MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #1 • Regular Surfaces • Due Date: 25/02/2018, 11:59PM (Optional)

Remark: *All assignments are optional but you are strongly recommended to work on them to keep yourself on track. You are welcome to submit any part of your homework to the Canvas system by the deadline. Follow the instructions posted on Canvas. The instructor and TA will give you some feedback as soon as possible.*

1. Consider a smooth map $\mathsf{F}(u,v) : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ where $\mathcal{U}$ is an open set. Denote:

$$\mathsf{F}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

Show that the following are equivalent:

(a) $\dfrac{\partial \mathsf{F}}{\partial u} \times \dfrac{\partial \mathsf{F}}{\partial v} \neq 0$ for any $(u,v) \in \mathcal{U}$.

(b) $\left\{\dfrac{\partial \mathsf{F}}{\partial u}, \dfrac{\partial \mathsf{F}}{\partial v}\right\}$ are linearly independent for any $(u,v) \in \mathcal{U}$.

(c) The Jacobian matrix:

$$\frac{\partial(x,y,z)}{\partial(u,v)} := \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$

has a trivial null-space for any $(u,v) \in \mathcal{U}$.

(d) For any $(u,v) \in \mathcal{U}$, at least one of the following Jacobian matrices is invertible:

$$\frac{\partial(x,y)}{\partial(u,v)} \quad \frac{\partial(y,z)}{\partial(u,v)} \quad \frac{\partial(z,x)}{\partial(u,v)}$$

(e) The matrix:

$$[g] := \begin{bmatrix} \mathsf{F}_u \cdot \mathsf{F}_u & \mathsf{F}_u \cdot \mathsf{F}_v \\ \mathsf{F}_v \cdot \mathsf{F}_u & \mathsf{F}_v \cdot \mathsf{F}_v \end{bmatrix}$$

is positive definite[1] for any $(u,v) \in \mathcal{U}$.

Solution: We will show (a) $\iff$ (e), and then (a) $\implies$ (b) $\iff$(c) $\implies$ (d) $\implies$ (a).
Denote θ to be the angle between $\frac{\partial \mathsf{F}}{\partial u}$ and $\frac{\partial \mathsf{F}}{\partial v}$. To show (a) $\iff$ (e), we observe that:

$$\begin{aligned} \det[g] &= \left|\frac{\partial \mathsf{F}}{\partial u}\right|^2 \left|\frac{\partial \mathsf{F}}{\partial v}\right|^2 - \left(\frac{\partial \mathsf{F}}{\partial u} \cdot \frac{\partial \mathsf{F}}{\partial v}\right)^2 = \left|\frac{\partial \mathsf{F}}{\partial u}\right|^2 \left|\frac{\partial \mathsf{F}}{\partial v}\right|^2 - \left|\frac{\partial \mathsf{F}}{\partial u}\right|^2 \left|\frac{\partial \mathsf{F}}{\partial v}\right|^2 \cos^2\theta \\ &= \left|\frac{\partial \mathsf{F}}{\partial u}\right|^2 \left|\frac{\partial \mathsf{F}}{\partial v}\right|^2 \sin^2\theta = \left|\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}\right|^2 \end{aligned}$$

Therefore, $\mathsf{F}_u \times \mathsf{F}_v \neq 0$ at (u,v) if and only if $\det[g] > 0$ at (u,v). Furthermore, $[g]$ is symmetric so its eigenvalues λ_1 and λ_2 must be real. Since $\mathrm{Tr}[g] = |\mathsf{F}_u|^2 + |\mathsf{F}_v|^2 \geq 0$, so we always have $\lambda_1 + \lambda_2 \geq 0$. As a result, we know that $\det[g] = \lambda_1\lambda_2 > 0$ if and only if $\lambda_1 > 0$ and $\lambda_2 > 0$. It proves (a) $\iff$ (e).

(a) $\implies$ (b): If $\mathsf{F}_u \times \mathsf{F}_v \neq 0$, then the angle θ between F_u and F_v is neither 0 nor π, and that F_u and F_v are non-zero vectors. Therefore $\{\mathsf{F}_u, \mathsf{F}_v\}$ are non-zero, non-parallel vectors, and equivalently they are linearly independent.

[1] A symmetric matrix being positive definite means all of its eigenvalues are positive.

