$du^i|_p : T_pM \to \mathbb{R}$

$du^i\left(\frac{\partial}{\partial u_j}\right) = \delta_j^{\,i}$

$B(X,Y) : T_pM \times T_pM \to \mathbb{R}$

Bilinear $\quad B(X_1+X_2, Y) = B(X_1,Y) + B(X_2,Y)$

similar for Y.

V vector space.

$V^* \ni T,S : V \to \mathbb{R}$

$T\otimes S : V\times V \longrightarrow \mathbb{R}$

$(T\otimes S)(X,Y) := T(X)\cdot S(Y)$ $\leftarrow$ Exercise: $T\otimes S$ is a bilinear map.

$T_1, \dots, T_k \in V^*$

$T_1 \otimes \cdots \otimes T_k : \underbrace{V\times\cdots\times V}_{k} \longrightarrow \mathbb{R}$

$(T_1\otimes\cdots\otimes T_k)(X_1,\dots,X_k) := \prod_{i=1}^{k} T_i(X_i).$

$$\left.\begin{aligned}&((T_1\otimes T_2)\otimes T_3)((X_1,X_2),X_3)\\ &= T_1\otimes(T_2\otimes T_3)(X_1,(X_2,X_3))\end{aligned}\right\}\text{ check.}$$

e.g. $\delta : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ (dot product.)

$T\mathbb{R}^3 = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}.$

$T^*\mathbb{R}^3 = \text{span}\{dx, dy, dz\}$

Claim $\delta = dx\otimes dx + dy\otimes dy + dz\otimes dz$ ✓

$\delta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \hat{i}\cdot\hat{j} = 0$

$(dx\otimes dx)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 1\cdot 0 = 0$

$(dx\otimes dx)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 1$

$(dx\otimes dx + dy\otimes dy + dz\otimes dz)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 1$

$(dx\otimes dx + dy\otimes dy + dz\otimes dz)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0$

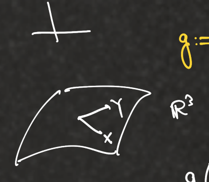
$x = r\cos\theta \qquad dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$

$y = r\sin\theta \qquad dy = \quad — \quad = \sin\theta\, dr + r\cos\theta\, d\theta$

$dx\otimes dx + dy\otimes dy = (\cos\theta\, dr - r\sin\theta\, d\theta)\otimes(\cos\theta\, dr - r\sin\theta\, d\theta)$
$\quad + (\sin\theta\, dr + r\cos\theta\, d\theta)\otimes(\sin\theta\, dr + r\cos\theta\, d\theta)$
$\quad = dr\otimes dr + r^2\, d\theta\otimes d\theta.$

$\Sigma^2 \subset \mathbb{R}^3$

$\uparrow F(u_1, u_2)$



$\text{span}\left\{\frac{\partial F}{\partial u_i}(p)\right\}_{i=1}^{2} = T_p\Sigma^2.$

$g := \delta|_{T\Sigma^2} : T_p\Sigma^2 \times T_p\Sigma^2 \longrightarrow \mathbb{R}.$

$\mathbb{R}^3 \qquad g(X,Y) := \underbrace{X\cdot Y}_{\text{in } \mathbb{R}^3}.$

$g\left(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right) =: g_{ij}$ $\leftarrow$ First Fundamental form.

$\iint \left|\frac{\partial F}{\partial u_1}\times\frac{\partial F}{\partial u_2}\right| du_1\, du_2$

$= \sqrt{\det[g]} \qquad [g] = \begin{bmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$

$g = \sum_{i,j=1}^{2} g_{ij}\, du^i \otimes du^j = \sum_{i,j=1}^{2} g_{ij}\, du^i du^j$

Verify: $g\left(\frac{\partial F}{\partial u_k}, \frac{\partial F}{\partial u_l}\right) = \sum_{i,j=1}^{2} g_{ij}\, du^i\otimes du^j\left(\frac{\partial F}{\partial u_k}, \frac{\partial F}{\partial u_l}\right)$

