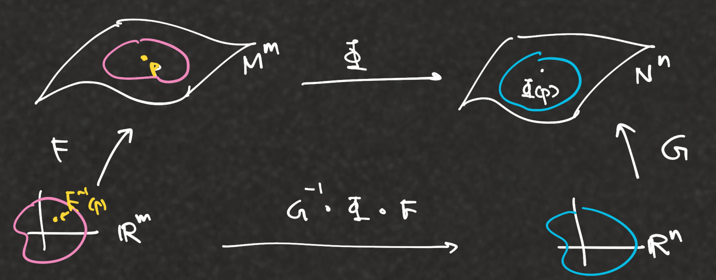
MATH 4033
LECTURES 10-11
WEEK 04



Φ is C^k at $p\in M$

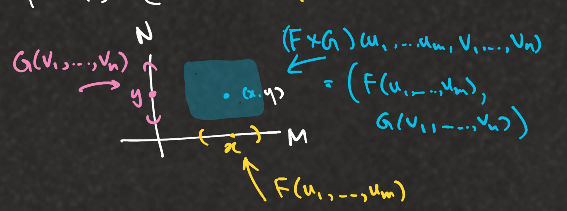
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists F:\mathcal{U}\to\mathcal{O}\ni p$ of M

$G:\tilde{\mathcal{U}}\to\tilde{\mathcal{O}}\ni\Phi(p)$ of N

s.t. $G^{-1}\circ\Phi\circ F$ is C^k at $F^{-1}(p)$

$$\begin{array}{ccc} M\times N & \xrightarrow{\pi} & M \\ (x,y) & \longmapsto & x \end{array}$$

$F\times G \nearrow \qquad \nwarrow F$



$F^{-1}\circ\pi\circ(F\times G)(u_1,\dots,u_m,v_1,\dots,v_n)$

$=F^{-1}\circ\pi\big(F(u_1,\dots,u_m),\,G(v_1,\dots,v_n)\big)$

$=F^{-1}\big(F(u_1,\dots,u_m)\big)=(u_1,\dots,u_m)$

C^∞.

e.g. 2.19

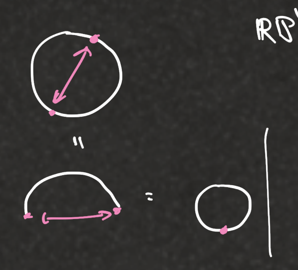
$\mathbb{S}^3=\{(x_1,\dots,x_4):\ x_1^2+x_2^2+x_3^2+x_4^2=1\}.$

$\mathbb{CP}^1=\{[z_1:z_2]:|z_1|^2+|z_2|^2\neq 0\}$

$\Phi(x_1,\dots,x_4)=[x_1+ix_2:x_3+ix_4]$

$\Phi:\mathbb{S}^3\to\mathbb{CP}^1$

e.g. $\mathbb{RP}^1$ diffeo. to $\mathbb{S}^1$.



$\mathbb{RP}^1=\{[x_1:x_2]:x_1^2+x_2^2\neq 0\}.$

$=\{[x_1:x_2]:\ x_1\neq 0\}$

$\amalg\{[x_1:x_2]:\ x_1=0\}$

$=\{[1:w]:w\in\mathbb{R}\}.=\mathbb{R}$

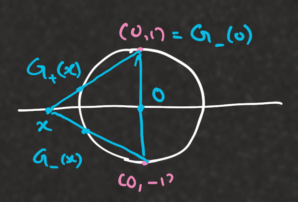
$\amalg\{[0:1]\}\qquad=\{pt\}.$

• pt.

