Chapter 3 : $\otimes$ and $\wedge$

V = finite-dim. vector space over $\mathbb{R}$.
$= \text{span}\{\underbrace{e_1, \ldots, e_n}_{\text{basis}}\}$

$V^* =$ set of all linear maps $f: V \to \mathbb{R}$.

e.g. $V = \mathbb{R}^n$
$f(x_1, \ldots, x_n) = x_1$, $f \in V^*$.
$f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. $f \in V^*$.
| $\vec{x}_0 \in \mathbb{R}^n$ fixed. $f(\vec{x}) = \vec{x} \cdot \vec{x}_0$, $f \in V^*$.

e.g. $V = C[0,1]$. (dim $= \infty$)
$I(f) = \int_0^1 f(x)\,dx$, $I \in C[0,1]^*$.
$g \in C[0,1]$ fixed. $J(f) = \int_0^1 f(x) g(x)\,dx$, $J \in C[0,1]^*$.

V basis $\{e_1, \ldots, e_n\}$.

$e_i^* \in V^*$. $e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$e_2^*(e_1) = 0$
$e_2^*(e_2) = 1$
$e_2^*(6e_1 + 8e_2 + 9e_3) = 8$.

$\underbrace{\{e_1^*, e_2^*, \ldots, e_n^*\}}_{\text{dual basis for } V^*}$ is a basis for V^*. (Exercise)

$T_pM = \text{span}\left\{\frac{\partial}{\partial u_1}(p), \ldots, \frac{\partial}{\partial u_n}(p)\right\}$.

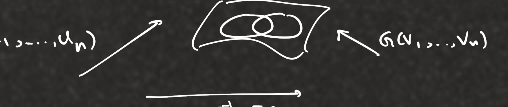
T_p^*M = dual space of T_pM.
$\uparrow$ cotangent space at $p \in M$.

$T_p^*M = \text{span}\{\underbrace{du^1|_p, \ldots, du^n|_p}_{\text{dual basis}}\}$

$-dt^2 + dx^2 + dy^2 + dz^2$
$-dt \otimes dt$

$du^i\left(\frac{\partial}{\partial u_j}\right) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

e.g. $\mathbb{R}^4 (t, x, y, z)$
$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$
$dt\left(\frac{\partial}{\partial t}\right) = 1$, $dt\left(\frac{\partial}{\partial x}\right) = 0 \ldots$



$(v_1, \ldots, v_n) = G^{-1} \circ F(u_1, \ldots, u_n)$

$\frac{\partial}{\partial v_i} = \sum_j \frac{\partial u_j}{\partial v_i} \frac{\partial}{\partial u_j}$ ✓

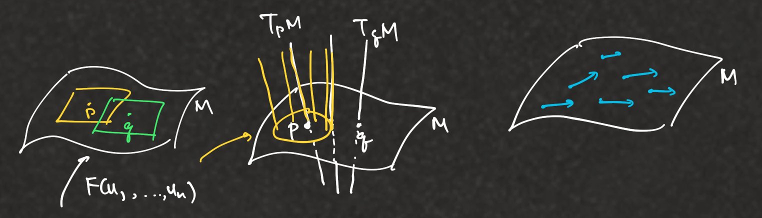
$dv^i\left(\frac{\partial}{\partial u_j}\right)$
$= dv^i\left(\sum_{k=1}^n \frac{\partial v_k}{\partial u_j} \frac{\partial}{\partial v_k}\right)$
$= \sum_{k=1}^n \frac{\partial v_k}{\partial u_j} \delta_{ik} = \frac{\partial v_i}{\partial u_j}$

$\Rightarrow \boxed{dv^i = \sum_j \frac{\partial v_i}{\partial u_j} du^j}$

V , V^*
$\alpha \in V^*$
Known:
$\alpha(e_i) = a_i$
$\Rightarrow \underbrace{\alpha = \sum_{i=1}^n a_i e_i^*}$
$\alpha(e_j) = \sum_{i=1}^n a_i \delta_{ij} = a_j$

e.g. $\mathbb{R}^2$
(x, y) , (r, θ)
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\,dr - r\sin\theta\,d\theta$
$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta\,dr + r\cos\theta\,d\theta$.

tangent bundle.
TM = disjoint union of all tangent spaces of M.
$= \bigcup_{p \in M} (\{p\} \times T_pM) = \{(p, V_p) : p \in M, V_p \in T_pM\}$.



