Summary:

$f: M \to \mathbb{R}$, $\quad f \in \Lambda^0 T^*M$

$\uparrow F(u_1,\dots,u_n)$

$$df = \sum_{i=1}^n \frac{\partial f}{\partial u_i} du^i$$

$$\omega = \sum \omega_{i_1\dots i_k}\, du^{i_1}\wedge\cdots\wedge du^{i_k}$$

$$d\omega = \sum d\omega_{i_1\dots i_k}\wedge du^{i_1}\wedge\cdots\wedge du^{i_k} = \sum \sum_j \frac{\partial \omega_{i_1\dots i_k}}{\partial u_j} du^j \wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$$

Important fact: $d^2 = 0$

Differential Form on $\mathbb{R}^3$	Multivariable Calculus
$f(x,y,z)$	$f(x,y,z)$
$\omega = P\,dx + Q\,dy + R\,dz$	$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$
$\beta = A\,dy\wedge dz + B\,dz\wedge dx + C\,dx\wedge dy$	$\mathbf{G} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$
df	∇f
$d\omega$	$\nabla\times\mathbf{F}$
$d\beta$	$\nabla\cdot\mathbf{G}$
$d^2 f = 0$	$\nabla\times\nabla f = 0$
$d^2\omega = 0$	$\nabla\cdot(\nabla\times\mathbf{F}) = 0$

e.g. $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $\mathbb{R}^2\setminus\{0\}$

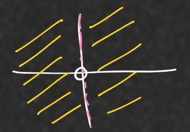
$$d\omega = \left(\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)dy\right)\wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)dx\wedge dy$$

$$= -\frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2}dy\wedge dx + \frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}dx\wedge dy$$

$$= \left(-\frac{x^2-y^2}{(x^2+y^2)^2}dy\wedge dx\right) + \frac{y^2-x^2}{(x^2+y^2)^2}dx\wedge dy = 0.$$

ω is closed

$d\left(\tan^{-1}\frac{y}{x}\right) = \omega$



$\Phi: M \to N$

$\omega \in T^*N$, $\quad (\Phi^*\omega) \in T^*M$

$(\Phi^*\omega)(X) = \omega(\Phi_* X)$, in TM

$G^{-1}\circ\Phi\circ F(u_i) = (v_j)$

$\Phi^* dv^j = \sum_i \frac{\partial v_j}{\partial u_i} du^i$

$T_1, T_2, \dots, T_k \in T^*N$

$T_1\otimes\cdots\otimes T_k \in T^*N\otimes\cdots\otimes T^*N : TN\times\cdots\times TN \to \mathbb{R}$.

$\Phi: M\to N$, $\quad \Phi^*(T_1\otimes\cdots\otimes T_k) := (\Phi^*T_1)\otimes\cdots\otimes(\Phi^*T_k)$

$\Sigma^2 \subset \mathbb{R}^3$ regular surface

$\iota: \Sigma^2 \to \mathbb{R}^3$, $p \mapsto p$ $\quad F(u_1,u_2) = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$

$\iota^*(dx) = \frac{\partial x}{\partial u_1}du^1 + \frac{\partial x}{\partial u_2}du^2$

$\iota^*(dx\otimes dx) = (\iota^* dx)\otimes(\iota^* dx)$

$= \left(\frac{\partial x}{\partial u_1}du^1 + \frac{\partial x}{\partial u_2}du^2\right)\otimes\left(\frac{\partial x}{\partial u_1}du^1 + \frac{\partial x}{\partial u_2}du^2\right)$

$= \left(\frac{\partial x}{\partial u_1}\right)^2 du^1\otimes du^1 + \frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}du^1\otimes du^2 + \frac{\partial x}{\partial u_2}\frac{\partial x}{\partial u_1}du^2\otimes du^1 + \left(\frac{\partial x}{\partial u_2}\right)^2 du^2\otimes du^2$

Check: $\iota^*(dx\otimes dx + dy\otimes dy + dz\otimes dz) = g$ first fundamental form