$\quad = \sum_{i,j=1}^{2} g_{ij}\,\delta_{ik}\,\delta_{jl} = g_{kl}$

$T : V\times V \to \mathbb{R}$

$T(e_i, e_j) = a_{ij} \implies T = \sum_{i,j=1}^{n} a_{ij}\, e_i^*\otimes e_j^*$

$dx\otimes dx = dx\,dx = (dx)^2 = dx^2$

$\mathbb{R}^4\,(t,x,y,z)$

$-c^2dt^2 + dx^2 + dy^2 + dz^2 : \mathbb{R}^4\times\mathbb{R}^4 \to \mathbb{R}$

$\mathbb{R}^4\,(t, r, \theta, \varphi)$

$= -c^2dt^2 + dr^2 + r^2(d\varphi^2 + \sin^2\varphi\, d\theta^2)$

$-\left(1-\frac{2m}{r}\right)c^2dt^2 + \frac{1}{1-\frac{2m}{r}}dr^2 + r^2(d\varphi^2 + \sin^2\varphi\, d\theta^2)$ ($d\theta^2$: $d\theta\otimes d\theta$) $\to$ Schwarzschild blackhole.

$: T\mathbb{R}^4\times T\mathbb{R}^4 \to \mathbb{R}.$

$T : V \to V$ linear. $\qquad V = \text{span}\{e_i\}_{i=1}^n, \quad V^* = \text{span}\{e_i^*\}_{i=1}^n$

Given: $T(e_i) = \sum_{j=1}^{n} A_i^{\,j} e_j$

$T = \sum_{i,j=1}^{n} A_i^{\,j}\, \underbrace{e_i^*}_{\text{in } V^*}\otimes \underbrace{e_j}_{\text{in } V} \rightarrow$

Verify:
$T(e_k) = \sum_{i,j=1}^{n} A_i^{\,j}\, e_i^*\otimes e_j\,(e_k)$
$\quad = \sum_{i,j=1}^{n} A_i^{\,j}\,\delta_{ik}\, e_j$
$\quad = \sum_{j=1}^{n} A_k^{\,j}\, e_j$

$(e_i^*\otimes e_j)(\underset{\text{in } V}{X}) = \underbrace{e_i^*(X)}_{\text{in } \mathbb{R}}\, \underbrace{e_j}_{\text{in } V}$

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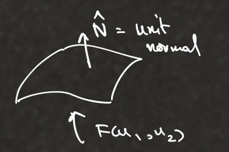
e.g. $\Sigma^2 \subset \mathbb{R}^3$ regular surface



$\frac{\partial \hat{N}}{\partial u_j} := \frac{\partial}{\partial u_j}(\hat{N}\circ F) \in T\Sigma^2.$

$\hat{N}\cdot\hat{N} = 1$

$\frac{\partial}{\partial u_j}(\hat{N}\cdot\hat{N}) = 0$

$2\frac{\partial \hat{N}}{\partial u_j}\cdot\hat{N} = 0$

$\underbrace{\frac{\partial \hat{N}}{\partial u_j}}_{\parallel\; (\hat{N}_*)\left(\frac{\partial F}{\partial u_j}\right)} = \sum_{i=1}^{2} h_j^{\,i}\frac{\partial F}{\partial u_i}$

$\hat{N}_* = \sum_{i=1}^{2} h_j^{\,i}\, du^j\otimes\frac{\partial F}{\partial u_i}$ $\leftarrow$ Weingarten map, shape operator

$\hat{N}_* : T\Sigma^2 \to T\Sigma^2$

$A : T\Sigma\times T\Sigma \to \mathbb{R}$

$A(X,Y) := \hat{N}_*(X)\cdot Y$

$\uparrow$ second fundamental form.

