If $\Phi: M \to N$ is a diffeomorphism,
then $(\Phi_*)_p$ is invertible.

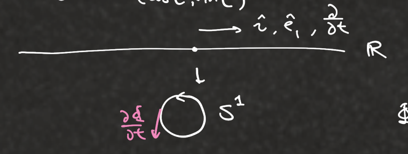
Q: Converse ?

A: No!

$\Phi: \mathbb{R} \to \mathbb{S}^1$, t

$t \mapsto (\cos t, \sin t)$

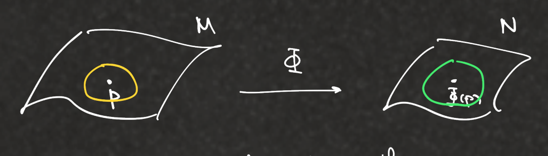
$\Phi_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial \Phi}{\partial t} = (\underline{-\sin t, \cos t}) \neq 0.$



$[\Phi_*] \neq 0.$

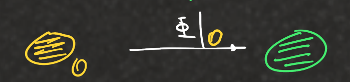
1×1

Φ_* invertible.



Φ is a local diffeomorphism if

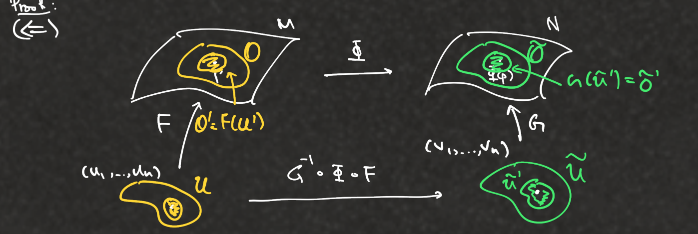
$\forall p \in M$, $\exists\ \mathcal{O} \ni p$ open set in M,
$\tilde{\mathcal{O}} \ni \Phi(p)$ open set in N,

s.t.  is diffeomorphism.

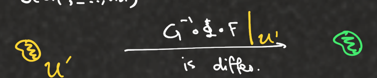
Inverse Function Theorem:

$\Phi: M \to N$ is a local diffeomorphism near $p \in M$
$\Leftrightarrow$ $(\Phi_*)_p: T_pM \to T_{\Phi(p)}N$ is invertible ←

Proof: (⇐)



$$[(\Phi_*)_p] = \frac{\partial(v_1, \dots, v_n)}{\partial(u_1, \dots, u_n)} = D(G^{-1}\circ\Phi\circ F) \xrightarrow{\text{given}} \det D(G^{-1}\circ\Phi\circ F) \neq 0.$$



Claim: $\Phi|_{\mathcal{O}'}: \mathcal{O}' \to \tilde{\mathcal{O}}'$ is diffeo.

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e.g. $\Phi: \mathbb{S}^2 \to \mathbb{RP}^2$

$\Phi(x_1, x_2, x_3) = [x_1 : x_2 : x_3]$

Claim: Φ is a local diffeo.

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "\Phi"] & \mathbb{RP}^2 \\
(u_1, u_2, \sqrt{1-u_1^2-u_2^2}) \arrow[u, "F(u_1,u_2)"] & G(v_1,v_2) = [v_1 : v_2 : 1] \arrow[u]
\end{tikzcd}$$

$$G^{-1}\circ\Phi\circ F(u_1,u_2) = G^{-1}\circ\Phi\left(u_1, u_2, \sqrt{1-u_1^2-u_2^2}\right)$$
$$= G^{-1}\left(\left[u_1 : u_2 : \sqrt{1-u_1^2-u_2^2}\right]\right)$$
$$= G^{-1}\left(\left[\frac{u_1}{\sqrt{1-u_1^2-u_2^2}} : \frac{u_2}{\sqrt{1-u_1^2-u_2^2}} : 1\right]\right)$$

$$(v_1, v_2) = \left(\frac{u_1}{\sqrt{1-u_1^2-u_2^2}}, \frac{u_2}{\sqrt{1-u_1^2-u_2^2}}\right)$$

$$\det \frac{\partial(v_1, v_2)}{\partial(u_1, u_2)} = \frac{1}{(1-u_1^2-u_2^2)^2} \neq 0.$$ ← exercise.

