MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #3 • Tensors and Differential Forms • Due Date: 08/04/2018, 11:59PM

1. Show that the Lie derivative of a 1-form

$$\alpha = \sum_i \alpha_i \, du^i$$

along a vector field $X = \sum_j X^j \frac{\partial}{\partial u_j}$ is locally expressed as:

$$\mathcal{L}_X \alpha = \sum_{i,j} \left(X^i \frac{\partial \alpha_j}{\partial u_i} + \alpha_i \frac{\partial X^i}{\partial u_j} \right) du^j.$$

2. Consider a smooth manifold M^n and the following symmetric $(2,0)$-tensors on M:

$$g = \sum_{i,j} g_{ij} \, du^i \otimes du^j \qquad\qquad h = \sum_{i,j} h_{ij} \, du^i \otimes du^j$$

where $(u_1, \ldots, u_n)$ are local coordinates of M. Suppose the matrix $[g_{ij}]$ is positive definite (all its eigenvalues are positive).

(a) Let $(v_1, \ldots, v_n)$ be another local coordinates system, and the local expression of g in terms of this system be:

$$g = \sum_{i,j} \widetilde{g}_{ij} \, dv^i \otimes dv^j.$$

Express $\widetilde{g}_{ij}$ in terms of g_{ij}, and then show that the matrix $[\widetilde{g}_{ij}]$ is also symmetric and positive definite.

(b) Show that the quantity $\dfrac{\det[h_{ij}]}{\det[g_{ij}]}$ is independent of local coordinates, i.e.

$$\frac{\det[h_{ij}]}{\det[g_{ij}]} = \frac{\det[\widetilde{h}_{ij}]}{\det[\widetilde{g}_{ij}]}.$$

3. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

(a) Express ω in spherical coordinates (ρ, θ, φ) defined by:

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}$$

(b) Compute $d\omega$ in both rectangular and spherical coordinates.

(c) Let $\mathbb{S}^2$ be the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Consider the inclusion map $\iota : \mathbb{S}^2 \to \mathbb{R}^3$. Compute the pull-back $\iota^*\omega$, and express it in terms of spherical coordinates.

(d) Let $\delta = dx \otimes dx + dy \otimes dy + dz \otimes dz$ be a symmetric $(2,0)$-tensor on $\mathbb{R}^3$. Compute $\iota^*\delta$, and express it in terms of spherical coordinates.

4. The purpose of this exercise is to show that any closed 1-form ω on $\mathbb{R}^3$ must be exact. Let

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

be a closed 1-form on $\mathbb{R}^3$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by:

$$f(x,y,z) = \int_{t=0}^{t=1} \left(xP(tx,ty,tz) + yQ(tx,ty,tz) + zR(tx,ty,tz)\right)\,dt.$$

(a) Show that $df = \omega$.

(b) Point out exactly where you have used the fact that $d\omega = 0$ in your solution to (a).

(c) Explain why your solution to (a) would fail if the domain of ω is $\mathbb{R}^3\backslash\{p\}$ (where p is a fixed point in $\mathbb{R}^3$).

5. Consider $\mathbb{R}^4$ with coordinates (t,x,y,z), which is also denoted as (x_0,x_1,x_2,x_3) in this problem. Denote $*$ to be the Minkowski Hodge-star operator on $\mathbb{R}^4$ (see P.104).

(a) Compute each of the following:

$$\begin{array}{ccc} *(dt\wedge dx) & *(dt\wedge dy) & *(dt\wedge dz) \\ *(dx\wedge dy) & *(dy\wedge dz) & *(dz\wedge dx) \end{array}$$

(b) The four Maxwell's equations are a set of partial differential equations that form the foundation of electromagnetism. Denote the components of the electric field E, magnetic field B, and current density J by

$$\begin{aligned} \mathsf{E} &= E_x\mathsf{i} + E_y\mathsf{j} + E_z\mathsf{k} \\ \mathsf{B} &= B_x\mathsf{i} + B_y\mathsf{j} + B_z\mathsf{k} \\ \mathsf{J} &= j_x\mathsf{i} + j_y\mathsf{j} + j_z\mathsf{k} \end{aligned}$$

All components of E, B and J are considered to be time-dependent. Denote ρ to be the charge density. The four Maxwell's equations assert that:

$$\begin{array}{cc} \nabla\cdot\mathsf{E} = \rho & \nabla\cdot\mathsf{B} = 0 \\ \nabla\times\mathsf{E} = -\dfrac{\partial\mathsf{B}}{\partial t} & \nabla\times\mathsf{B} = \mathsf{J} + \dfrac{\partial\mathsf{E}}{\partial t} \end{array}$$

We are going to convert the Maxwell's equations using the language of differential forms. We define the following analogue of E, B, J and ρ using differential forms:

$$\begin{aligned} E &= E_x\,dx + E_y\,dy + E_z\,dz \\ B &= B_x\,dy\wedge dz + B_y\,dz\wedge dx + B_z\,dx\wedge dy \\ J &= (j_x\,dy\wedge dz + j_y\,dz\wedge dx + j_z\,dx\wedge dy)\wedge dt + \rho\,dx\wedge dy\wedge dz \end{aligned}$$

Define the 2-form F by $F := B + E\wedge dt$. Show that the four Maxwell's equations can be rewritten in an elegant way as:

$$dF = 0 \qquad\qquad d(*F) = J.$$