder

$$\tilde{F}^{-1}\circ F(\theta, z) = \begin{cases} (\theta, z) & \text{if } \theta \in (0,\pi) \\ (\theta - 2\pi, z) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

e.g. $\mathbb{RP}^2 = \{[x_0 : x_1 : x_2] : x_i$ not all zero$\}$

$F_0 \nearrow \qquad \uparrow F_1 \qquad \nwarrow F_2$

$F_0(x_1, x_2) = [1 : x_1 : x_2] \qquad F_1(y_0, y_2) = [y_0 : 1 : y_2]$

If $F_0(x_1, x_2) = F_1(y_0, y_2)$ $\quad [\frac{1}{x_1} : 1 : \frac{x_2}{x_1}]$

$\Rightarrow [1 : x_1 : x_2] = [y_0 : 1 : y_2]$

$x_1 \neq 0$

Domain $F_1^{-1}\circ F_0$: $\{x_1 \neq 0\}$

$y_0 = \frac{1}{x_1}$

$y_2 = \frac{x_2}{x_1}$

$$\Rightarrow \frac{\partial(y_0, y_1)}{\partial(x_1, x_2)} = \begin{array}{cc|c} -\frac{1}{x_1^2} & 0 & y_0 \\ -\frac{x_2}{x_1^2} & \frac{1}{x_1} & y_1 \\ \partial/\partial x_1 & \partial/\partial x_2 & \end{array}$$

$$\det\frac{\partial(y_0, y_1)}{\partial(x_1, x_2)} = -\frac{1}{x_1^3}$$

Prop: 4.25 M^n orientable $\Longleftrightarrow \exists$ non-vanishing n-form Ω.

Proof why $\mathbb{RP}^2$ is non-orientable:

Assume $\mathbb{RP}^2$ is orientable $\Rightarrow \exists\ \Omega$ 2-form on $\mathbb{RP}^2$ s.t. $\Omega(p) \neq 0 \quad \forall p \in \mathbb{RP}^2$.

let $\Omega = f\, dx^1 \wedge dx^2$. $\quad$ +

$\Omega\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = f$

$dx^1 \otimes dx^2 - dx^2 \otimes dx^1$

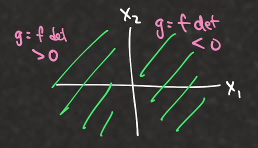
$$\Omega\left(\frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_1}\right) = f\,(dx^1\wedge dx^2)\left(\frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_1}\right)$$

$$= f\left(\frac{\partial x_1}{\partial y_0}\frac{\partial x_2}{\partial y_1} - \frac{\partial x_2}{\partial y_0}\frac{\partial x_1}{\partial y_1}\right)$$

$$= f\det\frac{\partial(x_1, x_2)}{\partial(y_0, y_1)}.$$

$$\Omega = f\det\frac{\partial(x_1, x_2)}{\partial(y_0, y_1)}\,dy^0 \wedge dy^1.$$

$= g\, dy^0 \wedge dy^1$



1°: M can be covered by one parametrization.

$F(u_1, \ldots, u_n) : \mathcal{U} \to M$

$\Omega = f\, du^1 \wedge \cdots \wedge du^n$

$$\int_M \Omega := \underbrace{\int_{\mathcal{U}} f\, du^1 \cdots du^n}_{\text{multivariable calculus integration.}} \quad \text{(basically).}$$

$$= \underbrace{\int_M f\, du^1 \wedge \cdots \wedge du^n}_{\text{integration of } n\text{-form}}$$

Issue: $f\, dx\wedge dy = -f\, dy \wedge dx$

$\int_M f\, dx\wedge dy := \int_{\mathbb{R}^2} f\, dx\, dy$

$\neq$??

$\int_M -f\, dy\wedge dx = \int_{\mathbb{R}^2} -f\, dy\, dx$

Pick: (x,y) as the order

then $\int_M x^2 y\, dy\wedge dx = \int_M -x^2 y\, dx \wedge dy \leftarrow (4033)$

$= \int_{\mathbb{R}^2} -x^2 y\, dx\, dy \leftarrow (2033).$

2°: Independent of coordinates?