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} m \qquad \longleftrightarrow \qquad T(x) = Ax.$$

n ↓

$$A\vec{x} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + \dots - x_n\vec{v}_n$$

T is 1-1
$\Leftrightarrow$ Ker(T) = null(A) = {0}.
$\Leftrightarrow$ RREF(A) has no 'free' col.

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A\vec{x} = \vec{0} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 = 0, \quad x_2 = 0$$

$$\text{RREF}(A) = \begin{bmatrix} ① & 0 & 6 & 8 & 9 \\ 0 & ① & 7 & 7 & 7 \end{bmatrix}$$

$x_1 + 6x_3 + 8x_4 + 9x_5 = 0$

$x_2 + 7x_3 + 7x_4 + 7x_5 = 0$

$T(x) = Ax$ is NOT 1-1.

$T(x) = Ax$ is onto $\Leftrightarrow$ $A\vec{x} = \vec{b}$ has solution for any $\vec{b}$.

$T: \mathbb{R}^n \to \mathbb{R}^m$ $\Leftrightarrow$ $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^m$.

$$A = \begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix} \qquad \dim \operatorname{span}\{v_1, \dots, v_5\} = 3$$

$$\text{RREF}(A) = \begin{bmatrix} ① & 5 & 0 & 0 & 6 \\ 0 & 0 & ① & 0 & 8 \\ 0 & 0 & 0 & ① & 9 \end{bmatrix}$$

Linearly indep.

$\Phi: M \to N$

- Φ is an immersion at $p \in M$
  $\overset{\text{def}}{\Leftrightarrow}$ $(\Phi_*)_p$ is 1-1 (injective)
- Φ is a submersion at $p \in M$
  $\overset{\text{def}}{\Leftrightarrow}$ $(\Phi_*)_p$ is onto (surjective)

$\Sigma \subset \mathbb{R}^3$ regular surface

Claim: $\iota: \Sigma \to \mathbb{R}^3$ is an immersion. $\quad \iota$ = iota \iota
$x \mapsto x$

$$\begin{tikzcd}
\Sigma \arrow[r, "\iota"] & \mathbb{R}^3 \\
\arrow[u, "F(u_1,u_2)"] & \mathbb{R}^3 \arrow[u, "id"']
\end{tikzcd}$$

$= (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$

$$id^{-1}\circ\iota\circ F(u_1,u_2) = id^{-1}\circ\iota(x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$$
$$= (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$$

$$[\iota_*] = \frac{\partial(x,y,z)}{\partial(u_1,u_2)} = \begin{bmatrix} x_{u_1} & x_{u_2} \\ y_{u_1} & y_{u_2} \\ z_{u_1} & z_{u_2} \end{bmatrix} \sim \begin{bmatrix} ① & 0 \\ 0 & ① \\ 0 & 0 \end{bmatrix} \cdots \qquad \frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} \neq 0. \quad + \frac{\partial(x,y)}{\partial(u_1,u_2)}\hat{k} \neq 0.$$

e.g. $\pi: M \times N \to M$
$(p,q) \mapsto p$

$$\begin{tikzcd}
M \times N \arrow[r, "\pi"] & M \\
(u_1, \dots, u_m, v_1, \dots, v_n) \arrow[u, "F\times G"] & \arrow[u, "F"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& M \\
(u_1, \dots, u_m) \arrow[ur, "F"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
N \\
(v_1, \dots, v_n) \arrow[u, "G"']
\end{tikzcd}$$

$$F^{-1}\circ\pi\circ(F\times G)(u_1, \dots, u_m, v_1, \dots, v_n) = (u_1, \dots, u_m)$$

$$[\pi_*] = \frac{\partial(u_1, \dots, u_m)}{\partial(u_1, \dots, u_m, v_1, \dots, v_n)} = \begin{bmatrix} \frac{\partial u_i}{\partial u_j} & \cancel{\frac{\partial u_i}{\partial v_j}} \end{bmatrix}\begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix}$$
$u_1 \cdots u_m \ v_1 \cdots v_n$

$$= \begin{bmatrix} ① & & & \\ & \ddots & & 0 \\ & & ① & \end{bmatrix}$$

$(\pi_*)_{(p,q)}$ is surjective
$\Rightarrow$ π is a submersion at $(p,q) \in M \times N$.