$dx\otimes\frac{\partial}{\partial y} = \left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right)\otimes\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\right)$

$= \cdots \quad dr\otimes\frac{\partial}{\partial r} \quad dr\otimes\frac{\partial}{\partial \theta} \quad + \cdots$

$T : V\times V \to V$ (cross product) $\qquad V = \mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}.$ ($\hat{i}, \hat{j}, \hat{k}$; $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$)

$T(e_1,e_2) = e_3, \quad T(e_2,e_1) = -e_3$

$T(e_2,e_3) = e_1, \dots$

$T = e_1^*\otimes e_2^*\otimes e_3 - e_2^*\otimes e_1^*\otimes e_3 + \cdots$

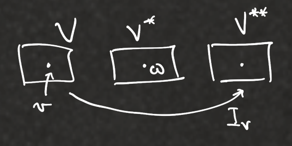
$\quad = dx\otimes dy\otimes\frac{\partial}{\partial z} - dy\otimes dx\otimes\frac{\partial}{\partial z} + \cdots$

$e_1^*\otimes\cdots e_k^*\otimes e_l : \underbrace{V\times\cdots\times V}_{k}\times V^* \longrightarrow \mathbb{R}$

$e_1^*\otimes\cdots\otimes e_k^*\otimes e_l\,(X_1,\dots,X_k,\omega)$

$= e_1^*(X_1)\cdots e_k^*(X_k)\cdot e_l(\omega)$

$\underbrace{I_{e_l}}_{\text{in } V^{**}}(\omega) = \omega(e_l)$, $\omega \in V^*$



$I_v : V^* \to \mathbb{R}$

$I_v(\omega) := \omega(v) \in \mathbb{R}.$

$T, S \in V^*, \quad V = \text{span}\{e_i\}_{i=1}^n$

$T\wedge S := T\otimes S - S\otimes T$

$S\wedge T = S\otimes T - T\otimes S = -T\wedge S$

$\det : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$

$\left(\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix}\right) \longmapsto \begin{vmatrix}a & c\\ b & d\end{vmatrix} \in \mathbb{R}$

$\det(X,Y) = -\det(Y,X).$

$\det(e_1,e_1) = 0 \qquad \det(e_2,e_1) = -1$

$\det(e_1,e_2) = \det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 1 \qquad \det(e_2,e_2) = 0$

$\det = 1\cdot e_1^*\otimes e_2^* - 1\, e_2^*\otimes e_1^* = e_1^*\wedge e_2^*.$

$T\wedge S\wedge Y \neq (T\wedge S)\otimes Y - Y\otimes(T\wedge S)$

Old MacDonald had a farm: $e^i\wedge e^i = 0$

$T_1, \dots, T_k \in V^*$

$T_1\wedge\cdots\wedge T_k := \sum_{\sigma\in S_k} \text{sign}(\sigma)\, T_{\sigma(1)}\otimes T_{\sigma(2)}\otimes\cdots\otimes T_{\sigma(k)}.$ ($\text{sign}(\sigma) = (-1)^{\#\text{ of transpositions of }\sigma}$)

$\sigma = (123) = (12)(23) \qquad \text{sgn}(\sigma) = (-1)^2 = 1.$

$\sigma' = (123)(34) = (12)(23)(34) \qquad \text{sgn}(\sigma') = -1.$

τ transposition

$T_{\tau(1)}\wedge\cdots\wedge T_{\tau(k)} = \sum_{\sigma\in S_k}\text{sign}(\sigma)\, T_{\sigma(\tau(1))}\otimes T_{\sigma(\tau(2))}\otimes\cdots\otimes T_{\sigma(\tau(k))}$

$\sigma' = \sigma\circ\tau \qquad = \sum_{\sigma'\in S_k} -\text{sign}(\sigma')\, T_{\sigma'(1)}\otimes T_{\sigma'(2)}\otimes\cdots\otimes T_{\sigma'(k)}$

$\text{sign}(\sigma') = -\text{sign}(\sigma) \qquad = -T_1\wedge\cdots\wedge T_k$

$\underset{k\text{-form}}{\omega}, \underset{r\text{-form}}{\eta} \qquad \omega\wedge\eta \overset{\text{generally}}{\neq} -\eta\wedge\omega.$

$\omega\wedge\eta = (-1)^{kr}\,\eta\wedge\omega.$

$M \quad (u_1,\dots,u_n) \qquad T_pM = \text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$