$\mathbb{R}$

$$\mathbb{RP}^1\xrightarrow{\Phi}\mathbb{S}^1$$

$F_1(u)=[1:u]$, $F_2(v)=[v:1]$, G_+, G_-, $\mathbb{R}$



$$\begin{cases}\Phi([1:u]):=G_+(u)\\ \Phi([0:1]):=(0,1)\end{cases}$$

$G_-^{-1}\circ\Phi\circ F_2(v)$

$=G_-^{-1}\circ\Phi([v:1])$

$=\begin{cases}G_-^{-1}(0,1) & \text{if } v=0\\ G_-^{-1}(G_+(\frac{1}{v})) & \text{if } v\neq 0\end{cases}$ $\qquad [v:1]=[1:\frac{1}{v}].$

$=\begin{cases}0 & \text{if } v=0\\ v & \text{if } v\neq 0.\end{cases}$

$=v.$ $\qquad C^\infty$

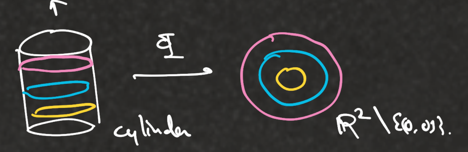
After checking $G_+^{-1}\circ\Phi\circ F_1$, $G_+^{-1}\circ\Phi\circ F_2$, $G_-^{-1}\circ\Phi\circ F_1$ are C^∞

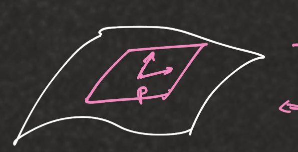
$\Rightarrow$ Φ is C^∞.

After checking their inverses are C^∞ $\Rightarrow$ Φ^{-1} is C^∞.

$\Rightarrow$ Φ is diffeomorphism b/w $\mathbb{RP}^1$ and $\mathbb{S}^1$.

e.g. 2.23



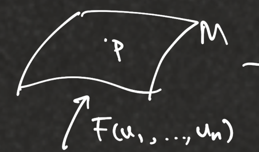


$\frac{\partial F}{\partial u_1}\times\frac{\partial F}{\partial u_2}\neq 0$

$\Leftrightarrow$ span$\{\frac{\partial F}{\partial u_1}(p),\frac{\partial F}{\partial u_2}(p)\}$ 2-d.

$\cancel{\frac{\partial F}{\partial u_i}(p)}\longleftrightarrow\frac{\partial}{\partial u_i}(p)$ tangent vector.

$T_pM^n:=\text{span}\{\frac{\partial}{\partial u_1}(p),\dots,\frac{\partial}{\partial u_n}(p)\}.$



$\xrightarrow{f}\mathbb{R}.$ $\qquad \frac{\partial f}{\partial u_i}(p):=\frac{\partial}{\partial u_i}(f\circ F)\Big|_{F^{-1}(p)}.$

f is C^∞ at p $\overset{\text{def}}{\Longleftrightarrow}$ $\exists F:\mathcal{U}\to\mathcal{O}\ni p$ s.t. $f\circ F$ is C^∞ at $F^{-1}(p)$.

$\frac{\partial}{\partial u_i}(p):C^\infty(M,\mathbb{R})\longrightarrow\mathbb{R}$

$f\longmapsto\frac{\partial f}{\partial u_i}(p).$

$a\frac{\partial}{\partial u_i}(p)+b\frac{\partial}{\partial u_j}(p):C^\infty(M,\mathbb{R})\to\mathbb{R}$

$f\mapsto a\frac{\partial f}{\partial u_i}(p)+b\frac{\partial f}{\partial u_j}(p).$

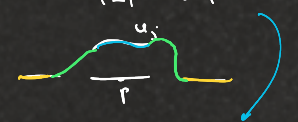
$T_pM:=\text{span}\{\frac{\partial}{\partial u_i}(p)\}_{i=1}^n$

Claim: span$\{\frac{\partial}{\partial u_i}(p)\}_{i=1}^n$ is indep. of local coordinates.