$= \left(\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2\right)du^1\otimes du^1 + \cdots$

$= \left|\frac{\partial F}{\partial u_1}\right|^2 du^1\otimes du^1 + \frac{\partial F}{\partial u_1}\cdot\frac{\partial F}{\partial u_2}du^1\otimes du^2 + \frac{\partial F}{\partial u_2}\cdot\frac{\partial F}{\partial u_1}du^2\otimes du^1 + \left|\frac{\partial F}{\partial u_2}\right|^2 du^2\otimes du^2$

$g_{ij} = (\iota^*(\varepsilon))\left(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right) = \frac{\partial F}{\partial u_i}\cdot\frac{\partial F}{\partial u_j}$

$\Phi^*(f\,du^i\otimes du^j) = (f\circ\Phi)\,\Phi^*du^i\otimes\Phi^*du^j$ $\quad (= \Phi^* f)$

$f: N\to\mathbb{R}$, $\quad f\circ\Phi: M\to\mathbb{R}$, $\quad \Phi: M\to N$

$dx\wedge dy = dx\otimes dy - dy\otimes dx$

$\Phi: M \to N$, $\quad (u_1,\dots,u_m)$, $(v_1,\dots,v_n)$

$\Phi^*(f\,dv^1\wedge\cdots\wedge dv^k) = \Phi^* f\,\Phi^*dv^1\wedge\cdots\wedge\Phi^*dv^k$

Prop. 3.56 (special case)

$\Phi: M^n \to N^n$

$G^{-1}\circ\Phi\circ F(u_i) = (v_j)$

$$\Phi^*(dv^1\wedge\cdots\wedge dv^n) = \det\frac{\partial(v_1,\dots,v_n)}{\partial(u_1,\dots,u_n)}du^1\wedge\cdots\wedge du^n$$

$\{du^1,\dots,du^n\}$

$\dim \Lambda^k T^*M = \begin{cases} C^n_k & \text{if } 0\le k\le n \\ 0 & \text{if } k>n\end{cases}$

$\dim \Lambda^n T^*M = 1$

Prop. 3.57

$\Phi: M^m \to N^n$

then $\Phi^* d_N = d_M \Phi^*$ (in T^*N)

Cor: $\Phi^*(d\eta) = d(\Phi^*\eta)$

If $d\omega = 0$ $\Rightarrow d(\Phi^*\omega) = \Phi^*(d\omega) = 0$

Proof (1-form)

$\omega = \sum_i \omega_i dv^i$ $\quad M \to N$, $F(u_i)$, $G(v_j)$

$\Phi^*(d\omega) = \Phi^*\left(\sum_{i,j}\frac{\partial\omega_i}{\partial v_j}dv^j\wedge dv^i\right)$

$= \sum_{i,j,k,l}\left(\frac{\partial\omega_i}{\partial v_j}\circ\Phi\right)\frac{\partial v_j}{\partial u_k}du^k\wedge\frac{\partial v_i}{\partial u_l}du^l$ $\quad \Phi^*dv^j = \sum_k \frac{\partial v_j}{\partial u_k}du^k$

$= \sum_{i,j,k,l}\left(\frac{\partial\omega_i}{\partial v_j}\circ\Phi\right)\frac{\partial v_j}{\partial u_k}\frac{\partial v_i}{\partial u_l}du^k\wedge du^l$

$d(\Phi^*\omega) = d\left(\sum_{i,j}(\omega_i\circ\Phi)\frac{\partial v_i}{\partial u_j}du^j\right)$ $\quad \Phi^* dv^i$

$= \sum_{i,j,k}\frac{\partial}{\partial u_k}\left((\omega_i\circ\Phi)\frac{\partial v_i}{\partial u_j}\right)du^k\wedge du^j$