$\quad (v_1,\dots,v_n) \qquad T_p^*M = \text{span}\{du^i|_p\}_{i=1}^n$

$du^1\wedge\cdots\wedge du^n \qquad$ P.86

$= \det\frac{\partial(u_1,\dots,u_n)}{\partial(v_1,\dots,v_n)}\, dv^1\wedge\cdots\wedge dv^n$

$\underset{k}{\omega} = \sum\omega_{i_1\cdots i_k}\, e_{i_1}^*\wedge\cdots\wedge e_{i_k}^*, \quad \eta = \sum\eta_{j_1 j_2\cdots j_l}\, e_{j_1}^*\wedge\cdots\wedge e_{j_l}^*$

$\omega\wedge\eta = \sum\omega_{i_1\cdots i_k}\,\eta_{j_1\cdots j_l}\, e_{i_1}^*\wedge\cdots\wedge e_{i_k}^*\wedge e_{j_1}^*\wedge\cdots\wedge e_{j_l}^*$

$\parallel$

$(-1)^{kl}\,\eta\wedge\omega$

$\omega\wedge\omega \qquad e^i\wedge e^i = 0.$

$\omega = e^1\wedge e^2 + e^3\wedge e^4$

$\omega\wedge\omega = (e^1\wedge e^2 + e^3\wedge e^4)\wedge(e^1\wedge e^2 + e^3\wedge e^4)$

$\quad = e^1\wedge e^2\wedge e^3\wedge e^4 + e^3\wedge e^4\wedge e^1\wedge e^2$

$\quad = 2\, e^1\wedge e^2\wedge e^3\wedge e^4$

e.g. $\begin{cases} x = r\cos\theta\\ y = r\sin\theta\end{cases}$

$dx\wedge dy = \left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right)\wedge\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$

$\quad = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta}\, dr\wedge d\theta + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\, d\theta\wedge dr$

$\quad = \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) dr\wedge d\theta$

$\quad = r\, dr\wedge d\theta$

$d : \Lambda^k T^*M \longrightarrow \Lambda^{k+1}T^*M$

$f : M^n \to \mathbb{R}$ scalar function $\qquad$ local coord $F(u_1,\dots,u_n)$

$df := \sum_{i=1}^{n}\frac{\partial f}{\partial u_i}du^i \qquad G(v_1,\dots,v_n)$

$\overset{?}{=} \sum_{i=1}^{n}\frac{\partial f}{\partial v_i}dv^i$

Check:

$\sum_{i=1}^{n}\frac{\partial f}{\partial u_i}du^i = \sum_{i=1}^{n}\frac{\partial f}{\partial u_i}\left(\sum_{j=1}^{n}\frac{\partial u_i}{\partial v_j}dv^j\right)$

$\quad = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial v_j}\right)dv^j = \sum_{j=1}^{n}\frac{\partial f}{\partial v_j}dv^j$

$df \quad \frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_n} \qquad du^i \leftrightarrow \hat{e}_i \in \mathbb{R}^n.$

$\updownarrow$

$\nabla f = \left(\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_n}\right) = \sum_{j=1}^{n}\frac{\partial f}{\partial u_j}\hat{e}_j$

$\omega = \sum_{i=1}^{n}\omega_i\, du^i \qquad$ 1-form

$d\omega := \sum_{i=1}^{n}\Big(\underbrace{\sum_{j=1}^{n}\frac{\partial\omega_i}{\partial u_j}du^j}_{d\omega_i}\Big)\wedge du^i \qquad$ 2-form

$\quad = \sum_{1\le i\ne j\le n}\frac{\partial\omega_i}{\partial u_j}du^j\wedge du^i$

$\quad = \sum_{1\le i<j\le n}\frac{\partial\omega_i}{\partial u_j}du^j\wedge du^i + \sum_{1\le j<i\le n}\frac{\partial\omega_i}{\partial u_j}du^j\wedge du^i$

$\quad = \sum_{1\le j<i\le n}\frac{\partial\omega_j}{\partial u_i}du^i\wedge du^j - \sum_{1\le j<i\le n}\frac{\partial\omega_i}{\partial u_j}du^i\wedge du^j$