Proof: $\{\frac{\partial}{\partial u_i}(p)\}_{i=1}^n$ linearly indep. — (*)

Suppose $\sum_{i=1}^n a_i\frac{\partial}{\partial u_i}(p)=0.\Rightarrow\sum_{i=1}^n a_i\frac{\partial f}{\partial u_i}(p)=0\quad\forall f\in C^\infty(M,\mathbb{R}).$

Choose: $f=\begin{cases}u_j & \text{near } p\\ 0 & \text{away from } p.\end{cases}$



$\sum_{i=1}^n a_i\frac{\partial u_j}{\partial u_i}(p)=0\Rightarrow\sum_{i=1}^n a_i\delta_{ij}=0$

$\Rightarrow a_j=0.$

j arbitrary $\Rightarrow$ $a_1,\dots,a_n=0.$ $\Rightarrow$ proves (*).

span$\{\frac{\partial}{\partial v_i}(p)\}_{i=1}^n\overset{?}{\subset}$ span$\{\frac{\partial}{\partial u_i}(p)\}_{i=1}^n$

$\frac{\partial f}{\partial v_i}=\sum_{j=1}^n\frac{\partial f}{\partial u_j}\frac{\partial u_j}{\partial v_i}\quad\forall f\in C^\infty(M,\mathbb{R})\Rightarrow\frac{\partial}{\partial v_i}=\sum_{j=1}^n\frac{\partial u_j}{\partial v_i}\frac{\partial}{\partial u_j}$

$\in\text{span}\{\frac{\partial}{\partial u_j}\}_{j=1}^n$

$\Rightarrow\underbrace{\text{span}\{\frac{\partial}{\partial v_i}(p)\}_{i=1}^n}_{\dim=n}<\underbrace{\text{span}\{\frac{\partial}{\partial u_j}(p)\}_{j=1}^n}_{\dim=n}.$

$\Rightarrow\text{span}\{\frac{\partial}{\partial v_i}(p)\}_{i=1}^n=\text{span}\{\frac{\partial}{\partial u_i}(p)\}_{i=1}^n.$

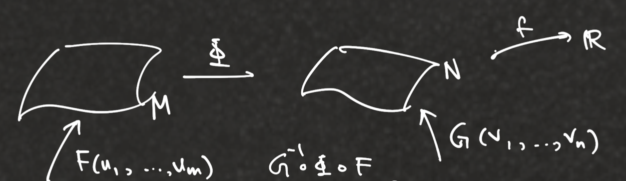
$$M\xrightarrow{\Phi}N\xrightarrow{f}\mathbb{R}$$

$\uparrow F\ (u_1,\dots,u_m)$, $\uparrow (v_1,\dots,v_n)$

$\frac{\partial\Phi}{\partial u_i}=\frac{\partial}{\partial u_i}(\Phi\circ F).$ $\quad\in T_{\Phi(p)}N$ $\qquad$ (regular surfaces in $\mathbb{R}^3$)

$\left(\frac{\partial\Phi}{\partial u_i}\right)(f):=\frac{\partial}{\partial u_i}(f\circ\Phi\circ F)$

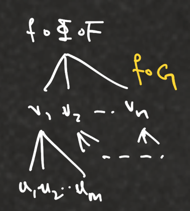
$f:N\to\mathbb{R}$



$\left(\frac{\partial\Phi}{\partial u_i}\right)(f):=\frac{\partial}{\partial u_i}(f\circ\Phi\circ F)$

$=\frac{\partial}{\partial u_i}\big((f\circ G)\circ(G^{-1}\circ\Phi\circ F)\big)$

$\uparrow$ $f\circ G(v_1,\dots,v_n)$



$\frac{\partial f}{\partial v_j}$

$=\sum_{j=1}^n\frac{\partial(f\circ G)}{\partial v_j}\frac{\partial v_j}{\partial u_i}$

$=\sum_{j=1}^n\frac{\partial v_j}{\partial u_i}\frac{\partial f}{\partial v_j}\qquad\forall f\in C^1(N,\mathbb{R})$

