MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #2 • Abstract Manifolds • Due Date: 11/03/2018, 11:59PM

Instructions: *"Outsource" all topological issues to MATH 4225. Make fair use of the word "similarly" to reduce your workload.*

1. Suppose Σ is a n-dimensional topological manifold in $\mathbb{R}^{n+1}$ (where $n \geq 2$) equipped with a family of local parametrizations $\{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to \Sigma\}$ covering the whole Σ and satisfying the following conditions:

   1. Each F_α is smooth as a map from $\mathcal{U}_\alpha$ to $\mathbb{R}^{n+1}$
   2. Each F_α is a homeomorphism onto its image $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$.
   3. Regarding $(x_1, \ldots, x_{n+1}) = \mathsf{F}_\alpha(u_1, \ldots, u_n)$, the Jacobian matrix

   $$\frac{\partial(x_1, \ldots, x_{n+1})}{\partial(u_1, \ldots, u_n)}$$

   at every $(u_1, \ldots, u_n) \in \mathcal{U}_\alpha$ has a trivial null-space.

   (a) Show that Σ is an n-dimensional smooth manifold.

Solution: The three conditions are basically higher dimensional analogue of regular surfaces in $\mathbb{R}^3$. The crucial part is to show the transition map $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ between any pair of overlapping parametrizations is smooth. We mimic what was done in the regular surface case, namely making use of the non-zero component of the normal vector to apply the implicit function theorem.

First note that columns of the Jacobian matrix $\frac{\partial(x_1,\ldots,x_{n+1})}{\partial(u_1,\ldots,u_n)}$ are vectors $\{\frac{\partial \mathsf{F}}{\partial u_1}, \ldots, \frac{\partial \mathsf{F}}{\partial u_n}\}$. Condition (3) shows these vectors are linearly independent, hence they span an n-dimensional subspace in $\mathbb{R}^{n+1}$. Take any non-zero vector N orthogonal to all of $\frac{\partial \mathsf{F}}{\partial u_i}$'s. Then, the $n+1$ vectors $\{\frac{\partial \mathsf{F}}{\partial u_1}, \ldots, \frac{\partial \mathsf{F}}{\partial u_n}, \mathsf{N}\}$ are linearly independent. Then the determinant of the following $(n+1) \times (n+1)$ matrix:

$$\det \begin{bmatrix} \frac{\partial \mathsf{F}}{\partial u_1} & \cdots & \frac{\partial \mathsf{F}}{\partial u_n} & \mathsf{N} \end{bmatrix} \neq 0$$

Express $\mathsf{N} = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$, then by co-factor expansion we have:

$$\sum_{i=1}^{n+1} (-1)^i y_i \det \frac{\partial(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})}{\partial(u_1, \cdots, u_n)} \neq 0.$$

Therefore, at least one of the determinants

$$\det \frac{\partial(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})}{\partial(u_1, \cdots, u_n)} \neq 0$$

Consider $\pi_i : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map

$$(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}).$$

Then, the composition $\pi_i \circ \mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is locally invertible. The rest of the proof then goes through as in the regular surface case.

(b) Show further that Σ is a submanifold of $\mathbb{R}^{n+1}$.

Solution: The key observation is that

$$[\iota_*] = \frac{\partial(x_1, \ldots, x_{n+1})}{\partial(u_1, \ldots, u_n)}$$

which has trivial null-space. It is equivalent to saying that ι_* is injective, and so ι is an immersion.

2. The complex projective plane $\mathbb{CP}^1$ is defined as follows:

$$\mathbb{CP}^1 := \{[z_0 : z_1] : (z_0, z_1) \neq (0,0)\}.$$

Here z_0, z_1 are complex numbers, and we declare $[z_0 : z_1] = [w_0 : w_1]$ if and only if $(z_0, z_1) = \lambda(w_0, w_1)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

(a) Show that $\mathbb{CP}^1$ is a smooth manifold of (real) dimension 2.

Solution: We define parameterizations

$$\begin{aligned} \mathsf{F}_0 : \quad \mathbb{R}^2 &\to \{[z_0 : z_1] : z_0 \neq 0\} = \mathcal{O}_0 \\ (u_1, u_2) &\mapsto [1 : u_1 + iu_2] \\ \mathsf{F}_1 : \quad \mathbb{R}^2 &\to \{[z_0 : z_1] : z_1 \neq 0\} = \mathcal{O}_1 \\ (v_1, v_2) &\mapsto [v_1 + iv_2 : 1]. \end{aligned}$$

Then $\mathcal{O}_0 \bigcup \mathcal{O}_1 = \mathbb{CP}^1$.
Next, we check the smoothness of transition maps. On $\mathbb{R}^2\backslash\{(0,0)\}$

$$\begin{aligned} \mathsf{F}_1{}^{-1} \circ \mathsf{F}_0(u_1, u_2) &= \mathsf{F}_1{}^{-1}\left([1 : u_1 + iu_2]\right) \\ &= \mathsf{F}_1{}^{-1}\left([\frac{1}{u_1 + iu_2} : 1]\right) \\ &= \mathsf{F}_1{}^{-1}\left([\frac{u_1}{u_1^2 + u_2^2} + i\frac{-u_2}{u_1^2 + u_2^2} : 1]\right) \\ &= \left(\frac{u_1}{u_1^2 + u_2^2}, \frac{-u_2}{u_1^2 + u_2^2}\right) \end{aligned}$$

which is smooth. Similarly, $\mathsf{F}_0{}^{-1} \circ \mathsf{F}_1$ is also smooth. Hence, $\mathbb{CP}^1$ is a smooth manifold of (real) dimension 2.