$= \sum_{i,j,k}\frac{\partial(\omega_i\circ\Phi)}{\partial u_k}\frac{\partial v_i}{\partial u_j}du^k\wedge du^j + (\omega_i\circ\Phi)\frac{\partial^2 v_i}{\partial u_k\partial u_j}du^k\wedge du^j$

symmetric $\quad$ alternating

$= \sum_{i,j,k,l}\frac{\partial\omega_i}{\partial v_l}\frac{\partial v_l}{\partial u_k}\frac{\partial v_i}{\partial u_j}du^k\wedge du^j$

$(f\circ g)'(x) = f'(g(x))\,g'(x)$

$\Sigma^2\subset\mathbb{R}^3$ regular surface

$\uparrow F(u,v) = (x(u,v), y(u,v), z(u,v))$

$\beta = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy \leftrightarrow P\mathbf{i}+Q\mathbf{j}+R\mathbf{k} = \vec{V}$

2-form in $\mathbb{R}^3$

$\iota^*\beta = P\,\iota^*dy\wedge\iota^*dz + Q\,\iota^*dz\wedge\iota^*dx + R\,\iota^*dx\wedge\iota^*dy$

$= P\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)\wedge\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right) + \cdots$

$= P\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)du\wedge dv + \cdots$

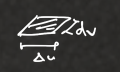
$= P\det\frac{\partial(y,z)}{\partial(u,v)}du\wedge dv + Q\det\frac{\partial(z,x)}{\partial(u,v)}du\wedge dv + R\det\frac{\partial(x,y)}{\partial(u,v)}du\wedge dv$

$= (P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\left(\det\frac{\partial(y,z)}{\partial(u,v)}\mathbf{i} + \det\frac{\partial(z,x)}{\partial(u,v)}\mathbf{j} + \det\frac{\partial(x,y)}{\partial(u,v)}\mathbf{k}\right)du\wedge dv$

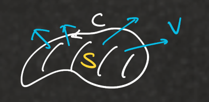
$= \vec{V}\cdot\left(\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right)du\wedge dv$ $\quad \left(\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right)$

$= \vec{V}\cdot\frac{\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}}{\left|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right|}\left|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right|du\wedge dv$

"=" $\vec{V}\cdot\hat{n}\,d\sigma$



Stokes' Theorem.



$\vec{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\quad d\vec{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k}$

$$\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\nabla\times\vec{V})\cdot\hat{n}\,d\sigma$$

$\vec{V} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$

$\updownarrow$

$\alpha = P\,dx + Q\,dy + R\,dz$

$\oint_C P\,dx + Q\,dy + R\,dz = \iint_S \iota^*(d\alpha) = \iint_S d(\iota^*\alpha)$

$= \oint_C \alpha$

β 2-form, $\quad \iota^*\beta = \vec{X}\cdot\hat{n}\,d\sigma$

$\nabla\times\vec{V} \leftrightarrow d\alpha$

$\mathbf{i} \leftrightarrow dy\wedge dz$, $\quad \mathbf{j}\leftrightarrow dz\wedge dx$, $\quad \mathbf{k}\leftrightarrow dx\wedge dy$

$\nabla\times\vec{V}\leftrightarrow d\alpha$

$$\oint_C\alpha = \iint_S d\alpha, \qquad C = \partial S$$

Divergence Theorem

$$\oiint_S \vec{V}\cdot\hat{n}\,d\sigma = \iiint_D \nabla\cdot\vec{V}\,dV \quad (dx\,dy\,dz)$$

$\vec{V} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k} \leftrightarrow \beta = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$

$\vec{V}\cdot\hat{n}\,d\sigma = \iota^*\beta$ $\quad d\beta = (\nabla\cdot\vec{V})\,dx\wedge dy\wedge dz$

LHS $= \oiint_S \iota^*\beta$ $\quad$ RHS $= \iiint_D d\beta$

$\Rightarrow \oiint_S \iota^*\beta = \iiint_D d\beta$

$$\oiint_{\partial D}\beta = \iiint_D d\beta$$

$\vec{E}, \vec{B}, \vec{J}$ $\qquad \vec{E}(x,y,z,t)$

$\nabla\cdot\vec{E} = \rho$ (charge density) $\qquad \nabla\cdot\vec{B} = 0$

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\qquad \nabla\times\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}$