(b) $\iff$ (c): It follows immediately from the observation that:

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff a\mathsf{F}_u + b\mathsf{F}_v = 0.$$

(c) $\implies$ (d): If (c) holds, then (b) holds as well and so $\{\mathsf{F}_u, \mathsf{F}_v\}$ span a two dimensional subspace of $\mathbb{R}^3$. Pick *any* vector $\mathsf{x} = (a, b, c)^T$ such that $\{\mathsf{F}_u, \mathsf{F}_v, \mathsf{x}\}$ are linearly independent vectors in $\mathbb{R}^3$ (for example $\mathsf{F}_u \times \mathsf{F}_v$ will do the job). Then the matrix:

$$\begin{bmatrix} x_u & x_v & a \\ y_u & y_v & b \\ z_u & z_v & c \end{bmatrix}$$

is invertible. Consider its determinant and its co-factor expansion:

$$0 \neq \det \begin{bmatrix} x_u & x_v & a \\ y_u & y_v & b \\ z_u & z_v & c \end{bmatrix} = a \det \frac{\partial(y,z)}{\partial(u,v)} + b \det \frac{\partial(z,x)}{\partial(u,v)} + c \det \frac{\partial(x,y)}{\partial(u,v)}.$$

Clearly, at least one of the determinants:

$$\det \frac{\partial(y,z)}{\partial(u,v)} \qquad \det \frac{\partial(z,x)}{\partial(u,v)} \qquad \det \frac{\partial(x,y)}{\partial(u,v)}$$

must be non-zero. It proves (d).

(d) $\implies$ (a): It follows immediately from the cross-product formula:

$$\mathsf{F}_u \times \mathsf{F}_v = \left(\det \frac{\partial(y,z)}{\partial(u,v)}, \det \frac{\partial(z,x)}{\partial(u,v)}, \det \frac{\partial(x,y)}{\partial(u,v)} \right).$$

2. Let A be a 3×3 matrix with real entries $[a_{ij}]$. Consider the set

$$\Sigma := \{\mathsf{x} \in \mathbb{R}^3 : \mathsf{x} \cdot A\mathsf{x} = 1\}.$$

Here $\mathsf{x} \in \mathbb{R}^3$ is regarded as a column vector.

(a) Show that Σ is a regular surface whenever $\Sigma \neq \emptyset$.

Solution: Define $g(\mathsf{x}) := \mathsf{x} \cdot A\mathsf{x}$ and let $\{\mathsf{e}_i\}_{i=1,2,3}$ be the standard basis in $\mathbb{R}^3$. For any $\mathsf{x}_0 \in g^{-1}(1)$, i.e. $\mathsf{x}_0 \cdot A\mathsf{x}_0 = 1$.

$$\begin{aligned} \frac{\partial}{\partial x_i} g(\mathsf{x}_0) &= \mathsf{e}_i \cdot A\mathsf{x}_0 + \mathsf{x}_0 \cdot A\mathsf{e}_i \\ &= \mathsf{e}_i \cdot A\mathsf{x}_0 + A\mathsf{e}_i \cdot \mathsf{x}_0 \\ &= \mathsf{e}_i \cdot A\mathsf{x}_0 + \mathsf{e}_i \cdot A^T\mathsf{x}_0 \\ &= \mathsf{e}_i \cdot (A + A^T)\mathsf{x} \end{aligned}$$

If $\dfrac{\partial}{\partial x_i} g(\mathsf{x}_0) = 0$ for all $i = 1, 2, 3$, then $(A + A^T)\mathsf{x}_0 = 0$ which implies

$$\mathsf{x}_0 \cdot (A + A^T)\mathsf{x}_0 = 0.$$

However,

$$\begin{aligned} \mathsf{x}_0 \cdot (A + A^T)\mathsf{x}_0 &= \mathsf{x}_0 \cdot A\mathsf{x}_0 + \mathsf{x}_0 \cdot A^T \mathsf{x}_0 \\ &= \mathsf{x}_0 \cdot A\mathsf{x}_0 + A\mathsf{x}_0 \cdot \mathsf{x}_0 \\ &= 1 + 1 \\ &= 2 \neq 0. \end{aligned}$$

Hence, $\nabla g(\mathsf{x}_0) = \left(\dfrac{\partial}{\partial x_1} g(\mathsf{x}_0), \dfrac{\partial}{\partial x_2} g(\mathsf{x}_0), \dfrac{\partial}{\partial x_3} g(\mathsf{x}_0) \right) \neq 0$. By Theorem 1.6 in the lecture note, $\Sigma = g^{-1}(1)$ is a regular surface.

(b) Suppose further that $A = P^T D P$ for some orthogonal matrix P (i.e. $P^T P = I$) and diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

i. Show that Σ is diffeomorphic to $\mathbb{S}^2$ if $\lambda_i > 0$ for all i.

Solution: Given $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, denote $\sqrt{D} = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})$. Define a linear transformation:

$$\Phi(\mathsf{x}) := \sqrt{D} P \mathsf{x}.$$

Then for any $\mathsf{x} \in \Sigma$,

$$|\sqrt{D}P\mathsf{x}|^2 = (\sqrt{D}P\mathsf{x})^T \sqrt{D}P\mathsf{x} = \mathsf{x}^T P^T \sqrt{D}\sqrt{D}P\mathsf{x} = \mathsf{x}^T P^T D P \mathsf{x} = \mathsf{x} \cdot A\mathsf{x} = 1.$$