e.g. $\Phi: \mathbb{C}^2\setminus\{0\} \to \mathbb{CP}^1$.
$(x_1+iy_1, x_2+iy_2) \mapsto [x_1+iy_1 : x_2+iy_2]$.

$$\begin{tikzcd}
\mathbb{C}^2\setminus\{0\} \arrow[r, "\Phi"] & \mathbb{CP}^1 \\
(x_1, y_1, x_2, y_2) \arrow[u, "F"] & (u_1, u_2) \arrow[u, "{G(u_1,u_2) = [1 : u_1+iu_2]}"']
\end{tikzcd}$$

$$G^{-1}\circ\Phi\circ F(x_1, y_1, x_2, y_2) = G^{-1}([x_1+iy_1 : x_2+iy_2])$$
$$= G^{-1}\left(\left[1 : \frac{x_2+y_2 i}{x_1+y_1 i}\right]\right)$$
$$= \left(\underbrace{\operatorname{Re}\left(\frac{x_2+y_2 i}{x_1+y_1 i}\right)}_{u_1}, \underbrace{\operatorname{Im}\left(\frac{x_2+y_2 i}{x_1+y_1 i}\right)}_{u_2}\right)$$

Check:

$$D(G^{-1}\circ\Phi\circ F) = \frac{\partial(u_1, u_2)}{\partial(x_1, y_1, x_2, y_2)}$$
$$= \begin{bmatrix} \operatorname{Re}(A) & -\operatorname{Im}(A) & \operatorname{Re}(B) & -\operatorname{Im}(B) \\ \operatorname{Im}(A) & \operatorname{Re}(A) & \operatorname{Im}(B) & \operatorname{Re}(B) \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}$$

det = $|A|^2$

where $A, B \neq 0$ in $\mathbb{C}$.

---

e.g. $f: \mathbb{R}^3 \to \mathbb{R}$

$\Sigma = f^{-1}(c) = \{(x,y,z) : f(x,y,z) = c\} \neq \phi.$

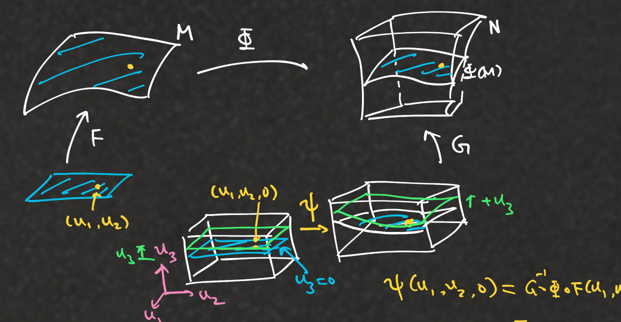
$\nabla f \neq 0 \ \ \forall p \in \Sigma$ $\Rightarrow$ Σ is a regular surface.
⇕
f is a submersion at $p \in \Sigma$.

$$[f_*] = \frac{\partial(f)}{\partial(x,y,z)} = \left[\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y} \ \frac{\partial f}{\partial z}\right] \quad f(x,y,z)$$

Echolen form

$\nabla f \neq 0$ non-zero row.

Immersion Theorem: (2.42)

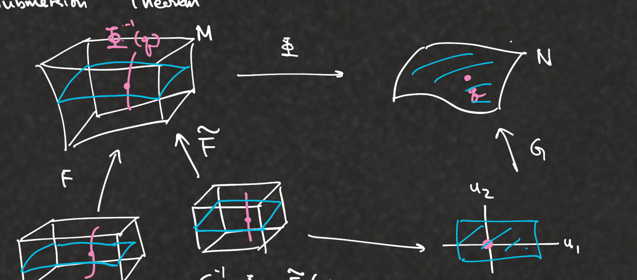


$\psi(u_1, u_2, 0) = G^{-1}\circ\Phi\circ F(u_1, u_2)$

$$\psi(u_1, u_2, u_3) = G^{-1}\circ\Phi\circ F(u_1, u_2) + (0, 0, u_3) \longrightarrow [\psi_*] = \left[\Phi_* \ \middle|\ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right]$$

$$\tilde{G} := G\circ\psi \Rightarrow \tilde{G}^{-1}\circ\Phi\circ F(u_1, u_2) = (G\circ\psi)^{-1}\circ\Phi\circ F(u_1, u_2)$$
$$= \psi^{-1}\circ G^{-1}\circ\Phi\circ F(u_1, u_2)$$
$$= \psi^{-1}\circ\psi(u_1, u_2, 0) = (u_1, u_2, 0)$$