$E = E_x\,dx + E_y\,dy + E_z\,dz \leftrightarrow \vec{E}$

$B = B_x\,dy\wedge dz + B_y\,dz\wedge dx + B_z\,dx\wedge dy \leftrightarrow \vec{B}$

let $F := B + E\wedge dt$. $\quad \mathbb{R}^4\ (x,y,z,t)$

$dF = 0$

$\otimes$	dx	dy	dz	dt
dx	0	B_z	B_y	E_x
dy	$-B_z$	0	B_x	E_y
dz	$-B_y$	$-B_x$	0	E_z
dt	$-E_x$	$-E_y$	$-E_z$	0

$*(dx\wedge dy) = ?$

$\Leftrightarrow (dx\wedge dy)\wedge ? = dt\wedge dx\wedge dy\wedge dz$

$dt\wedge dz$

$\Rightarrow *(dx\wedge dy) = dt\wedge dz$.

$*dx = ?$

$dx\wedge ? = dt\wedge dx\wedge dy\wedge dz$

$*dx = -dt\wedge dy\wedge dz$

$*(dt\wedge dz) = ?$

$\Leftrightarrow dt\wedge dz\wedge ? = -dt\wedge dx\wedge dy\wedge dz$

Minkowski Hodge star.

let $J = -(J_x\,dy\wedge dz + J_y\,dz\wedge dx + J_z\,dx\wedge dy)\wedge dt + \rho\,dx\wedge dy\wedge dz$

Maxwell's equations $\Leftrightarrow \begin{cases} dF = 0, \\ d(*F) = J.\end{cases}$

$X\in TM$, $\quad \omega\in\Lambda^k T^*M$ (k-form)

$i_X\omega\in\Lambda^{k-1}T^*M$.

$(i_X\omega)(\underbrace{Y_1,\dots,Y_{k-1}}_{\text{in } TM}) := \omega(X, Y_1,\dots,Y_{k-1})$

Cartan's Magic Formula

$\omega\in\Lambda^k T^*M$, $\quad X\in TM$

$\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega)$.

$\mathcal{L}_X(T\otimes S) := (\mathcal{L}_X T)\otimes S + T\otimes(\mathcal{L}_X S)$.

Cor: $\mathcal{L}_X\,d\eta = d\,i_X d\eta$ — (1)

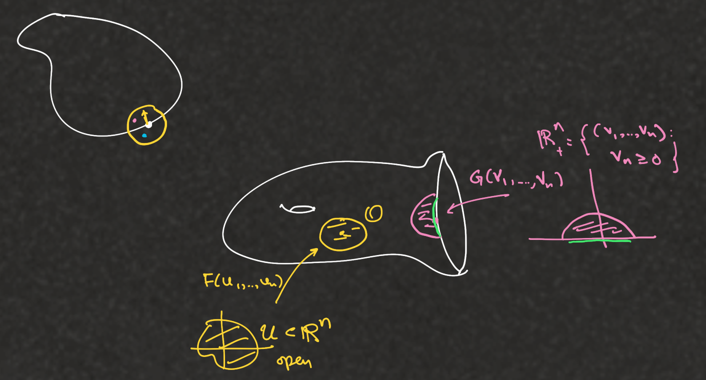
$d(\mathcal{L}_X\eta) = d(i_X d\eta + d\,i_X\eta) = d\,i_X d\eta$ — (2)

$\Rightarrow \mathcal{L}_X\,d\eta = d\,\mathcal{L}_X\eta$.

Cor: If $\omega\in\Lambda^1 T^*M$.