It implies $\sqrt{D}P\mathsf{x} \in \mathbb{S}^2$. Hence Φ restricts to a map

$$\Phi : \Sigma \to \mathbb{S}^2.$$

The inverse $\Phi^{-1} : \mathbb{S}^2 \to \Sigma$ exists and it is given by $\Phi^{-1}(\mathsf{y}) = P^T \sqrt{D}^{-1} \mathsf{y}$ (one can verify as above that if $|y| = 1$, then $\Phi^{-1}(\mathsf{y}) \in \Sigma$).
Since $\Phi(\mathsf{x}) = \sqrt{D}P\mathsf{x}$ is a linear transformation in $\mathbb{R}^3$, it is a smooth map between $\mathbb{R}^3$. To show it is smooth as a map from Σ to $\mathbb{S}^2$, we observe that both Σ and $\mathbb{S}^2$ are level surfaces of functions satisfying the condition stated in Theorem 1.6. In the proof of Theorem 1.6, we know that they can be locally parametrized as graphs of smooth functions.
Therefore, for any *graphical* local parametrizations F of Σ, and G of $\mathbb{S}^2$, the composition $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ is smooth since G^{-1} is just a projection map.
A similar argument applied to $\mathsf{F}^{-1} \circ \Phi^{-1} \circ \mathsf{G}$ shows Φ^{-1} is smooth. Hence Σ is diffeomorphic to $\mathbb{S}^2$.

ii. Show that Σ is diffeomorphic to the cylinder $x^2 + y^2 = 1$ if $\lambda_1, \lambda_2 > 0$, and $\lambda_3 = 0$.

Solution: Define a linear transformation

$$L(\mathsf{x}) := \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} P\mathsf{x}.$$

We first show it restricts to a map from Σ to the cylinder $C = \{x^2 + y^2 = 1\}$. For any $\mathsf{x} \in \Sigma$, denote

$$\mathsf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} := L\mathsf{x} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} P\mathsf{x}.$$

which implies:

$$\begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} P\mathsf{x}.$$

We need to show $\mathsf{z} \in C$. Observe that:

$$\begin{aligned} z_1^2 + z_2^2 &= \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} \\ &= \left(\begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} P\mathsf{x} \right)^T \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} P\mathsf{x} \\ &= \mathsf{x}^T P^T D P \mathsf{x} = \mathsf{x}^T A \mathsf{x} = 1, \end{aligned}$$

which implies z is on the cylinder.
Clearly, L is invertible and we have

$$L^{-1} = P^T \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For any $\mathsf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ lies on the cylinder, i.e. $y_1^2 + y_2^2 = 1$,

$$\begin{aligned} &\left(L^{-1}\mathsf{y}\right) \cdot P^T D P \left(L^{-1}\mathsf{y}\right) \\ &= \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} P \left(L^{-1}\mathsf{y}\right) \cdot \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} P \left(L^{-1}\mathsf{y}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &= y_1^2 + y_2^2 \\ &= 1\ , \end{aligned}$$

which implies $L^{-1}\mathrm{y} \in \Sigma$.
Since L is a linear transformation, L is smooth as a map between $\mathbb{R}^3$. By the same reason as in (b)(i), it restricts to a smooth map between Σ and C. Similarly for L^{-1}. Hence Σ is diffeomorphic to the cylinder $x^2 + y^2 = 1$.

3. Let $\mathbb{S}^2$ be the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Suppose $f : \mathbb{S}^2 \to (0, \infty)$ is a smooth, positive-valued function. Consider the set Σ defined by:

$$\Sigma := \{f(\mathrm{x})\,\mathrm{x} : \mathrm{x} \in \mathbb{S}^2\}.$$

(a) Suppose $\mathsf{F}(u, v) : \mathcal{U} \to \mathbb{S}^2$ is a smooth local parametrization of $\mathbb{S}^2$. Show that:

$$\begin{aligned} &\mathsf{G} : \mathcal{U} \to \Sigma \\ &(u, v) \mapsto f\big(\mathsf{F}(u, v)\big)\,\mathsf{F}(u, v) \end{aligned}$$

is a smooth local parametrization of Σ. Hence, show that Σ is a regular surface.

Solution: Condition (1): Since f is smooth, the composition $f \circ \mathsf{F} : \mathcal{U} \to \mathbb{R}$ is also smooth. Combining with the fact that $\mathsf{F} : \mathcal{U} \to \mathbb{R}^3$ is smooth, we conclude that G is a smooth function from $\mathcal{U}$ to $\mathbb{R}^3$.
Condition (2): We first show that G is injective. Given any $(u, v), (u', v') \in \mathcal{U}$ such that $\mathsf{G}(u, v) = \mathsf{G}(u', v')$, we have

$$f(\mathsf{F}(u, v))\mathsf{F}(u, v) = f(\mathsf{F}(u', v'))\mathsf{F}(u', v'). \tag{*}$$

Since $|\mathsf{F}| = 1$ (unit sphere) and $f > 0$, it follows from taking $||$ on (*) that:

$$|f(\mathsf{F}(u, v))\mathsf{F}(u, v)| = \left|f(\mathsf{F}(u', v'))\mathsf{F}(u', v')\right| \implies f(\mathsf{F}(u, v)) = f(\mathsf{F}(u', v')).$$