$\tilde{F}(\underbrace{u_1, \ldots, u_n, a^1, \ldots, a^n}_{2n}) = (\underbrace{F(u_1, \ldots, u_n)}_{\text{in } M}, a^1 \frac{\partial}{\partial u_1} + a^2 \frac{\partial}{\partial u_2} + \cdots + a^n \frac{\partial}{\partial u_n})$

TM is a smooth $2n$-manifold.

V is a vector field on M
$\overset{\text{def}}{\Longleftrightarrow}$ $V: M \to TM$ and $V(p) \in \{p\} \times T_pM$ $\forall p \in M$.
V is a C^k (or C^∞) vector field on M $\overset{\text{def}}{\Longleftrightarrow}$ $V: M \to TM$ is C^k (or C^∞).

$\Leftrightarrow \underbrace{\tilde{F}^{-1} \circ V \circ F}$ is C^k (or C^∞) $\Leftrightarrow a^i(u_1, \ldots, u_n)$ are C^k (C^∞) $\forall i$.

$\tilde{F}^{-1} \circ V \circ F(u_1, \ldots, u_n)$
$= \tilde{F}^{-1}(F(u_1, \ldots, u_n), \sum_{i=1}^n a^i \frac{\partial}{\partial u_i})$
$= (u_1, \ldots, u_n, a^1, \ldots, a^n)$.

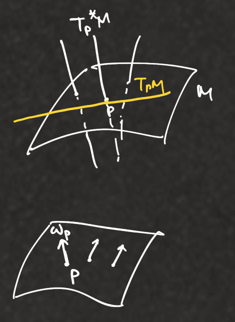
$V(F(u_1, \ldots, u_n))$
$= (\underbrace{F(u_1, \ldots, u_n)}_{p}, \sum_{i=1}^n a^i \frac{\partial}{\partial u_i})$

$a^i(u_1, \ldots, u_n)$ $\quad a^i \frac{\partial}{\partial u_i}$

e.g. $V = (x^2 + y^2) \frac{\partial}{\partial x} + (y + x) \frac{\partial}{\partial y}$ $(\mathbb{R}^2)$
C^∞.
$W = \sqrt{x^2 + y^2} \frac{\partial}{\partial x} + (x^2 + y^2)^{1/3} \frac{\partial}{\partial y}$ not C^1.

$du^i\left(\frac{\partial}{\partial u_j}\right)$ $\qquad \sum_i a_i du^i$

$T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$
$= \{(p, \omega_p) : p \in M, \omega_p \in T_p^*M\}$



Check: T^*M is a smooth $2n$-manifold
$\uparrow$ dim $M = n$.

$\begin{cases} \omega : M \to T^*M \\ \omega(p) \in \{p\} \times T_p^*M \end{cases}$
ω : differential 1-form.

If $\omega : M \to T^*M$ is C^k (or C^∞), then call it C^k (or C^∞) differential 1-form.

$\omega(F(u_1, \ldots, u_n)) = (F(u_1, \ldots, u_n), \sum_{i=1}^n a_i(u_1, \ldots, u_n)\, du^i)$

$\Phi_{*p} : T_pM \to T_{\Phi(p)}N$. (push-forward)

$\Phi^* : T^*N \to T^*M$ (pull-back)

$\Phi^*(\omega) : TM \to \mathbb{R}$
$\uparrow$ in T^*N

$(\Phi^*(\omega))(V) := \omega(\underbrace{\Phi_*(V)}_{\text{in } TN})$
V in TM; ω in T^*N; in TM