Submersion Theorem



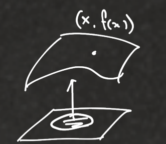
$$G^{-1}\circ\Phi\circ\tilde{F}(u_1, u_2, u_3) = (u_1, u_2)$$

Submanifold

$N \subset M$. smooth manifolds $\qquad N$ is a submanifold of M
$\overset{\text{def}}{\Leftrightarrow}$ $\iota: N \to M$ is an immersion.

e.g. $\Phi: M \to N$ any smooth map.

$\Gamma_\Phi := \{(p, \Phi(p)) : p \in M\} \subset M \times N.$



Claim: Γ_Φ is a submanifold of $M \times N$.

$$\begin{tikzcd}
\Gamma_\Phi \arrow[r, "\iota"] & M \times N \\
\arrow[u, "{H(u_1,\dots,u_m)}"] & \arrow[u, "F\times G"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& M & N \\
F(u_1,\dots,u_m) \arrow[ur] & & G(v_1,\dots,v_n) \arrow[u]
\end{tikzcd}$$

$= (F(u_1, \dots, u_m), \Phi\circ F(u_1, \dots, u_m))$

$$(F\times G)^{-1}\circ\iota\circ H(u_1, \dots, u_m)$$
$$= (F\times G)^{-1}(F(u_1, \dots, u_m), \Phi\circ F(u_1, \dots, u_m)) = (u_1, \dots, u_m, G^{-1}\circ\Phi\circ F(u_1, \dots, u_m))$$

$$D((F\times G)^{-1}\circ\iota\circ H)$$
$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \hline & * & \end{bmatrix} m+n \xrightarrow{\text{row reduction}} \begin{bmatrix} ① & & \\ & \ddots & \\ & & ① \\ \hline & 0 & \end{bmatrix}$$
m

$[\iota_*] = D((F\times G)^{-1}\circ\iota\circ H)$ $\Rightarrow$ ι_* is injective. $\quad\therefore$ Γ_Φ is a submfd of $M \times N$.

$$\begin{tikzcd}
z = \sqrt{x^2+y^2} & \mathbb{R}^3 \arrow[l, "\iota"'] \\
F(x,y) = (x, y, \sqrt{x^2+y^2}) \arrow[u] & \mathbb{R}^3 \arrow[u, "id"]
\end{tikzcd}$$

$G \quad G(x_1, x_2, x_3) = (x_1, x_2, x_3 + \sqrt{x_1^2+x_2^2})$

$G^{-1}\circ\iota\circ F(x,y)$
$= G^{-1}\circ\iota(x, y, \sqrt{x^2+y^2})$
$= (x, y, 0)$

$D(G^{-1}\circ\iota\circ F)$
$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ ✓

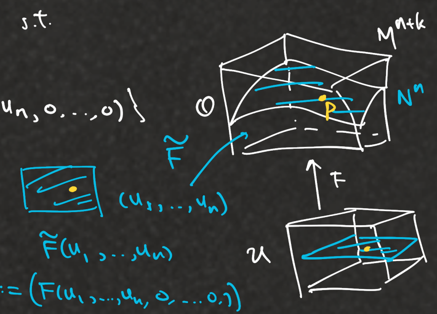
$$\underline{id^{-1}\circ\iota\circ F(x,y)} = id^{-1}\circ\iota(x, y, \sqrt{x^2+y^2})$$
$$= (x, y, \sqrt{x^2+y^2}).$$
↑
not C^∞ at (0,0).

ι is not an immersion at (0,0,0) ⇒  is NOT a submfd of $\mathbb{R}^3$.