From (*) again, we have $\mathsf{F}(u, v) = \mathsf{F}(u', v')$. As F is injective, we get $(u, v) = (u', v')$ showing that G is injective, and G^{-1} can be defined.
We next claim that $\mathsf{G}^{-1}(x, y, z) = \mathsf{F}^{-1}\left(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right)$. Given $(x, y, z) = \mathsf{G}(u, v)$, we need to solve (u, v) in terms of (x, y, z). By the definition of G, we get:

$$(x, y, z) = f(\mathsf{F}(u, v))\,\mathsf{F}(u, v).$$

Since $|\mathsf{F}| = 1$ (unit sphere) and $f > 0$, we get have:

$$\sqrt{x^2 + y^2 + z^2} = |(x, y, z)| = |f(\mathsf{F}(u, v))|\,|\mathsf{F}(u, v)| = f(\mathsf{F}(u, v)).$$

This implies $(x, y, z) = \sqrt{x^2 + y^2 + z^2}\,\mathsf{F}(u, v) \implies \mathsf{F}(u, v) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$.
Hence $(u, v) = \mathsf{F}^{-1}\left(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right)$ as desired. This verifies that $\mathsf{G}^{-1}(x, y, z) = \mathsf{F}^{-1}\left(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right)$, which is continuous as F^{-1} is so and that $(x, y, z) \neq 0$ for any $(x, y, z) \in \Sigma$.

Condition (3): Finally, we compute that:

$$\begin{aligned}
\frac{\partial \mathsf{G}}{\partial u} &= \frac{\partial (f\circ \mathsf{F})}{\partial u}\,\mathsf{F} + f(\mathsf{F}(u,v))\,\frac{\partial \mathsf{F}}{\partial u}\\
\frac{\partial \mathsf{G}}{\partial v} &= \frac{\partial (f\circ \mathsf{F})}{\partial v}\,\mathsf{F} + f(\mathsf{F}(u,v))\,\frac{\partial \mathsf{F}}{\partial v}\\
\frac{\partial \mathsf{G}}{\partial u}\times\frac{\partial \mathsf{G}}{\partial v} &= f\frac{\partial f}{\partial u}\,\mathsf{F}\times\frac{\partial \mathsf{F}}{\partial v} + f\frac{\partial f}{\partial v}\frac{\partial \mathsf{F}}{\partial u}\times\mathsf{F} + f^2\frac{\partial \mathsf{F}}{\partial u}\times\frac{\partial \mathsf{F}}{\partial v}.
\end{aligned}$$

Note that F parametrizes the unit sphere, so $\mathsf{F}_u\times\mathsf{F}_v$ and F are parallel (the former is a normal vector). Since they are non-zero vectors, we have $(\mathsf{F}_u\times\mathsf{F}_v)\cdot\mathsf{F}\neq 0$. On the other hand, $\mathsf{F}\cdot(\mathsf{F}\times\mathsf{F}_u) = \mathsf{F}\cdot(\mathsf{F}\times\mathsf{F}_v) = 0$. Finally, we get:

$$\mathsf{F}\cdot\left(\frac{\partial \mathsf{G}}{\partial u}\times\frac{\partial \mathsf{G}}{\partial v}\right) = f^2\mathsf{F}\cdot\left(\frac{\partial \mathsf{F}}{\partial u}\times\frac{\partial \mathsf{F}}{\partial v}\right)\neq 0.$$

Therefore, $\mathsf{G}_u\times\mathsf{G}_v\neq 0$, showing condition (3).

For any point $\mathsf{y}\in\Sigma$, there exists $\mathsf{x}\in\mathbb{S}^2$ such that $\mathsf{y} = f(\mathsf{x})\,\mathsf{x}$. Let F be a smooth local parametrization covering x, then its induced parametrization G will cover y. Hence, Σ is a regular surface.

(b) Let $\mathsf{F}_i(u,v) : \mathcal{U}_i\to\mathbb{S}^2$, where $i = 1,2$, be two overlapping smooth local parametrizations of $\mathbb{S}^2$, and $\mathsf{G}_i : \mathcal{U}_i\to\Sigma$ be the parametrization of Σ induced by F_i. Show that $\mathsf{G}_1^{-1}\circ\mathsf{G}_2 = \mathsf{F}_1^{-1}\circ\mathsf{F}_2$.