(b) Show that $\mathbb{CP}^1$ and the sphere $\mathbb{S}^2$ are diffeomorphic. [Hint: consider stereographic projections]

Solution: Define $\Phi : \mathbb{CP}^1 \to \mathbb{S}^2$ such that

$$\begin{cases} \Phi\left([1 : u_1 + iu_2]\right) = \left(\dfrac{2u_1}{u_1^2 + u_2^2 + 1}, \dfrac{2u_2}{u_1^2 + u_2^2 + 1}, \dfrac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}\right) \\ \Phi\left([0 : 1]\right) = (0, 0, 1). \end{cases}$$

Also, we define $\Psi : \mathbb{S}^2 \to \mathbb{CP}^1$ such that

$$\Psi(x_1, x_2, x_3) = \begin{cases} \left[1, \dfrac{x_1 + ix_2}{1 - x_3}\right] & \text{if } x_3 \neq 1, \\ [0, 1] & \text{if } x_3 = 1 \end{cases}.$$

Then,

$$\begin{aligned}
&\Phi \circ \Psi(x_1, x_2, x_3) \\
&= \begin{cases} \Phi\left(\left[1 : \dfrac{x_1 + ix_2}{1 - x_3}\right]\right) & \text{if } x_3 \neq 1, \\ \Phi([0 : 1]) & \text{if } x_3 = 1 \end{cases} \\
&= \begin{cases} \left(\dfrac{2x_1(1 - x_3)}{x_1^2 + x_2^2 + (1 - x_3)^2}, \dfrac{2x_2(1 - x_3)}{x_1^2 + x_2^2 + (1 - x_3)^2}, \dfrac{x_1^2 + x_2^2 - 1 - x_3^2 + 2x_3}{x_1^2 + x_2^2 + 1 + x_3^2 - 2x_3}\right) & \text{if } x_3 \neq 1, \\ (0, 0, 1) & \text{if } x_3 = 1 \end{cases} \\
&= (x_1, x_2, x_3)
\end{aligned}$$

where we have used the fact $x_1^2 + x_2^2 + x_3^2 = 1$ to simplify the expressions, and

$$\begin{aligned}
&\begin{cases} \Psi \circ \Phi([1 : u_1 + iu_2]) \\ \Psi \circ \Phi([0 : 1]) = (0, 0, 1) \end{cases} \\
&= \begin{cases} \Psi\left(\dfrac{2u_1}{u_1^2 + u_2^2 + 1}, \dfrac{2u_2}{u_1^2 + u_2^2 + 1}, \dfrac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}\right) \\ \Psi(0, 0, 1) \end{cases} \\
&= \begin{cases} [1 : u_1 + iu_2] \\ [0 : 1] \end{cases}.
\end{aligned}$$

Hence, $\Psi = \Phi^{-1}$.
Suppose

$$\mathsf{G}_0(u_1, u_2) = \left(\frac{2u_1}{u_1^2 + u_2^2 + 1}, \frac{2u_2}{u_1^2 + u_2^2 + 1}, \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}\right) : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{(0, 0, 1)\},$$

$$\mathsf{G}_1(v_1, v_2) = \left(\frac{2v_1}{v_1^2 + v_2^2 + 1}, \frac{2v_2}{v_1^2 + v_2^2 + 1}, \frac{1 - v_1^2 - v_2^2}{v_1^2 + v_2^2 + 1}\right) : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{(0, 0, -1)\}$$

are local parameterizations of $\mathbb{S}^2$.
On $\{\Phi \circ \mathsf{F}_0(\mathbb{R}^2)\} \bigcap \{\mathsf{G}_0(\mathbb{R}^2)\}$, we have

$$\begin{aligned}
{\mathsf{G}_0}^{-1} \circ \Phi \circ \mathsf{F}_0(u_1, u_2) &= {\mathsf{G}_0}^{-1} \circ \Phi([1 : u_1 + iu_2]) \\
&= {\mathsf{G}_0}^{-1}\left(\frac{2u_1}{u_1^2 + u_2^2 + 1}, \frac{2u_2}{u_1^2 + u_2^2 + 1}, \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}\right) \\
&= (u_1, u_2).
\end{aligned}$$

which means ${\mathsf{G}_0}^{-1} \circ \Phi \circ \mathsf{F}_0$ is identity map. It is easy to check for other parameterizations in a similar way. So, Φ is smooth.