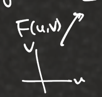
e.g. $\Phi : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$.
$\Phi(t) := (\cos t, \sin t)$.
$T\mathbb{R} = \text{span}\{\frac{\partial}{\partial t}\}$.
$\frac{\partial \Phi}{\partial t} = -\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}$

$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$

$(\Phi^*\omega)\left(\frac{\partial}{\partial t}\right) = \omega\left(\Phi_*\left(\frac{\partial}{\partial t}\right)\right)$
$= \omega\left(\frac{\partial \Phi}{\partial t}\right) = \omega\left(-\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}\right)$ $\qquad \alpha(e_j) = a_j$
$= \left(-\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy\right)\left(-\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}\right)$ $\qquad \Rightarrow \alpha = \sum_j a_j e_j^*$

$= +\frac{y \sin t}{x^2 + y^2} + \frac{x \cos t}{x^2 + y^2} = 1 \quad \Rightarrow \Phi^*\omega = dt$.

e.g. $\Sigma^2 \subset \mathbb{R}^3_{(x,y,z)}$ regular surface.



$\iota : \Sigma \to \mathbb{R}^3$ inclusion
$p \mapsto p$.

$F(u,v) = (x(u,v), y(u,v), z(u,v))$

$\iota^* dx \leftarrow$ in $T^*\Sigma$. $= \text{span}\{du, dv\}$.
$T^*\mathbb{R}^3 = \text{span}\{dx, dy, dz\}$

$(\iota^* dx)\left(\frac{\partial F}{\partial u}\right) = dx\left(\iota_* \frac{\partial F}{\partial u}\right) = dx\left(\frac{\partial F}{\partial u}\right) = dx\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)$
$= \frac{\partial x}{\partial u}$
$(\iota^* dx)\left(\frac{\partial F}{\partial v}\right) = \frac{\partial x}{\partial v}$
$\Rightarrow \iota^* dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & N \\ F \nearrow & & \nwarrow G \\ (u_1, \ldots, u_m) & \xrightarrow{G^{-1} \circ \Phi \circ F} & (v_1, \ldots, v_n) \end{array}$$

$(v_1, \ldots, v_n) = G^{-1} \circ \Phi \circ F(u_1, \ldots, u_m)$

$\frac{\partial (v_1, \ldots, v_n)}{\partial u_j} = \left(\frac{\partial v_1}{\partial u_j}, \ldots, \frac{\partial v_n}{\partial u_j}\right)$

$(\Phi^* dv^i)\left(\frac{\partial}{\partial u_j}\right) = dv^i\left(\Phi_* \frac{\partial}{\partial u_j}\right)$
$= dv^i\left(\frac{\partial \Phi}{\partial u_j}\right)$
$= dv^i\left(\frac{\partial v_1}{\partial u_j}\frac{\partial}{\partial v_1} + \cdots + \frac{\partial v_n}{\partial u_j}\frac{\partial}{\partial v_n}\right)$
$= \frac{\partial v_i}{\partial u_j}$

$\Rightarrow \Phi^* dv^i = \sum_{j=1}^m \frac{\partial v_i}{\partial u_j} du^j$

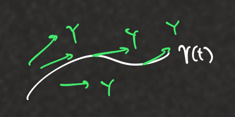
$[\Phi^*] = \left[\Phi^* dv^1 \cdots \Phi^* dv^n\right]_m^n = \begin{bmatrix} \frac{\partial v_1}{\partial u_1} & \cdots & \frac{\partial v_n}{\partial u_1} \\ \vdots & & \vdots \\ \frac{\partial v_1}{\partial u_m} & \cdots & \frac{\partial v_n}{\partial u_m} \end{bmatrix} \begin{array}{l} \leftarrow \frac{\partial}{\partial u_1} \\ \\ \leftarrow \frac{\partial}{\partial u_m} \end{array}$

$(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$ $\qquad = \left(\frac{\partial (v_1, \ldots, v_n)}{\partial (u_1, \ldots, u_m)}\right)^T$

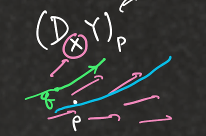
$|\psi\rangle \qquad \langle \psi|$
$|\psi\rangle = \sum_i \langle e_i | \psi\rangle |e_i\rangle$
$\langle \psi| = \sum_i \langle \psi | e_i\rangle \langle e_i|$
$T = \sum_i T(e_i) e_i^*$.