$\Rightarrow\frac{\partial\Phi}{\partial u_i}=\sum_{j=1}^n\frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}$ $\quad\in T_{\Phi(p)}N$

e.g. $\Phi:\mathbb{RP}^2\to\mathbb{RP}^3$

$\Phi([x_0:x_1:x_2]):=[x_0^2:x_1^2:x_2^2:x_0x_1].$

$$\mathbb{RP}^2\xrightarrow{\Phi}\mathbb{RP}^3$$

$F(u_1,u_2)=[1:u_1:u_2]$, $\qquad G^{-1}\circ\Phi\circ F$, $\qquad G(v_1,v_2,v_3)=[1:v_1:v_2:v_3].$

$G^{-1}\circ\Phi\circ F(u_1,u_2)=G^{-1}\circ\Phi([1:u_1:u_2])$

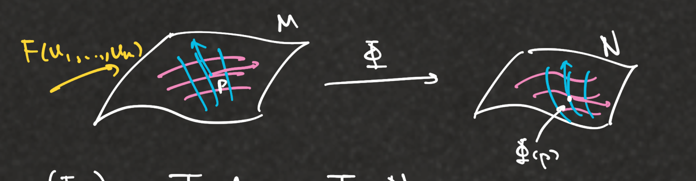
$=G^{-1}([1:u_1^2:u_2^2:u_1])$ $\qquad\frac{\partial\Phi}{\partial u_2}=2u_2\frac{\partial}{\partial v_2}$

$=(u_1^2,u_2^2,u_1)$

$(v_1,v_2,v_3)=(u_1^2,u_2^2,u_1)\to\frac{\partial}{\partial u_2}(u_1^2,u_2^2,u_1)=(0,2u_2,0)$

$\frac{\partial\Phi}{\partial u_1}=\sum_{j=1}^3\frac{\partial v_j}{\partial u_1}\frac{\partial}{\partial v_j}=\frac{\partial v_1}{\partial u_1}\frac{\partial}{\partial v_1}+\frac{\partial v_2}{\partial u_1}\frac{\partial}{\partial v_2}+\frac{\partial v_3}{\partial u_1}\frac{\partial}{\partial v_3}$

$=2u_1\frac{\partial}{\partial v_1}+0\frac{\partial}{\partial v_2}+1\frac{\partial}{\partial v_3}$



$(\Phi_*)_p:T_pM\longrightarrow T_{\Phi(p)}N$ linear.

$(\Phi_*)_p\left(\frac{\partial}{\partial u_i}(p)\right):=\frac{\partial\Phi}{\partial u_i}(\Phi(p))$ $\qquad\Phi_*\left(\frac{\partial}{\partial u_i}\right)=\frac{\partial\Phi}{\partial u_i}$

$=\sum_{j=1}^n\frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}$

$(v_1,\dots,v_n)=G^{-1}\circ\Phi\circ F(u_1,\dots,u_m)$

$$[(\Phi_*)_p]=\begin{bmatrix}\frac{\partial v_1}{\partial u_1} & \cdots & \frac{\partial v_1}{\partial u_m}\\ \frac{\partial v_2}{\partial u_1} & \cdots & \frac{\partial v_2}{\partial u_m}\\ \vdots & & \vdots\\ \frac{\partial v_n}{\partial u_1} & \cdots & \frac{\partial v_n}{\partial u_m}\end{bmatrix}=\frac{\partial(v_1,\dots,v_n)}{\partial(u_1,\dots,u_m)}.$$

$\uparrow(\Phi_*)_p(\frac{\partial}{\partial u_1})\qquad\uparrow(\Phi_*)_p(\frac{\partial}{\partial u_m})$

e.g. $$\begin{array}{ccc} M\times N & \xrightarrow{\pi} & M \\ (x,y) & \longmapsto & x \end{array}$$

$(F\times G)(\vec{u},\vec{v})=(F(\vec{u}),G(\vec{v}))\longrightarrow F(u_1,\dots,u_m)$