On $\{\Psi \circ \mathsf{G}_0(\mathbb{R}^2)\} \bigcap \{\mathsf{F}_0(\mathbb{R}^2)\}$, we have

$$\begin{aligned}\mathsf{F_0}^{-1} \circ \Psi \circ \mathsf{G}_0(u_1, u_2) &= \mathsf{F_0}^{-1} \circ \Psi \left(\frac{2u_1}{u_1^2 + u_2^2 + 1}, \frac{2u_2}{u_1^2 + u_2^2 + 1}, \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1} \right) \\ &= \mathsf{F_0}^{-1}[1 : u_1 + iu_2] \\ &= (u_1, u_2).\end{aligned}$$

which means $\mathsf{F_0}^{-1} \circ \Psi \circ \mathsf{G}_0$ is identity map. It is also easy to check for other parameterizations in a similar way. So, Ψ is smooth.
Therefore, $\mathbb{CP}^1$ and the sphere $\mathbb{S}^2$ are diffeomorphic.

3. Consider the following equivalence relation $\sim$ defined on $\mathbb{R}^2$:

$$(x, y) \sim (x', y') \quad \Longleftrightarrow \quad (x', y') = \big((-1)^n x + m,\, y + n\big) \text{ for some integers } m \text{ and } n.$$

(a) Sketch an edge-identified square to represent the quotient space $\mathbb{R}^2/\sim$.

(b) Consider the two parametrizations of $\mathbb{R}^2/\sim$:

$$\begin{aligned}\mathsf{G}_1 : (0,1) \times (0,1) &\to \mathbb{R}^2/\sim \\ (x,y) &\mapsto [(x,y)]\end{aligned} \qquad \begin{aligned}\mathsf{G}_2 : (0,1) \times (0.5,1.5) &\to \mathbb{R}^2/\sim \\ (x,y) &\mapsto [(x,y)]\end{aligned}$$

Find the transition map $\mathsf{G}_2^{-1} \circ \mathsf{G}_1$.

Solution: The domain of $\mathsf{G}_2^{-1} \circ \mathsf{G}_1$ is $(0,1) \times \{(0,0.5) \bigcup (0.5,1)\}$. From the equivalence relation, $[(x,y)] = [(1-x, y+1)]$. Hence, for any (x,y) in the domain, we have

$$\begin{aligned}\mathsf{G}_2^{-1} \circ \mathsf{G}_1(x,y) &= \begin{cases} \mathsf{G}_2^{-1}([(1-x, y+1)]) & \text{if } (x,y) \in (0,1) \times (0,0.5) \\ \mathsf{G}_2^{-1}([(x,y)]) & \text{if } (x,y) \in (0,1) \times (0.5,1) \end{cases} \\ &= \begin{cases} (1-x, y+1) & \text{if } (x,y) \in (0,1) \times (0,0.5) \\ (x,y) & \text{if } (x,y) \in (0,1) \times (0.5,1) \end{cases}\end{aligned}$$

(c) Write down a diffeomorphism between $\mathbb{R}^2/\sim$ and the Klein bottle K in $\mathbb{R}^4$ described in Example 2.16.

Solution: The diffeomorphism $\Phi : \mathbb{R}^2/\sim \to K$ is

$$\Phi([x,y]) = \begin{bmatrix} (\cos 2\pi x + 2)\cos 2\pi y \\ (\cos 2\pi x + 2)\sin 2\pi y \\ \sin 2\pi x \cos \pi y \\ \sin 2\pi x \sin \pi y \end{bmatrix}.$$

One can check that Φ is well-defined, bijective, and is a diffeomorphism. Detail is omitted here.

4. Consider the following subset of $\mathbb{R}^2 \times \mathbb{RP}^1$

$$M = \left\{ \Big((x_1, x_2), [y_1 : y_2] \Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \,\middle|\, x_1 y_2 = y_1 x_2 \right\}$$

(a) Show that M is a smooth 2-manifold by considering the following parametrizations:

$$\mathsf{F}(u_1, u_2) = \Big((u_1u_2,\ u_2), [u_1 : 1]\Big)$$
$$\mathsf{G}(v_1, v_2) = \Big((v_1,\ v_1v_2), [1 : v_2]\Big)$$

Solution:

$$\begin{aligned}
M &= \left\{ \Big((x_1, x_2), [y_1 : y_2]\Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \;\middle|\; x_1y_2 = y_1x_2,\ y_1 \neq 0 \right\} \\
&\quad \bigcup \left\{ \Big((x_1, x_2), [y_1 : y_2]\Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \;\middle|\; x_1y_2 = y_1x_2,\ y_2 \neq 0 \right\} \\
&= \left\{ \Big((x_1, x_1\frac{x_2}{x_1}), [1 : \frac{x_2}{x_1}]\Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \;\middle|\; x_1y_2 = y_1x_2,\ y_1 \neq 0 \right\} \\
&\quad \bigcup \left\{ \Big((x_2\frac{x_1}{x_2}, x_2), [\frac{x_1}{x_2} : 1]\Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \;\middle|\; x_1y_2 = y_1x_2,\ y_2 \neq 0 \right\}.
\end{aligned}$$