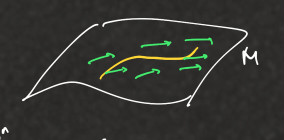
Lie derivatives

In $\mathbb{R}^n$,
$\gamma(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$
$Y : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$.



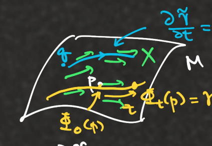
$D_{\gamma'(t)} Y = \frac{d}{dt} Y \circ \gamma$ $\qquad D_{\gamma'(t)} f = \frac{d}{dt} f \circ \gamma$
$= \lim_{h \to 0} \frac{Y(\gamma(t+h)) - Y(\gamma(t))}{h}$





If X is C^1 on $\mathbb{R}^n$ then given any $p \in \mathbb{R}^n$.
$\exists \gamma : I \to \mathbb{R}^n$ s.t.
$\gamma'(t) = X(\gamma(t))$ and $\gamma(0) = p$.
(existence theorem of ODE). MATH 4051.

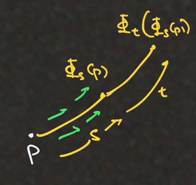
$(D_X Y)_p := (D_{\gamma'(t)} Y)_p$.



$\frac{\partial \tilde{\gamma}}{\partial t} = X(\tilde{\gamma}(t))$, $\tilde{\gamma}(0) = q$.
$\Phi_t(q)$

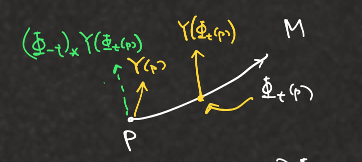
$\frac{\partial \gamma}{\partial t} = X(\gamma(t))$ and $\gamma(0) = p$.
$\Phi_t(p) := \gamma(t)$

$\boxed{\Phi_t(\Phi_s(p)) = \Phi_{t+s}(p)}$



Cor: $\Phi_t(\Phi_{-t}(p)) = \Phi_0(p) = p$
$\Phi_t \circ \Phi_{-t} = \text{id}$.

$(\mathcal{L}_X Y)_p$
$:= \frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_* (Y_{\Phi_t(p)})$



$\frac{\partial \Phi_t}{\partial t} = X \circ \Phi_t$.

Prop. 3.23 $\mathcal{L}_X Y = [X, Y]$. where X, Y are vector fields.

$X(Y(f)) - Y(X(f)) =: [X, Y]f$
$[X, Y] := XY - YX$.

$X = \sum_i X^i \frac{\partial}{\partial u_i}$, $Y = \sum_j Y^j \frac{\partial}{\partial u_j}$
$Y(f) = \sum_j Y^j \frac{\partial f}{\partial u_j}$
$X(Y(f)) = \sum_i X^i \frac{\partial}{\partial u_i}\left(\sum_j Y^j \frac{\partial f}{\partial u_j}\right) = \sum_{i,j} X^i \left(\frac{\partial Y^j}{\partial u_i}\frac{\partial f}{\partial u_j} + Y^j \frac{\partial^2 f}{\partial u_i \partial u_j}\right)$
$Y(X(f)) = \quad - \quad - \quad = \sum_{i,j} Y^i \left(\frac{\partial X^j}{\partial u_i}\frac{\partial f}{\partial u_j} + X^j \frac{\partial^2 f}{\partial u_i \partial u_j}\right)$

$[X, Y] = XY - YX \quad = \sum_{i,j}\left(X^i \frac{\partial Y^j}{\partial u_i} - Y^i \frac{\partial X^j}{\partial u_i}\right)\frac{\partial}{\partial u_j}$

$\left[\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right] = 0$.