Solution: From (a), we know that

$$\mathsf{G}^{-1}(x,y,z) = \mathsf{F}^{-1}\left(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right).$$

To prove the claim in this part, we consider:

$$\begin{aligned}
\mathsf{G}_1^{-1}\circ\mathsf{G}_2(u,v) &= \mathsf{G}_1^{-1}\Big(f(\mathsf{F}_2(u,v))\,\mathsf{F}_2(u,v)\Big)\\
&= \mathsf{F}_1^{-1}\left(\frac{f(\mathsf{F}_2(u,v))\,\mathsf{F}_2(u,v)}{|f(\mathsf{F}_2(u,v))\,\mathsf{F}_2(u,v)|}\right)\\
&= \mathsf{F}_1^{-1}\big(\mathsf{F}_2(u,v)\big) && \text{(since } f>0 \text{ and } |\mathsf{F}_2| = 1)\\
&= \mathsf{F}_1^{-1}\circ\mathsf{F}_2(u,v)
\end{aligned}$$

as desired. Concerning the domains of $\mathsf{G}_1^{-1}\circ\mathsf{G}_2$ and $\mathsf{F}_1^{-1}\circ\mathsf{F}_2$, we can check that they are the same:

$$\begin{aligned}
(u,v)\in\mathsf{G}_2^{-1}\big(\mathsf{G}_1(\mathcal{U}_1)\cap\mathsf{G}_2(\mathcal{U}_2)\big) &\iff \mathsf{G}_2(u,v)\in\mathsf{G}_1(\mathcal{U}_1)\cap\mathsf{G}_2(\mathcal{U}_2)\\
&\iff f(\mathsf{F}_2(u,v))\mathsf{F}_2(u,v)\in\mathsf{G}_i(\mathcal{U}_i)\ \text{ for } i = 1,2\\
&\iff \mathsf{F}_2(u,v)\in\mathsf{F}_i(\mathcal{U}_i)\ \text{ for } i = 1,2\\
&\iff (u,v)\in\mathsf{F}_2^{-1}\big(\mathsf{F}_1(\mathcal{U}_1)\cap\mathsf{F}_2(\mathcal{U}_2)\big).
\end{aligned}$$

Hence, their domains are the same. In the third step above, the $\Longrightarrow$-part follows from the fact that $f>0$ and $|\mathsf{F}_i| = 1$.

(c) Show that $\mathbb{S}^2$ and Σ are diffeomorphic. Write down the diffeomorphism explicitly.

Solution: Define $\Psi : \mathbb{S}^2 \to \Sigma$ by $\Psi(\mathsf{x}) = f(\mathsf{x})\,\mathsf{x}$. It is clearly surjective by the definition of Σ. To show it is injective, we consider $\mathsf{x}_1, \mathsf{x}_2 \in \mathbb{S}^2$ such that $f(\mathsf{x}_1)\,\mathsf{x}_1 = f(\mathsf{x}_2)\,\mathsf{x}_2$. Since both x_1 and x_2 are unit and $f > 0$, we get $f(\mathsf{x}_1)\,|\mathsf{x}_1| = f(\mathsf{x}_2)\,|\mathsf{x}_2| \implies f(\mathsf{x}_1) = f(\mathsf{x}_2)$. This shows $\mathsf{x}_1 = \mathsf{x}_2$ as well. This shows Ψ is injective.

Next we show Ψ and Ψ^{-1} are smooth. Given any smooth local parametrization $\mathsf{F} : \mathcal{U} \to \mathbb{S}^2$, and its induced parametrization G. Observing that $\Psi(\mathsf{F}(u,v)) = \mathsf{G}(u,v)$, we can verify that

$$\mathsf{G}^{-1} \circ \Psi \circ \mathsf{F}(u,v) = \mathsf{G}^{-1} \circ \mathsf{G}(u,v) = (u,v)$$
$$\mathsf{F}^{-1} \circ \Psi^{-1} \circ \mathsf{G}(u,v) = (\mathsf{G}^{-1} \circ \Psi \circ \mathsf{F})^{-1}(u,v) = (u,v).$$

The local expressions of both Ψ and Ψ^{-1} are the identity maps, which are clearly smooth. Therefore, Ψ is a diffeomorphism between $\mathbb{S}^2$ and Σ.

(d) Define a map $\Phi : \mathbb{R}^3 \backslash \{(0,0,0)\} \to \mathbb{R}^3 \backslash \{(0,0,0)\}$ by:

$$\Phi(\mathsf{x}) = \frac{\mathsf{x}}{|\mathsf{x}|^2}.$$

Denote $\Sigma^* := \Phi(\Sigma)$, i.e. the image of Σ under the map Φ.

i. Explain why Σ^* is also a regular surface.

Solution: By definition of Φ, we have:

$$\Sigma^* = \left\{\Phi\left(f(\mathsf{x})\mathsf{x}\right) : \mathsf{x} \in \mathbb{S}^2\right\} = \left\{\frac{1}{f(\mathsf{x})}\mathsf{x} : \mathsf{x} \in \mathbb{S}^2\right\}.$$

Hence Σ^* is simply Σ with f replaced by $1/f$. As f is a positive smooth function on $\mathbb{S}^2$, then so does $1/f$. By (a), Σ^* is also a regular surface.

ii. Let $\phi : \Sigma \to \Sigma^*$ be the restriction of Φ on Σ. Show that ϕ is a diffeomorphism.