Hence, $\mathsf{F}(\mathbb{R}^2)$ and $\mathsf{G}(\mathbb{R}^2)$ is a covering of M.
The domain of the transition map $\mathsf{G}^{-1} \circ \mathsf{F}$ is $\mathbb{R}^2 \backslash \{u_1 = 0\}$.

$$\begin{aligned}
\mathsf{G}^{-1} \circ \mathsf{F}(u_1, u_2) &= \mathsf{G}^{-1}\Big((u_1u_2,\ u_2), [u_1 : 1]\Big) \\
&= \mathsf{G}^{-1}\Big((u_1u_2,\ u_1u_2\frac{1}{u_1}), [1 : \frac{1}{u_1}]\Big) \\
&= \Big(u_1u_2, \frac{1}{u_1}\Big),
\end{aligned}$$

which is smooth. The domain of the transition map $\mathsf{F}^{-1} \circ \mathsf{G}$ is $\mathbb{R}^2 \backslash \{v_2 = 0\}$.

$$\begin{aligned}
\mathsf{F}^{-1} \circ \mathsf{G}(v_1, v_2) &= \mathsf{F}^{-1}\Big((v_1, v_1v_2), [1 : v_2]\Big) \\
&= \mathsf{F}^{-1}\Big((v_1v_2\frac{1}{v_2},\ v_1v_2), [\frac{1}{v_2} : 1]\Big) \\
&= \Big(\frac{1}{v_2}, v_1v_2\Big),
\end{aligned}$$

which is smooth.
Thus, M is a smooth 2-manifold.

(b) Consider the two projection maps $\pi_1 : M \to \mathbb{R}^2$ and $\pi_2 : M \to \mathbb{RP}^1$ defined by:

$$\pi_1\Big((x_1, x_2), [y_1 : y_2]\Big) = (x_1, x_2)$$
$$\pi_2\Big((x_1, x_2), [y_1 : y_2]\Big) = [y_1 : y_2]$$

i. Show that $\pi_1^{-1}(p)$ is either a point, or diffeomorphic to $\mathbb{RP}^1$.

Solution: *Case 1:* When $x_1 \neq 0$ and $x_2 \neq 0$.
Then $y_1 \neq 0$ and $y_2 \neq 0$. We have $[y_1 : y_2] = [x_1 : x_2]$. Hence,

$$\pi_1^{-1}(x_1, x_2) = \Big((x_1, x_2), [x_1 : x_2]\Big)$$

is a point.

Case 2: When $x_1 = 0$ and $x_2 \neq 0$.
Then $y_1 = 0$ and we have $[y_1 : y_2] = [0 : 1]$. Hence,

$$\pi_1^{-1}(x_1, x_2) = \Big((0, x_2), [0 : 1]\Big)$$

is a point.
Case 3: When $x_1 \neq 0$ and $x_2 = 0$.
Then $y_2 = 0$ and we have $[y_1 : y_2] = [1 : 0]$. Hence,

$$\pi_1^{-1}(x_1, x_2) = \Big((x_1, 0), [1 : 0]\Big)$$

is a point.
Case 4: When $x_1 = 0$ and $x_2 = 0$.
Then (y_1, y_2) can be any point in $\mathbb{R}^2$ but $(0, 0)$. Hence,

$$\pi_1^{-1}(x_1, x_2) = \Big((0, 0), [y_1 : y_2]\Big)$$

Which is diffeomorphic to $\mathbb{RP}^1$.

ii. Show that π_2 is a submersion.

Solution: Let

$$\begin{aligned} \mathsf{H}_1(t) &= [1 : t] : \mathbb{R} \to \{[x : y] | x \neq 0\} \\ \mathsf{H}_2(t) &= [t : 1] : \mathbb{R} \to \{[x : y] | y \neq 0\} \end{aligned}$$

be a local parameterizations of $\mathbb{RP}^1$.
On $\mathbb{R}^2$, $\mathsf{H}_2{}^{-1} \circ \pi_2 \circ \mathsf{F}(u_1, u_2) = u_1$,

$$\begin{aligned} [\pi_{2*}]_p &= \left[\frac{\partial}{\partial u_1} \Big(\mathsf{H}_2{}^{-1} \circ \pi_2 \circ \mathsf{F}\Big)\bigg|_{\mathsf{F}^{-1}(p)} \quad \frac{\partial}{\partial u_2} \Big(\mathsf{H}_2{}^{-1} \circ \pi_2 \circ \mathsf{F}\Big)\bigg|_{\mathsf{F}^{-1}(p)} \right] \\ &= [\, 1 \quad 0 \,] \\ &\neq 0, \end{aligned}$$

Hence, it is surjective.