$F(u_1, \ldots, u_n) : \mathcal{U} \to M$

$G(v_1, \ldots, v_n) : \mathcal{V} \to M$

Declare $(u_1, \ldots, u_n)$ as the order

$f\, du^1 \wedge \cdots \wedge du^n = g\, dv^1 \wedge \cdots \wedge dv^n$

Q: $\int f\, du^1 \cdots du^n \overset{?}{=} \int g\, dv^1 \cdots dv^n.$

$$f\, du^1\wedge\cdots\wedge du^n = \underbrace{f\det\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}}_{= g} dv^1\wedge\cdots\wedge dv^n.$$

$$\int g\, dv^1\wedge\cdots\wedge dv^n = \int f\det\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}\, dv^1\wedge\cdots\wedge dv^n$$

Need: $(v_1, \ldots, v_n)$ is the order of integration.

$$= \int f \det\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,\partial v_n)}\, dv^1 \cdots dv^n$$

Need: $\det\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)} > 0$

$= +\int f\, du^1 \cdots du^n = +\int f\, du^1\wedge\cdots\wedge du^n$

from 2023: or 3043

$$\int f\left|\det\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}\right| dv^1\cdots dv^n = \int f\, du^1\cdots du^n$$

$\mathbb{S}^2$ covered by one parametrization a.e.

e.g. $\omega = \iota^*(dx\wedge dy) \qquad \mathbb{S}^2 \subset \mathbb{R}^3$

$\iota^* dx = \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi = \cos\varphi\cos\theta\, d\varphi - \sin\varphi\sin\theta\, d\theta$

$\iota^* dy = --$

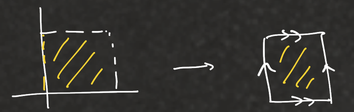
$\omega = (\iota^* dx)\wedge(\iota^* dy) = \sin\varphi\cos\varphi\, d\varphi\wedge d\theta.$

$$\int_{\mathbb{S}^2}\omega = \int \omega = \int \sin\varphi\cos\varphi\, d\varphi\wedge d\theta = \int_0^{2\pi}\int_0^{\pi}\sin\varphi\cos\varphi\, d\varphi\, d\theta$$

(φ, θ)

$(0,\pi) \quad (0,2\pi)$

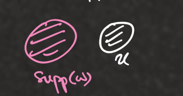
$= 0.$



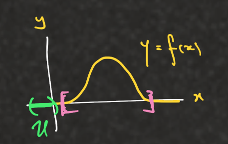
$\mathrm{supp}(f) := \overline{\{x : f(x) \neq 0\}}$

$\mathrm{supp}(\omega) := \overline{\{p \in M : \omega(p) \neq 0\}}$

If $\mathcal{U} \cap \mathrm{supp}(\omega) = \emptyset$



then $\int_{\mathcal{U}} \omega = 0$



If $\mathcal{U} \cap \mathrm{Supp}(f) = \emptyset$

then $f \equiv 0$ on $\mathcal{U}$.

$f^{(n)} \equiv 0$ on $\mathcal{U}$.

$\int_{\mathcal{U}} f\, dx = 0.$

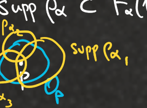
Partition of unity (Def. 4.18, P. 124)

$\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\}$ atlas of M.

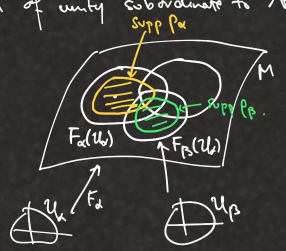
$\{\rho_\alpha : M \to [0,1]\}$ is a partition of unity subordinate to $\mathcal{A}$

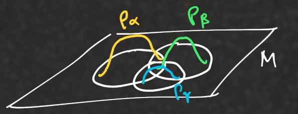
if ① $\mathrm{supp}\,\rho_\alpha \subset F_\alpha(\mathcal{U}_\alpha)$

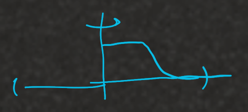
②



③ $\sum_{\text{all }\alpha}\rho_\alpha \equiv 1$ on M







$$\omega = \underbrace{\left(\sum_{\text{all }\alpha}\rho_\alpha\right)}_{=1}\omega = \sum_{\text{all }\alpha}(\rho_\alpha\omega) \quad ✓$$

$\mathrm{supp}(\rho_\alpha\omega) \subset \mathrm{supp}(\rho_\alpha) \subset F_\alpha(\mathcal{U}_\alpha)$

$\int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega$ ✓

$$\boxed{\int_M \omega := \sum_{\text{all }\alpha}\int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega}$$

If $\exists\ \{\psi_\alpha : M \to [0,\infty)\}$ s.t. $\mathrm{supp}\,\psi_\alpha \subset F_\alpha(\mathcal{U}_\alpha)\ \forall\alpha$ and $\forall p \in M$, $\exists\beta$ s.t. $\psi_\beta(p) \neq 0$.

then define: $\rho_\alpha := \frac{\psi_\alpha}{\sum_{\text{all }\alpha}\psi_\alpha}$

$\int_M \omega = \sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega$ is indep. of atlas, and indep. of the choice of ρ_α.