Solution: Parametrize $\mathbb{S}^2$ by $F(u,v)$ as in (b), and parametrize Σ and Σ^* respectively by:

$$\mathsf{G}_\Sigma(u,v) = f(\mathsf{F}(u,v))\mathsf{F}(u,v)$$
$$\mathsf{G}_{\Sigma^*}(u,v) = \frac{1}{f(\mathsf{F}(u,v))}\mathsf{F}(u,v)$$

By observing that

$$\phi(f(\mathsf{x})\mathsf{x}) = \frac{1}{f(\mathsf{x})}\mathsf{x},$$

one can easily check that:

$$\mathsf{G}_{\Sigma^*}^{-1} \circ \phi \circ \mathsf{G}_\Sigma(u,v) = (u,v).$$

Hence ϕ is smooth, and so does ϕ^{-1} (as the coordinate representation is also the identity).

iii. Show that for any $p \in \Sigma$, the tangent map $(\phi_*)_p : T_p\Sigma \to T_{\phi(p)}\Sigma^*$ at p is:

$$(\phi_*)_p(V) = \frac{|p|^2 V - 2(p \cdot V)p}{|p|^4}$$

where $|p|$ is the norm of p in $\mathbb{R}^3$, and $p \cdot V$ is the usual dot product of p and V in $\mathbb{R}^3$.

Solution: From (ii), the coordinate representation (with respect to parametrizations G_Σ and G_{Σ^*}) of ϕ is the identity. Hence, $(\phi_*)_p$ maps $\frac{\partial \mathsf{G}_\Sigma}{\partial u}(p)$ to $\frac{\partial \mathsf{G}_{\Sigma^*}}{\partial u}(\phi(p))$, and maps $\frac{\partial \mathsf{G}_\Sigma}{\partial v}(p)$ to $\frac{\partial \mathsf{G}_{\Sigma^*}}{\partial v}(\phi(p))$.
To prove the desired result, we observe that the map:

$$V \mapsto \frac{|p|^2 V - 2(p \cdot V)p}{|p|^4}$$

is linear (note that p is fixed), so it suffices to verify the desired result when V are basis vectors $\frac{\partial \mathsf{G}_\Sigma}{\partial u}(p)$ and $\frac{\partial \mathsf{G}_\Sigma}{\partial v}(p)$.

$$\begin{aligned}
(\phi_*)_p\left(\frac{\partial \mathsf{G}_\Sigma}{\partial u}(p)\right) &= \frac{\partial \mathsf{G}_{\Sigma^*}}{\partial u}(\phi(p)) \\
&= \frac{\partial\left(\frac{1}{f(\mathsf{F}(u,v))}\right)}{\partial u}\mathsf{F}(u,v) + \frac{1}{f(\mathsf{F}(u,v))}\frac{\partial \mathsf{F}}{\partial u} \\
&= -\frac{1}{f(\mathsf{F}(u,v))^2}\frac{\partial f}{\partial u}\mathsf{F}(u,v) + \frac{1}{f(\mathsf{F}(u,v))}\frac{\partial \mathsf{F}}{\partial u}
\end{aligned}$$

Here (u,v) is the local coordinate of p under the parametrization G, i.e. $\mathsf{G}(u,v) = p$, or equivalently, $f(\mathsf{F}(u,v))\mathsf{F}(u,v) = p$. In particular, $|p| = f(\mathsf{F}(u,v))$ as F is a parametrization of the unit sphere.
On the other hand, we have

$$\begin{aligned}
&\frac{|p|^2 \frac{\partial \mathsf{G}_\Sigma}{\partial u}(p) - 2\left(p \cdot \frac{\partial \mathsf{G}_\Sigma}{\partial u}(p)\right)p}{|p|^4} \\
&= \frac{1}{|p|^2}\left(\frac{\partial f}{\partial u}\mathsf{F}(u,v) + f(\mathsf{F}(u,v))\frac{\partial \mathsf{F}}{\partial u}\right) \\
&\quad - 2\frac{p}{|p|^4}f(\mathsf{F}(u,v))\mathsf{F}(u,v) \cdot \left(\frac{\partial f}{\partial u}\mathsf{F}(u,v) + f(\mathsf{F}(u,v))\frac{\partial \mathsf{F}}{\partial u}\right) \\
&= \frac{1}{f(\mathsf{F}(u,v))^2}\frac{\partial f}{\partial u}\mathsf{F}(u,v) + \frac{1}{f(\mathsf{F}(u,v))}\frac{\partial \mathsf{F}}{\partial u} - 2\frac{p}{|p|^4}f(\mathsf{F}(u,v))\frac{\partial f}{\partial u}.
\end{aligned}$$

Here we used the fact that $\mathsf{F} \cdot \mathsf{F} = 1$ and $\mathsf{F} \cdot \frac{\partial \mathsf{F}}{\partial u} = 0$ (for round spheres). Using the fact again that $p = f(\mathsf{F}(u,v))\mathsf{F}(u,v)$, one can complete the proof that:

$$\frac{|p|^2 \frac{\partial \mathsf{G}_\Sigma}{\partial u}(p) - 2\left(p \cdot \frac{\partial \mathsf{G}_\Sigma}{\partial u}(p)\right)p}{|p|^4} = -\frac{1}{f(\mathsf{F}(u,v))^2}\frac{\partial f}{\partial u}\mathsf{F}(u,v) + \frac{1}{f(\mathsf{F}(u,v))}\frac{\partial \mathsf{F}}{\partial u}.$$

This shows

$$(\phi_*)_p(V) = \frac{|p|^2 V - 2(p \cdot V)p}{|p|^4}$$

when $V = \frac{\partial G_\Sigma}{\partial u}$, and similarly it holds when $V = \frac{\partial G_\Sigma}{\partial v}$. By linearity, the desired result holds for any $V \in T_p\Sigma$.