On $\mathbb{R}^2$, $\mathsf{H_1}^{-1} \circ \pi_2 \circ \mathsf{G}(v_1, v_2) = v_2$,

$$\begin{aligned}[\pi_{2*}]_p &= \left[\frac{\partial}{\partial v_1} \left(\mathsf{H_1}^{-1} \circ \pi_2 \circ \mathsf{G} \right) \Big|_{\mathsf{G}^{-1}(p)} \quad \frac{\partial}{\partial v_2} \left(\mathsf{H_1}^{-1} \circ \pi_2 \circ \mathsf{G} \right) \Big|_{\mathsf{G}^{-1}(p)} \right] \\ &= [\ 0 \ \ 1\ ] \\ &\neq 0,\end{aligned}$$

Hence, it is surjective.
Thus, π_2 is a submersion.

5. The tangent bundle TM of a smooth n-manifold M is the disjoint union of all tangent spaces of M, i.e.

$$TM := \bigcup_{p \in M} \{p\} \times T_pM = \{(p, V_p) : p \in M \text{ and } V_p \in T_pM\}.$$

(a) Show that TM is a smooth $2n$-manifold. [Again, skip the topological parts, but show detail work of the differentiable parts.]

Solution: Given any local parametrization $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to \mathcal{O}$ of M, we define an induced local parametrization $\widetilde{\mathsf{F}} : \mathcal{U} \times \mathbb{R}^n \to TM$ of the tangent bundle TM by:

$$\widetilde{\mathsf{F}}(u_1, \ldots, u_n, a^1, \ldots, a^n) := \left(\mathsf{F}(u_1, \ldots, u_n),\ a^1 \frac{\partial}{\partial u_1} + \cdots + a^n \frac{\partial}{\partial u_n} \right) \in TM.$$

Suppose $\mathsf{G}(v_1, \ldots, v_n)$ is another local parametrization of M, and its induced parametrization is $\widetilde{\mathsf{G}}$ given by:

$$\widetilde{\mathsf{G}}(v_1, \ldots, v_n, b^1, \ldots, b^n) = \left(\mathsf{G}(v_1, \ldots, v_n),\ b^1 \frac{\partial}{\partial v_1} + \cdots + b^n \frac{\partial}{\partial v_n} \right).$$

Since $\frac{\partial}{\partial u_i} = \sum_k \frac{\partial v_k}{\partial u_i} \frac{\partial}{\partial v_k}$ by regarding $(v_1, \ldots, v_n) = \mathsf{G}^{-1} \circ \mathsf{F}(u_1, \ldots, u_n)$, we can find the transition map between $\widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{G}}$:

$$\begin{aligned}&\widetilde{\mathsf{G}}^{-1} \circ \widetilde{\mathsf{F}}(u_1, \ldots, u_n, a^1, \ldots, a^n) \\ &= \widetilde{\mathsf{G}}^{-1} \left(\mathsf{F}(u_1, \ldots, u_n),\ a^1 \frac{\partial}{\partial u_1} + \cdots + a^n \frac{\partial}{\partial u_n} \right) \\ &= \widetilde{\mathsf{G}}^{-1} \left(\mathsf{F}(u_1, \ldots, u_n),\ a^1 \sum_k \frac{\partial v_k}{\partial u_1} \frac{\partial}{\partial v_k} + \cdots + a^n \sum_k \frac{\partial v_k}{\partial u_n} \frac{\partial}{\partial v_k} \right) \\ &= \widetilde{\mathsf{G}}^{-1} \left(\mathsf{F}(u_1, \ldots, u_n),\ \sum_k \left(\sum_j a^j \frac{\partial v_k}{\partial u_j} \right) \frac{\partial}{\partial v_k} \right) \\ &= \left(\mathsf{G}^{-1} \circ \mathsf{F}(u_1, \ldots, u_n),\ \sum_j a^j \frac{\partial v_1}{\partial u_j}, \ldots, \sum_j a^j \frac{\partial v_n}{\partial u_j} \right)\end{aligned}$$

Since $\mathsf{G}^{-1} \circ \mathsf{F}$ is smooth, and hence each v_k is a smooth function of $(u_1, \ldots, u_n)$, we conclude that $\widetilde{\mathsf{G}}^{-1} \circ \widetilde{\mathsf{F}}$ is smooth as well. This shows the transition map $\widetilde{\mathsf{G}}^{-1} \circ \widetilde{\mathsf{F}}$ is smooth.

The induced atlas $\{\widetilde{\mathsf{F}}_\alpha : \mathcal{U}_\alpha \times \mathbb{R}^n \to TM\}$ cover the whole TM, as for each $(p, V_p) \in TM$, we can first cover $p \in M$ by a local parametrization $\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M$, then its induced local parametrization $\widetilde{\mathsf{F}}$ covers all pairs $(p, V) \in \{p\} \times T_pM$.

(b) Show that the map $\pi : TM \to M$ defined by $\pi(p, V_p) := p$ is a submersion.