$\quad = \sum_{1\le j<i\le n}\left(\frac{\partial\omega_j}{\partial u_i} - \frac{\partial\omega_i}{\partial u_j}\right)du^i\wedge du^j.$

$\Omega = \sum_{1\le i_1<\cdots<i_k\le n}\Omega_{i_1\cdots i_k}\, du^{i_1}\wedge\cdots\wedge du^{i_k}$

$d\Omega = \sum_{1\le i_1<\cdots<i_k\le n} d\Omega_{i_1\cdots i_k}\wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

---

e.g. $\alpha = P\,dx + Q\,dy + R\,dz$ on $\mathbb{R}^3$. $\leftarrow P\hat{i} + Q\hat{j} + R\hat{k} = \vec{V}$

$d\alpha = \underbrace{\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)}_{dP}\wedge dx$

$\quad + \left(\frac{\partial Q}{\partial x}dx + \;?\; + \frac{\partial Q}{\partial z}dz\right)\wedge dy$

$\quad + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \;?\;\right)\wedge dz$

$$\left.\begin{aligned} &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy \longleftrightarrow \hat{k}\\ &+ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\wedge dz \longleftrightarrow \hat{i}\\ &+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\wedge dx \longleftrightarrow \hat{j}\end{aligned}\right\} \longleftrightarrow \underset{\text{curl}(\vec{V})}{\nabla\times\vec{V}} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$\beta = P\,\underbrace{dy\wedge dz}_{\leftrightarrow\hat{i}} + Q\,\underbrace{dz\wedge dx}_{\leftrightarrow\hat{j}} + R\,\underbrace{dx\wedge dy}_{\leftrightarrow\hat{k}} \longleftrightarrow P\hat{i} + Q\hat{j} + R\hat{k} = \vec{W}$

$d\beta = \frac{\partial P}{\partial x}dx\wedge dy\wedge dz + \frac{\partial Q}{\partial y}dy\wedge dz\wedge dx + \frac{\partial R}{\partial z}dz\wedge dx\wedge dy$

$\quad = \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)}_{= \nabla\cdot\vec{W} = \text{div}(\vec{W})} dx\wedge dy\wedge dz$

$d(\underset{k\text{-form}}{\alpha}\wedge\underset{r\text{-form}}{\beta}) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$

<u>Theorem</u>: $d^2 = 0.$

$\Lambda^{k-1}T^*M \xrightarrow{d_{k-1}} \Lambda^k T^*M \xrightarrow{d_k} \Lambda^{k+1}T^*M$

$d_k\circ d_{k-1} = 0$

<u>Proof</u>: $(k=1)$

$df = \sum_{i=1}^{n}\frac{\partial f}{\partial u_i}du^i$

$d(df) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial u_j}\left(\frac{\partial f}{\partial u_i}\right)du^j\wedge du^i$

$\quad = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial}{\partial u_i}\left(\frac{\partial f}{\partial u_j}\right)du^i\wedge du^j$

$\quad = -\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial}{\partial u_j}\left(\frac{\partial f}{\partial u_i}\right)du^j\wedge du^i$

$\quad = -d(df)$

$\implies d(df) = 0.$

$\Lambda^{k-1}T^*M \xrightarrow{d_{k-1}} \Lambda^k T^*M \xrightarrow{d_k} \Lambda^{k+1}T^*M$

$d_k\circ d_{k-1} = 0$

$\text{Im}(d_{k-1}) \subset \text{Ker}(d_k)$

$\{d_{k-1}\eta : \eta\in\Lambda^{k-1}T^*M\} \qquad \{\omega\in\Lambda^k T^*M : d_k\omega = 0\}.$

$\text{Ker}(d_k)/\text{Im}(d_{k-1}) =: H^k_{dR}(M) \leftarrow$ de Rham cohomology group.

a topological invariant !!

If $\omega = d\eta$ $\exists\eta$, then say ω is <u>exact</u>.

If $d\omega = 0$, then say ω is <u>closed</u>.

ω exact $\Rightarrow$ ω closed

Converse: Not true in general.