4. Let F_+ and F_- be the stereographic parametrizations of the unit sphere $\mathbb{S}^2$ as discussed in Example 1.5 of the lecture notes. Here we regard $\mathbb{C}$ as $\mathbb{R}^2$ by identifying $z = u + iv \in \mathbb{C}$ with $(u, v) \in \mathbb{R}^2$. Then, $\mathsf{F}_+ : \mathbb{C} \to \mathbb{S}^2 \backslash \{(0,0,1)\}$ and its inverse can be expressed as:

$$\mathsf{F}_+(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \qquad \mathsf{F}_+^{-1}(x_1, x_2, x_3) = \frac{x_1 + x_2 i}{1 - x_3}$$

Here we use (x_1, x_2, x_3) for coordinates of $\mathbb{R}^3$ instead of (x, y, z) to avoid notation conflicts.

(a) Consider the south-pole stereographic parametrization $\mathsf{F}_- : \mathbb{C} \to \mathbb{S}^2 \backslash \{(0,0,-1)\}$. Find the explicit expressions of $\mathsf{F}_-(z)$, where $z \in \mathbb{C}$, and $\mathsf{F}_-^{-1}(x_1, x_2, x_3)$, where $(x_1, x_2, x_3) \in \mathbb{S}^2 \backslash \{(0,0,-1)\}$.

Solution:

$$\mathsf{F}_-(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right) \qquad \mathsf{F}_-^{-1}(x_1, x_2, x_3) = \frac{x_1 + x_2 i}{1 + x_3}$$

(b) Verify that:

$$\mathsf{F}_-^{-1} \circ \mathsf{F}_+(z) = \frac{1}{\bar{z}} \qquad \mathsf{F}_+^{-1} \circ \mathsf{F}_-(z) = \frac{1}{\bar{z}}.$$

State the domains on which they are defined.

Solution: By direct calculation,

$$\begin{aligned}
&\mathsf{F}_-^{-1} \circ \mathsf{F}_+(z) \\
&= \mathsf{F}_-^{-1} \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \\
&= \frac{\frac{2\mathrm{Re}(z)}{|z|^2+1} + \frac{2\mathrm{Im}(z)}{|z|^2+1} i}{1 + \frac{|z|^2-1}{|z|^2+1}} \\
&= \frac{\mathrm{Re}(z) + i\mathrm{Im}(z)}{|z|^2} \\
&= \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}
\end{aligned}$$

and its domain is $\mathbb{C}\backslash\{0\}$. Similarly,

$$\begin{aligned}
&\mathsf{F}_+^{-1} \circ \mathsf{F}_-(z) \\
&= \mathsf{F}_+^{-1}\left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right) \\
&= \frac{\frac{2\mathrm{Re}(z)}{|z|^2+1} + \frac{2\mathrm{Im}(z)}{|z|^2+1} i}{1 - \frac{1-|z|^2}{|z|^2+1}} \\
&= \frac{\mathrm{Re}(z) + i\mathrm{Im}(z)}{|z|^2} \\
&= \frac{1}{\bar{z}}
\end{aligned}$$

and its domain is $\mathbb{C}\backslash\{0\}$.

(c) Consider the complex-valued function $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$ such that $\alpha\delta \neq \beta\gamma$. Define a map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by:

$$\Phi(p) := \begin{cases} \mathsf{F}_+(\alpha/\gamma) & \text{if } p = (0,0,1) \\ (0,0,1) & \text{if } p = \mathsf{F}_+(-\delta/\gamma) \\ \mathsf{F}_+ \circ f \circ \mathsf{F}_+^{-1}(p) & \text{otherwise} \end{cases}$$

i. Show that Φ is bijective.