Solution: We need to find the tangent map π_* and show it is surjective. Suppose $\mathsf{F}(u_1, \ldots, u_n)$ is a local parametrization of M, then:

$$\begin{aligned}
&\mathsf{F}^{-1} \circ \pi \circ \widetilde{\mathsf{F}}(u_1, \ldots, u_n, a^1, \ldots, a^n) \\
&= \mathsf{F}^{-1} \circ \pi\left(\mathsf{F}(u_1, \ldots, u_n), a^1 \frac{\partial}{\partial u_1} + \cdots + a^n \frac{\partial}{\partial u_n}\right) \\
&= \mathsf{F}^{-1}\left(\mathsf{F}(u_1, \ldots, u_n)\right) \\
&= (u_1, \ldots, u_n)
\end{aligned}$$

Hence, the Jacobian of $\mathsf{F}^{-1} \circ \pi \circ \widetilde{\mathsf{F}}$ is given by:

$$[\pi_*] = D(\mathsf{F}^{-1} \circ \pi \circ \widetilde{\mathsf{F}}) = \frac{\partial(u_1, \ldots, u_n)}{\partial(u_1, \ldots, u_n, a^1, \ldots, a^n)} = [I_n \ \ 0],$$

which has full rank. Hence $(\pi_*)_p : T_{(p,V)}(TM) \to T_pM$ is surjective for any $(p, V) \in TM$.

(c) Define the subset $\Sigma_0 := \{(p, 0_p) \in TM : p \in M\}$ where 0_p is the zero vector in T_pM. This set Σ_0 is called the zero section of the tangent bundle. Show that Σ_0 is a smooth n-manifold diffeomorphic to M, and that it is a submanifold of TM.

Solution: Given any local parametrization $\mathsf{F}(u_1, \ldots, u_n)$ of M, we define an induced local parametrization $\overline{\mathsf{F}}(u_1, \ldots, u_n)$ of Σ_0 by:

$$\overline{\mathsf{F}}(u_1, \ldots, u_n) = \left(\mathsf{F}(u_1, \ldots, u_n), 0\frac{\partial}{\partial u_1} + \cdots + 0\frac{\partial}{\partial u_n}\right).$$

Clearly, given another local parametrization $\mathsf{G}(v_1, \ldots, v_n)$, the transition of the induced parametrizations of Σ_0 is given by:

$$\overline{\mathsf{G}}^{-1} \circ \overline{\mathsf{F}}(u_1, \ldots, u_n) = \mathsf{G}^{-1} \circ \mathsf{F}(u_1, \ldots, u_n),$$

which is smooth. It is clear that these induced parametrizations cover the whole Σ_0. This shows Σ_0 is a smooth manifold.
To show Σ_0 is diffeomorphic to M, we define $\Phi : M \to \Sigma_0$ by:

$$\Phi(p) := (p, 0_p).$$

Then clearly $\overline{\mathsf{F}}^{-1} \circ \Phi \circ \mathsf{F}(u_1, \ldots, u_n) = (u_1, \ldots, u_n)$, which is smooth. The map Φ is bijective with $\Phi^{-1}(p, 0_p) = p$. The inverse Φ^{-1} is smooth too as $\mathsf{F} \circ \Phi^{-1} \circ \overline{\mathsf{F}}(u_1, \ldots, u_n) = (u_1, \ldots, u_n)$. This concludes M and Σ_0 are diffeomorphic.

To show Σ_0 is a submanifold of TM, we need to show that $\iota : \Sigma_0 \to TM$ is an immersion.

$$\begin{aligned}
&\widetilde{\mathsf{F}}^{-1} \circ \iota \circ \overline{\mathsf{F}}(u_1, \ldots, u_n) \\
&= \widetilde{\mathsf{F}}^{-1} \circ \iota \left(\mathsf{F}(u_1, \ldots, u_n), \, 0\frac{\partial}{\partial u_1} + \cdots + 0\frac{\partial}{\partial u_n} \right) \\
&= \widetilde{\mathsf{F}}^{-1} \left(\mathsf{F}(u_1, \ldots, u_n), \, 0\frac{\partial}{\partial u_1} + \cdots + 0\frac{\partial}{\partial u_n} \right) \\
&= (u_1, \ldots, u_n, 0, \ldots, 0).
\end{aligned}$$

Hence, the tangent map ι_* is presented by the matrix:

$$[\iota_*] = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

whose columns are linearly independent. Therefore, ι_* is injective.

(d) Now suppose M is just a C^k-manifold (where $k \geq 2$), then TM is a $C^{\text{what?}}$-manifold?

Solution: According to (a), the transition maps of TM are given by:

$$\widetilde{\mathsf{G}}^{-1} \circ \widetilde{\mathsf{F}}(u_1, \ldots, u_n, a^1, \ldots, a^n) = \left(\mathsf{G}^{-1} \circ \mathsf{F}(u_1, \ldots, u_n), \sum_j a^j \frac{\partial v_1}{\partial u_j}, \ldots, \sum_j a^j \frac{\partial v_n}{\partial u_j} \right).$$

If M is just a C^k-manifold, then $(v_1, \ldots, v_n) = \mathsf{G}^{-1} \circ \mathsf{F}(u_1, \ldots, u_n)$ is C^k, and hence each $\frac{\partial v_j}{\partial u_i}$ is C^{k-1}. Hence TM is a C^{k-1}-manifold.