$\mathcal{A} = \{F_\alpha : \mathcal{U}_\alpha \to M\} \leftarrow \{\rho_\alpha : M \to [0,1]\}$

$\mathcal{A}' = \{G_\beta : \mathcal{V}_\beta \to M\} \leftarrow \{\sigma_\beta : M \to [0,1]\}.$

WANT: $\sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega = \sum_\beta \int_{G_\beta(\mathcal{V}_\beta)} \sigma_\beta\omega.$

$$\sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega = \sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} \left(\sum_\beta \sigma_\beta\right)\rho_\alpha\omega$$

$\sum_\beta \sigma_\beta = 1.$

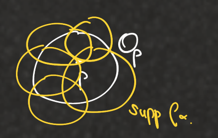
$$= \sum_\alpha\sum_\beta \int_{F_\alpha(\mathcal{U}_\alpha)} \sigma_\beta\rho_\alpha\omega$$

$\sigma_\beta = 0$ outside $G_\beta(\mathcal{V}_\beta)$

$$= \sum_\alpha\sum_\beta \int_{F_\alpha(\mathcal{U}_\alpha)\cap G_\beta(\mathcal{V}_\beta)} \sigma_\beta\rho_\alpha\omega$$

$\sum_\beta \int_{G_\beta(\mathcal{V}_\beta)} \sigma_\beta\omega \quad =$

For non compact manifold, just assume $\mathrm{supp}(\omega)$ is compact



$\sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha\omega$

M is orientable

$\overset{\text{def}}{\Longleftrightarrow} \exists$ an oriented atlas $\{F_\alpha : \mathcal{U}_\alpha \to M\}$ s.t. $\leftarrow \mathcal{A}$

$\det D(F_\beta^{-1}\circ F_\alpha) > 0 \quad \forall\alpha,\beta$ on the overlap.

Prop. 4.25 $\Longleftrightarrow \exists$ a non-vanishing n-form Ω on M $\leftarrow$ global (where $n = \dim M$).

$\forall p \in M.\ \Omega(p) \neq 0.$

Proof:

$(\Rightarrow)$ $\{\rho_\alpha : M \to [0,1]\}$ partition of unity subordinate to $\mathcal{A}$.

$F_\alpha(u_\alpha^1, \ldots, u_\alpha^n)$ local coordinates

$\eta_\alpha = du_\alpha^1 \wedge\cdots\wedge du_\alpha^n \neq 0$ on $F_\alpha(\mathcal{U}_\alpha)$

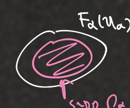
$\Omega := \sum_\alpha \rho_\alpha\eta_\alpha \to$ Justify: $\forall p \in M.\ \Omega(p) \neq 0.$ Fix $(u_\alpha^1, \ldots, u_\alpha^n)$

η_α on $F_\alpha(\mathcal{U}_\alpha)$

$$\rho_\alpha\eta_\alpha(p) = \begin{cases} \rho_\alpha\eta_\alpha(p) & \text{if } p \in F_\alpha(\mathcal{U}_\alpha) \\ 0 & \text{if } p \notin F_\alpha(\mathcal{U}_\alpha). \end{cases}$$

$\equiv 0$ outside $F_\alpha(\mathcal{U}_\alpha)$

$$\eta_\beta = du_\beta^1\wedge\cdots\wedge du_\beta^n = \det\frac{\partial(u_\beta^1,\ldots,u_\beta^n)}{\partial(u_\alpha^1,\ldots,u_\alpha^n)}\underbrace{du_\alpha^1\wedge\cdots\wedge du_\alpha^n}_{\eta_\alpha}$$



$$\Omega(p) = \sum_\beta \rho_\beta\eta_\beta(p) = \left(\sum_\beta \rho_\beta \det\frac{\partial(u_\beta^1,\ldots,u_\beta^n)}{\partial(u_\alpha^1,\ldots,u_\alpha^n)}\right)\eta_\alpha(p)$$

⊕ $\quad$ + $\quad \exists\beta$ s.t. $\rho_\beta(p) > 0.$