Immersion Thm

⇒ If $N \subset M$ submanifold, then at every $p \in N$
$\exists\ F: \mathcal{U} \to \underset{\substack{\cup \\ p}}{\mathcal{O}} \subset M$ s.t.

$$N \cap \mathcal{O} = \{F(u_1, \dots, u_n, 0, \dots, 0)\}$$

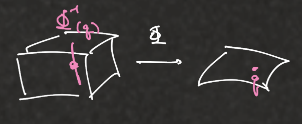


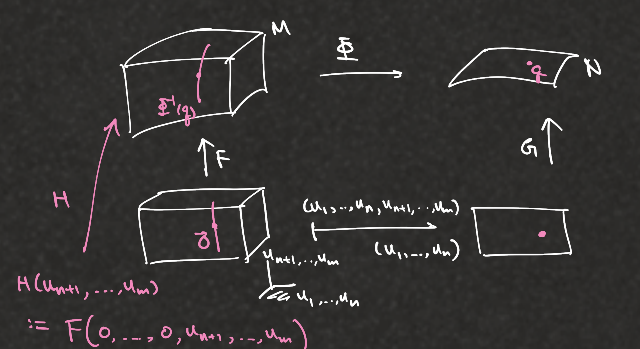
$\tilde{F}(u_1, \dots, u_n)$
$:= (F(u_1, \dots, u_n, 0, \dots, 0))$

Regular value Theorem (2.52)

$\Phi: M^m \to N^n$, $q \in N$. $\underline{\Phi^{-1}(q)} \neq \phi$.
If Φ is a submersion at every $p \in \Phi^{-1}(q)$,
then $\Phi^{-1}(q)$ is a submanifold of M
with dim = dim M − dim N.

e.g. $f: \mathbb{R}^{③} \to \mathbb{R}^1$
f submersion on $f^{-1}(c)$
$\Leftrightarrow$ $\nabla f \neq 0$ on $f^{-1}(c)$.
$f^{-1}(c)$ regular surface (dim = 3 − 1 = 2).





$H(u_{n+1}, \dots, u_m)$
$:= F(0, \dots, 0, u_{n+1}, \dots, u_m)$

$$\begin{tikzcd}
\Phi^{-1}(q) \arrow[r, "\iota"] & M \\
\arrow[u, "H"] & \arrow[u, "F"']
\end{tikzcd}$$

$$F^{-1}\circ\iota\circ H(u_{n+1}, \dots, u_m)$$
$$= F^{-1}\circ(F(0, \dots, 0, u_{n+1}, \dots, u_m))$$
$$= (0, \dots, 0, u_{n+1}, \dots, u_m)$$

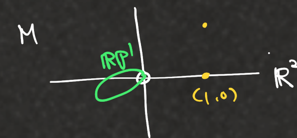
e.g. $M = \{((x_1, x_2), [y_1 : y_2]) \in \mathbb{R}^2 \times \mathbb{RP}^1 : x_1 y_2 = x_2 y_1\}$

$= \{((x_1, x_2), [y_1 : y_2]) : \underline{(x_1, x_2) \neq (0,0)}, x_1 y_2 = x_2 y_1\}$
$\amalg \{((0,0), [y_1 : y_2])\}$

if $x_1 \neq 0$: $y_2 = \frac{x_2}{x_1} y_1$ ⇒ $[y_1 : y_2] = [\cancel{y_1} : \frac{x_2}{x_1}\cancel{y_1}] = [x_1 : x_2]$

if $x_2 \neq 0$: $[y_1 : y_2] = [x_1 : x_2]$

$= \{((x_1, x_2), [x_1 : x_2]) : (x_1, x_2) \neq (0,0)\}$ ↔ $\mathbb{R}^2\setminus 0$
$\amalg \{((0,0), [y_1 : y_2]) : [y_1 : y_2] \in \mathbb{RP}^1\}$ ↔ $\mathbb{RP}^1$



= $\mathbb{R}^2$ blown up at a point.

$\pi((x_1, x_2), [y_1 : y_2]) = (x_1, x_2).$
$\pi: M \to \mathbb{R}^2$, $\pi^{-1}(0,0) \cong \mathbb{RP}^1$
$\pi^{-1}(1,0) = \{((1,0), [1:0])\}.$

G group, C^∞ smooth

$\mu: G \times G \to G$, $(g, h) \mapsto g * h$ $\qquad \nu: G \to G$, $g \mapsto g^{-1}$

G = Lie group $\overset{\text{def}}{\Leftrightarrow}$ μ, ν are C^∞.

GL(n, R), O(n), SO(n), …

Lie(G) $:= T_e G$. Lie algebra of G