$d\omega(X,Y) = (i_X d\omega)(Y) = (\mathcal{L}_X\omega - d\,i_X\omega)(Y)$

$= \cdots = X(\omega(Y)) - Y(\omega(X)) - \omega(\mathcal{L}_X Y)$

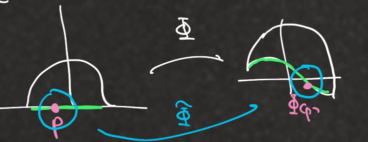


C^∞ of maps from $\mathbb{R}^n_+$:

$\Phi: \mathbb{R}^n_+ \to \mathbb{R}^m$ is C^k at $p\in\{\underbrace{(x_1,\dots,x_{n-1},0)}_{\mathbb{R}^{n-1}}\}$

$\Leftrightarrow \exists\ \tilde\Phi: B_\varepsilon(p)\to\mathbb{R}^m$, C^k

s.t. $\tilde\Phi(x) = \Phi(x)$ $\quad \forall x\in B_\varepsilon(p)\cap\mathbb{R}^n_+$



e.g. $\mathbb{R}^2_+ = \{(x,y): y\ge 0\}$.

$\Phi_1(x,y) = \sin x + \cos y : \mathbb{R}^2_+\to\mathbb{R}$ $\leftarrow\ \therefore\ C^\infty$ at $(x,0)$

$\tilde\Phi_1(x,y) = \sin x + \cos y : \mathbb{R}^2\to\mathbb{R}$, C^∞

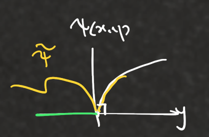
$\Phi_2(x,y) = |y|$: $\mathbb{R}^2_+\to\mathbb{R}$. C^∞ $\quad \leftarrow (x,y)\ y\ge 0$.

$\tilde\Phi_2(x,y) := y : \mathbb{R}^2\to\mathbb{R}$ $\quad C^\infty$ $\quad \tilde\Phi_2(x,y) = y = |y| = \Phi_2(x,y)$ $\quad \uparrow y\ge 0$

non-example:

$\psi(x,y) = \sqrt{y} : \mathbb{R}^2_+\to\mathbb{R}$

is C^1 at $(x,0)$.



(Manifold with boundary)

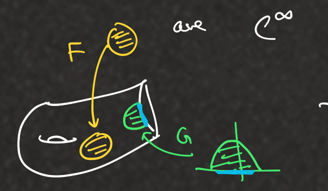
M is C^∞ manifold with boundary

$\overset{\text{def}}{\Leftrightarrow}$ $\exists$ two families of parametrizations

$\{F_\alpha: \mathcal{U}_\alpha\subset\mathbb{R}^n\to M\}$ and $\{G_\beta: V_\beta\subset\mathbb{R}^n_+\to M\}$

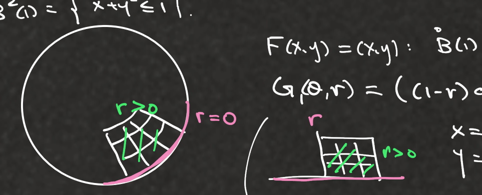
s.t. $\left(\bigcup_\alpha F_\alpha(\mathcal{U}_\alpha)\right)\cup\left(\bigcup_\beta G_\beta(V_\beta)\right) = M$

and $F_\alpha^{-1}\circ F_\beta$, $F_\alpha^{-1}\circ G_\beta$, $G_\alpha^{-1}\circ F_\beta$, $G_\alpha^{-1}\circ G_\beta$ are C^∞ $\forall\alpha,\beta$.



$\partial M := \bigcup_\beta\{G(v_1,\dots,v_{n-1},0)\}$.

$\bar{B}^2(1) = \{x^2+y^2\le 1\}$.



$F(x,y) = (x,y) : \mathring{B}(1) = \{x^2+y^2<1\}\to\mathring{B}(1)$

$G(\theta,r) = ((1-r)\cos\theta, (1-r)\sin\theta) : (0,2\pi)\times[0,1)$ $\to \bar{B}(1)$

$x = r\cos\theta$, $\quad y = r\sin\theta$

$G_2(\theta,r) = ((1-r)\cos\theta, (1-r)\sin\theta) : (-\pi,\pi)\times[0,1)$

Prop. 4.6

$f: M^m\to\mathbb{R}$ smooth, assume $f^{-1}(c)\ne\emptyset$.