6. A *Lie group* G is a smooth manifold such that multiplication and inverse maps

$$\begin{aligned} \mu : G \times G &\to G & \qquad \nu : G &\to G \\ (g, h) &\mapsto gh & g &\mapsto g^{-1} \end{aligned}$$

are both smooth (C^∞) maps. As an example, $\mathrm{GL}(n, \mathbb{R})$ is a Lie group since it is an open subset of $M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, hence it can be globally parametrized using coordinates of $\mathbb{R}^{n^2}$. The multiplication map is given by products and sums of coordinates in $\mathbb{R}^{n^2}$, hence it is smooth. The inverse map is smooth too by the Cramer's rule $A^{-1} = \frac{1}{\det(A)}\mathrm{adj(A)}$ and that $\det(A) \neq 0$ for any $A \in \mathrm{GL}(n, \mathbb{R})$.

(a) Recall that $T_{(e,e)}(G \times G)$ can be identified with $T_eG \oplus T_eG = \{(X, Y) : X, Y \in T_eG\}$.

i. Show that the tangent map of μ at (e, e) is given by:

$$(\mu_*)_{(e,e)}(X, Y) = X + Y.$$

Solution: Let $\mathsf{F}(u_1, \ldots, u_n)$ be a local parametrization of G covering e, then

$$(\mathsf{F} \times \mathsf{F})(u_1, \ldots, u_n, v_1, \ldots, v_n) = \big(\mathsf{F}(u_1, \ldots, u_n), \mathsf{F}(v_1, \ldots, v_n)\big)$$

parametrizes $G \times G$ near (e, e). Parametrize G near $\mu(e, e) = e$ by the same parametrization $\mathsf{F}(w_1, \ldots, w_n)$, with different coordinates label to avoid confusion.
In order to compute μ_*, we first need to find $\mathsf{F}^{-1} \circ \mu \circ (\mathsf{F} \times \mathsf{F})$:

$$\begin{aligned} &\mathsf{F}^{-1} \circ \mu \circ (\mathsf{F} \times \mathsf{F})(u_1, \ldots, u_n, v_1, \ldots, v_n) \\ &= \mathsf{F}^{-1} \circ \mu\big(\mathsf{F}(u_1, \ldots, u_n), \mathsf{F}(v_1, \ldots, v_n)\big) \\ &= \mathsf{F}^{-1}\big(\mathsf{F}(u_1, \ldots, u_n)\mathsf{F}(v_1, \ldots, v_n)\big) \end{aligned}$$

We need to compute:

$$\frac{\partial}{\partial u_i}\mathsf{F}^{-1}\big(\mathsf{F}(u_1, \ldots, u_n)\mathsf{F}(v_1, \ldots, v_n)\big)$$
$$\frac{\partial}{\partial v_j}\mathsf{F}^{-1}\big(\mathsf{F}(u_1, \ldots, u_n)\mathsf{F}(v_1, \ldots, v_n)\big)$$

at $(u_1, \ldots, u_n, v_1, \ldots, v_n) = (\mathsf{F}^{-1}(e), \mathsf{F}^{-1}(e))$. When computing the partial derivative by u_i, we can first put $(v_1, \ldots, v_n) = \mathsf{F}^{-1}(e)$ before differentiation. This gives:

$$\begin{aligned} &\frac{\partial}{\partial u_i}\mathsf{F}^{-1}\big(\mathsf{F}(u_1, \ldots, u_n)\mathsf{F}(v_1, \ldots, v_n)\big)\Big|_{(\mathsf{F}^{-1}(e), \mathsf{F}^{-1}(e))} \\ &= \frac{\partial}{\partial u_i}\mathsf{F}^{-1}(\mathsf{F}(u_1, \ldots, u_n)\, e)\Big|_{\mathsf{F}^{-1}(e)} \\ &= \frac{\partial}{\partial u_i}(u_1, \ldots, u_n)\Big|_{\mathsf{F}^{-1}(e)} \\ &= (0, \ldots, \underbrace{1}_{i}, \ldots, 0) \end{aligned}$$

Hence, $(\mu_*)_{(e,e)}\left(\frac{\partial}{\partial u_i}, 0\right) = \frac{\partial \mu}{\partial u_i} = \frac{\partial}{\partial w_i}$. Similarly:

$$\frac{\partial}{\partial v_i}\mathsf{F}^{-1}\big(\mathsf{F}(u_1, \ldots, u_n)\mathsf{F}(v_1, \ldots, v_n)\big)\Big|_{(\mathsf{F}^{-1}(e), \mathsf{F}^{-1}(e))} = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$$

and so $(\mu_*)_{(e,e)}\left(0, \frac{\partial}{\partial v_i}\right) = \frac{\partial \mu}{\partial v_i} = \frac{\partial}{\partial w_i}$.
Given any $(X, Y) \in T_{(e,e)}(G \times G)$, express them in local coordinates:

$$X = \sum_i X^i \frac{\partial}{\partial u_i} \quad \text{and} \quad Y = \sum_j Y^j \frac{\partial}{\partial v_j}.$$

Then:

$$\begin{aligned} (\mu_*)_{(e,e)}(X, Y) &= (\mu_*)_{(e,e)}\left(\sum_i X^i \frac{\partial}{\partial u_i}, \sum_j Y^j \frac{\partial}{\partial v_j}\right) \\ &= \sum_i X^i \frac{\partial}{\partial w_i} + \sum_j Y^j \frac{\partial}{\partial w_j} = X + Y \end{aligned}$$

ii. Show that μ is a submersion at (e, e).