Solution: We claim that the following map is the inverse of Φ:

$$\Psi(p) := \begin{cases} \mathsf{F}_+(-\delta/\gamma) & \text{if } p = (0,0,1) \\ (0,0,1) & \text{if } p = \mathsf{F}_+(\alpha/\gamma) \\ \mathsf{F}_+ \circ g \circ \mathsf{F}_+^{-1}(p) & \text{otherwise} \end{cases}$$

where $g : \mathbb{C}\backslash\{\alpha/\gamma\} \to \mathbb{C}\backslash\{-\delta/\gamma\}$ is defined by

$$g(w) = \frac{\beta - \delta w}{\gamma w - \alpha}.$$

By direct computations, one can check $f \circ g(w) = w$ and $g \circ f(z) = z$. Hence, $f^{-1} = g$. Next we verify that:

$$\Psi(\Phi(p)) = \begin{cases} \Psi \circ \mathsf{F}_+(\alpha/\gamma) = (0,0,1) = p & \text{if } p = (0,0,1) \\ \Psi(0,0,1) = \mathsf{F}_+(\alpha/\gamma) = p & \text{if } p = \mathsf{F}_+(-\delta/\gamma) \\ \mathsf{F}_+ \circ g \circ \mathsf{F}_+^{-1}(\mathsf{F}_+ \circ f \circ \mathsf{F}_+^{-1}(p)) = p & \text{otherwise} \end{cases}$$

Similarly, one can also verify that $\Phi(\Psi(p)) = p$ for any $p \in \mathbb{S}^2$. Hence Φ is bijective with inverse Ψ.

ii. Find an explicit expression of each of the following:

$$\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_+(z) \qquad \mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_+(z) \qquad \mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_-(z) \qquad \mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-(z)$$

State the domain of each of them.

Solution: By direct calculations, we have

$$\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_+(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

The domain of $\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_+(z)$ is $\mathbb{C}\backslash\{-\delta/\gamma\}$.

$$\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_+(z) = \frac{\overline{\gamma z} + \overline{\delta}}{\overline{\alpha z} + \overline{\beta}}$$

The domain of $\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_+(z)$ is $\mathbb{C}\backslash\{-\beta/\alpha\}$.

$$\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_-(z) = \frac{\alpha + \beta\overline{z}}{\gamma + \delta\overline{z}}$$

The domain of $\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_-(z)$ is $\mathbb{C}\backslash\{-\overline{\gamma}/\overline{\delta}\}$.

$$\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-(z) = \frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}$$

The domain of $\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-(z)$ is $\mathbb{C}\backslash\{-\overline{\alpha}/\overline{\beta}\}$.

iii. Show that Φ is smooth at the point $(0,0,1)$.

Solution: We pick F_- as a smooth local parametrization and $(0,0,1) = \mathsf{F}_-(0)$. By part ii, we have

$$\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-(z) = \frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}$$

which is holomorphic on $\mathbb{C}\backslash\{-\overline{\alpha}/\overline{\beta}\}$, so it is smooth. Since $\mathbb{S}^2$ is a regular surface, by Proposition 1.11 in the lecture note, the transition maps are smooth. Hence, for any local parametrization G , $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{G}(z)$ is smooth. Therefore, Φ is smooth at the point $(0,0,1)$.

iv. Show that tangent map Φ_* at $(0,0,1)$ is invertible.

Solution: A matrix representation of Φ_* at $(0,0,1)$ is the Jacobian matrix of the map:

$$\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-(z) = \frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}$$

at $z = 0$ (since $\mathsf{F}_-(0) = (0,0,1)$).
Write $z = u + iv$, we need to compute

$$\frac{\partial}{\partial u}\left(\frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}\right) \qquad \text{and} \qquad \frac{\partial}{\partial v}\left(\frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}\right)$$

at $(u,v) = (0,0)$.

Observing that $\frac{\partial z}{\partial u} = 1$ and $\frac{\partial z}{\partial v} = i$, we can directly compute that:

$$U := \frac{\partial}{\partial u}\left(\frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}\right)\bigg|_{z=0} = \frac{(\overline{\alpha} + \overline{\beta}z)\overline{\delta} - (\overline{\gamma} + \overline{\delta}z)\overline{\beta}}{(\overline{\alpha} + \overline{\beta}z)^2}\bigg|_{z=0} = \frac{\overline{\alpha\delta - \beta\gamma}}{\overline{\alpha}^2}$$

$$V := \frac{\partial}{\partial v}\left(\frac{\overline{\gamma} + \overline{\delta}z}{\overline{\alpha} + \overline{\beta}z}\right)\bigg|_{z=0} = \frac{(\overline{\alpha} + \overline{\beta}z)\overline{\delta}i - (\overline{\gamma} + \overline{\delta}z)\overline{\beta}i}{(\overline{\alpha} + \overline{\beta}z)^2}\bigg|_{z=0} = \frac{\overline{\alpha\delta - \beta\gamma}}{\overline{\alpha}^2}i.$$

For simplicity, denote

$$Z := \frac{\overline{\alpha\delta - \beta\gamma}}{\overline{\alpha}^2} \neq 0,$$

then $U = \mathrm{Re}(Z) + i\mathrm{Im}(Z)$ and $V = -\mathrm{Im}(Z) + i\mathrm{Re}(Z)$, and hence

$$[\Phi_*]_{(0,0,1)} = \begin{bmatrix} \mathrm{Re}(Z) & -\mathrm{Im}(Z) \\ \mathrm{Im}(Z) & \mathrm{Re}(Z) \end{bmatrix},$$

whose determinant is $|Z|^2 \neq 0$.
Therefore, Φ_* is invertible at $(0,0,1)$.