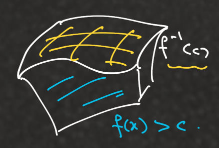
$\uparrow \partial M = \emptyset$ or $\ne\emptyset$.

$\Sigma := \{x\in M^m : f(x)\ge c\}$.

If f is a submersion at $\forall p\in f^{-1}(c)$,

then Σ is a m-manifold with boundary.

$\partial\Sigma = f^{-1}(c)$.



e.g. $\bar{B}^2 =$  $\quad f: \mathbb{R}^2\to\mathbb{R}$

$\| \qquad f(\vec{x}) = 1 - |\vec{x}|^2$.

$\{\vec{x}: f(\vec{x})\ge 0\}$

$f^{-1}(0) = \{|\vec{x}|=1\}$. $\qquad \nabla f = (-2x, -2y)$

$[f_*] = \frac{\partial(f)}{\partial(x,y)} = [-2x\ \ -2y]$.

$\ne [0\ \ 0]$ on $f^{-1}(0)$

$\bar{B}^2$ $\qquad f$ is submersion at $p\in f^{-1}(0)$

$\Rightarrow \{f(x)\ge 0\}$ is a 2-manifold with boundary

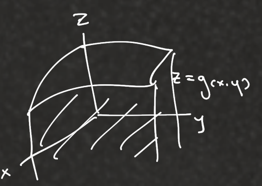
$\partial\bar{B}^2 = f^{-1}(0) = \{|x|=1\}$

e.g. $g: \mathbb{R}^2\to\mathbb{R}$ smooth function

$D = \{(x,y,z): z\le g(x,y)\}$

exercise: manifold with boundary

$\partial D = \{z = g(x,y)\}$.



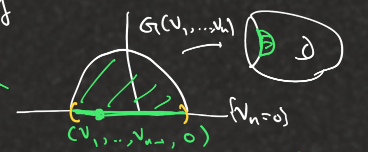
M^n manifold with boundary

∂M^n $(n-1)$-manifold.

$\partial(\partial M^n) = \emptyset$.

$\updownarrow$

$d^2 = 0$.



$\tilde{G}(v_1,\dots,v_{n-1}) : V\subset\mathbb{R}^{n-1} \to \partial M$ (open)

$:= G(v_1,\dots,v_{n-1},0)$

Tangent space $\quad M^n$, $\partial M\ne\emptyset$.

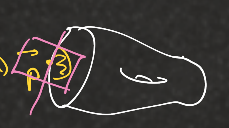
$p\in\partial M$

still

$T_pM = \text{span}\left\{\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_n}\right\}$

$= \left\{a_1\frac{\partial}{\partial v_1}+\cdots+a_n\frac{\partial}{\partial v_n}\right\}$

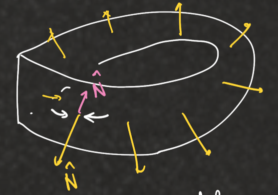
any real #.



$V = \sum_{i=1}^n V^i\frac{\partial}{\partial v_i}$. $\quad$ say V is outward-pointing if $V^n<0$

say V is inward-pointing if $V^n>0$.

$V^n = 0 \Leftrightarrow V\in T_p(\partial M)$.



Σ Orientable surface $\overset{\text{def}}{\Leftrightarrow} \exists\ \hat{N}:\Sigma\to\mathbb{R}^3$ continuous.

$\subset\mathbb{R}^3$ $\qquad$ unit normal

M^n manifold is orientable

$\overset{\text{def}}{\Leftrightarrow}$ $\exists$ atlas $\{F_\alpha: \mathcal{U}_\alpha\to M\}$

s.t. $\det D(F_\alpha^{-1}\circ F_\beta) > 0$.

Prop. $\Leftrightarrow \exists$ n-form Ω s.t. $\Omega(p)\ne 0$ $\forall p\in M$.