Solution: From (a)(i), the matrix representation of μ_* at (e, e) is given by $[I_n \quad I_n]$, which has full rank. Therefore, $(\mu_*)_{(e,e)}$ is surjective and so μ is a submersion at (e, e).

(b) Show that the tangent map of ν at e is given by:

$$(\nu_*)_e\,(X) = -X.$$

[Hint for part (a): when taking partial derivative $\frac{\partial f}{\partial u}$ at $(u, v) = (u_0, v_0)$, it is OK to substitute $v = v_0$ first, and then differentiate $f(u, v_0)$ by u. It is possible to prove (b) using the result from (a)i and the manifold chain rule in an appropriate way.]

Solution: Parametrize the domain G by $\mathsf{F}(u_1, \ldots, u_n)$ near e, and parametrize the target G using the same parametrization $\mathsf{F}(w_1, \ldots, w_n)$ near $\nu(e) = e$. We use different coordinate labels to avoid confusion.

Denote:

$$(w_1, \ldots, w_n) = \mathsf{F}^{-1} \circ \nu \circ \mathsf{F}(u_1, \ldots, u_n),$$

then we need to find the Jacobian matrix at $\mathsf{F}^{-1}(e)$:

$$\frac{\partial(w_1, \ldots, w_n)}{\partial(u_1, \ldots, u_n)}.$$

Observe that:

$$\mathsf{F}(w_1, \ldots, w_n)\mathsf{F}(u_1, \ldots, u_n) = e \quad \Longrightarrow \quad \mathsf{F}^{-1}\big(\mathsf{F}(w_1, \ldots, w_n)\mathsf{F}(u_1, \ldots, u_n)\big) = \mathsf{F}^{-1}(e).$$

Regarding w_j's are functions of u_i's, we can apply the chain rule to differentiate both sides by u_i:

$$\begin{aligned} \frac{\partial}{\partial u_i} \underbrace{\mathsf{F}^{-1}\big(\mathsf{F}(w_1, \ldots, w_n)\mathsf{F}(u_1, \ldots, u_n)\big)}_{=:\mathcal{F}} &= (0, \ldots, 0) \\ \sum_k \frac{\partial \mathcal{F}}{\partial w_k}\frac{\partial w_k}{\partial u_i} + \frac{\partial \mathcal{F}}{\partial u_i} &= (0, \ldots, 0). \end{aligned}$$

Similar to (a)(i), $\dfrac{\partial \mathcal{F}}{\partial w_k} = (0, \ldots, \underbrace{1}_{k}, \ldots, 0) = \mathsf{e}_k$ and $\dfrac{\partial \mathcal{F}}{\partial u_i} = (0, \ldots, \underbrace{1}_{i}, \ldots, 0) = \mathsf{e}_i$ at the point $(\mathsf{F}^{-1}(e), \mathsf{F}^{-1}(e))$. Combining with above, we get:

$$\left(\frac{\partial w_1}{\partial u_i}, \ldots, \frac{\partial w_n}{\partial u_i}\right) + (0, \ldots, \underbrace{1}_{i}, \ldots, 0) = (0, \ldots, 0),$$

which implies $\dfrac{\partial w_j}{\partial u_i} = -\delta_{ij}$. Therefore, we have at $\mathsf{F}^{-1}(e)$:

$$D(\mathsf{F}^{-1} \circ \nu \circ \mathsf{F}) = -I_n$$

or equivalently, $(\nu_*)_e = -\mathrm{id}$.

Alternatively, one can show the same result by defining $\nu \times \mathrm{id} : G \to G \times G$ by $(\nu \times \mathrm{id})(g) = (\nu(g), g)$ and then considering the composition:

$$\mu \circ (\nu \times \mathrm{id})(g) = \mu \circ (\nu(g), g) = \nu(g)\, g = e.$$

Since it holds for any $g \in G$, the composition $\mu \circ (\nu \times \mathrm{id})$ is a constant map.

By the (manifold) chain rule, we have for any $X \in T_e G$:

$$\big(\mu \circ (\nu \times \mathrm{id})\big)_*(X) = 0 \quad \Longrightarrow \quad \mu_* \circ (\nu \times \mathrm{id})_*(X) = 0.$$

It can be shown (detail omitted) that $(\nu \times \mathrm{id})_*(X) = (\nu_*(X), X)$. From (a), we then have at e:

$$0 = \mu_* \circ (\nu \times \mathrm{id})_*(X) = \mu_*(\nu_*(X), X) = \nu_*(X) + X,$$

which implies $\nu_*(X) = -X$ at e.