Sketch of proof of $\mathcal{L}_X Y = [X, Y]$. $\qquad F(u_1, \ldots, u_n)$

$(\mathcal{L}_X Y)_p := \frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_* (Y_{\Phi_t(p)})$ $\qquad \frac{\partial \Phi_t(p)}{\partial t} = X(\Phi_t(p))$

$F^{-1} \circ \Phi_t \circ F(u_1, \ldots, u_n)$ $\qquad X = \sum_i X^i \frac{\partial}{\partial u_i}$, $Y = \sum_j Y^j \frac{\partial}{\partial u_j}$
$= (v_t^1, \ldots, v_t^n)$
$\frac{\partial \Phi_t}{\partial t} = \sum_i \frac{\partial v_t^i}{\partial t} \frac{\partial}{\partial u_i}(\Phi_t(p))$
$= \sum_i X^i \frac{\partial}{\partial u_i}$

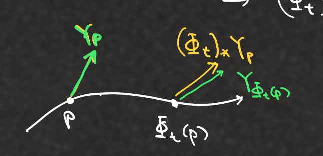
$Y^j(\Phi_t(p))$
$(\Phi_{-t})_*(Y_{\Phi_t(p)}) = (\Phi_{-t})_*\left(\sum_j Y^j \frac{\partial}{\partial u_j}\right)$
$= \sum_j Y^j \frac{\partial (\Phi_{-t})}{\partial u_j}$
$= \sum_j Y^j \left(\sum_i \frac{\partial v_{-t}^i}{\partial u_j} \frac{\partial}{\partial u_i}\right)$

$F^{-1} \circ \Phi_{-t} \circ F(u_1, \ldots, u_n)$
$= (v_{-t}^1, \ldots, v_{-t}^n)$

Next: $\frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_*(Y_{\Phi_t(p)}) = \; \cdot \; - \; -$

$(\mathcal{L}_X Y)_p := \frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_*(Y_{\Phi_t(p)}) \overset{\text{prop}}{=} [X, Y]_p$.

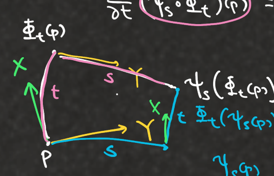
$\Phi_0 = \text{id}$
If $\underline{[X, Y] \equiv 0}$, then $(\Phi_{-t})_*(Y_{\Phi_t(p)}) = (\Phi_0)_*(Y_{\Phi_0(p)})$
$\forall t \qquad = Y_p$
$\Rightarrow (\Phi_t)_*((\Phi_{-t})_* Y_{\Phi_t(p)}) = (\Phi_t)_* Y_p$
$Y_{\Phi_t(p)} = (\Phi_t)_* Y_p$



$\frac{\partial \Phi_t(p)}{\partial t} = X(\Phi_t(p))$
$\frac{\partial \Psi_s(p)}{\partial s} = Y(\Psi_s(p))$

If $[X, Y] \equiv 0$. then $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$.

Proof: $\frac{\partial}{\partial t}(\Phi_t \circ \Psi_s)(p) = \frac{\partial}{\partial t}\Phi_t(\Psi_s(p)) = X(\Phi_t(\Psi_s(p)))$
$\frac{\partial}{\partial t}(\Psi_s \circ \Phi_t)(p) = \; - \; \cdots = X(\Psi_s(\Phi_t(p)))$



$F(s, t) := \Phi_t \circ \Psi_s(p)$
$= \Psi_s \circ \Phi_t(p)$.

$\frac{\partial F}{\partial s} = Y$
$\frac{\partial F}{\partial t} = X$

$\mathcal{L}_X \alpha$ $\leftarrow$ 1-form
X: $\nwarrow$ tangent vector field

$:= \frac{d}{dt}\Big|_{t=0} \underbrace{\Phi_t^* \alpha_{\Phi_t(p)}}_{\text{in } T_p^*M}$