$F(u_1,\dots,u_m)\nearrow M$ $\qquad N\nwarrow G(v_1,\dots,v_n)$

$F^{-1}\circ\pi\circ(F\times G)(u_1,\dots,u_m,v_1,\dots,v_n)$

$=F^{-1}\circ\big(\pi(F(u_1,\dots,u_m),G(v_1,\dots,v_n))\big)$

$=F^{-1}\big(F(u_1,\dots,u_m)\big)$

$=(u_1,\dots,u_m)$

$\frac{\partial}{\partial u_i}(u_1,\dots,u_m)=(0,\dots,\underset{i}{1},\dots,0)\Rightarrow\frac{\partial\pi}{\partial u_i}=\frac{\partial}{\partial u_i}$

$\frac{\partial}{\partial v_j}(u_1,\dots,u_m)=(0,\dots,\dots,0)\Rightarrow\frac{\partial\pi}{\partial v_j}=0$

$\pi_*\left(\frac{\partial}{\partial u_i}\right)=\frac{\partial\pi}{\partial u_i}=\frac{\partial}{\partial u_i},\quad\pi_*\left(\frac{\partial}{\partial v_j}\right)=\frac{\partial\pi}{\partial v_j}=0.$

$T_p(M\times N)=\text{span}\{\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_m},\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_n}\}$

$=\text{span}\{\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_m}\}\oplus\text{span}\{\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_n}\}$

$=T_pM\oplus T_pN.$ $\qquad\oplus\quad\otimes$

$\pi_*\left(\sum_i a_i\frac{\partial}{\partial u_i}+\sum_j b_j\frac{\partial}{\partial v_j}\right)=\sum_i a_i\frac{\partial}{\partial u_i}$

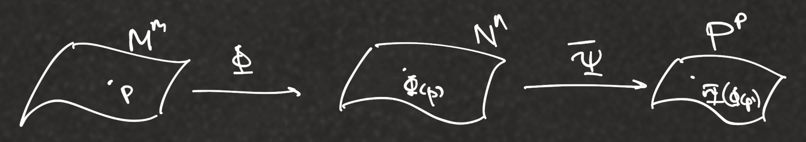
Chain rule.

$$\mathbb{R}^m\xrightarrow{\Phi}\mathbb{R}^n\xrightarrow{\Psi}\mathbb{R}^k$$

$(u_1,\dots,u_m)\quad(v_1,\dots,v_n)\quad(w_1,\dots,w_k)$

$$\frac{\partial w_i}{\partial u_j}=\sum_{l=1}^n\frac{\partial w_i}{\partial v_l}\frac{\partial v_l}{\partial u_j}\qquad\forall i,j$$

$$\frac{\partial(w_1,\dots,w_k)}{\partial(u_1,\dots,u_m)}=\frac{\partial(w_1,\dots,w_k)}{\partial(v_1,\dots,v_n)}\frac{\partial(v_1,\dots,v_n)}{\partial(u_1,\dots,u_m)}.$$



$\Phi_*:T_pM\to T_{\Phi(p)}N,\qquad\Psi_*:T_{\Phi(p)}N\longrightarrow T_{\Psi(\Phi(p))}P$

Chain rule: $\boxed{\Psi_*\circ\Phi_*=(\Psi\circ\Phi)_*}$

Immediate corollary: If $\Phi:M\to N$ is a diffeomorphism.

then $(\Phi_*)_p$ is invertible

$\Rightarrow\dim T_pM=\dim T_{\Phi(p)}N$

$\Rightarrow\dim M=\dim N.$

$$M^m\xrightarrow{\Phi}N^n\xrightarrow{\Psi}P^p$$

$(u_1,\dots,u_m)\quad(v_1,\dots,v_n)\quad(w_1,\dots,w_p).$

$(\Psi_*\circ\Phi_*)\left(\frac{\partial}{\partial u_i}\right)=\Psi_*\left(\frac{\partial\Phi}{\partial u_i}\right)=\Psi_*\left(\sum_j\frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}\right)$

$=\sum_j\frac{\partial v_j}{\partial u_i}\frac{\partial\Psi}{\partial v_j}=\sum_j\sum_k\frac{\partial v_j}{\partial u_i}\frac{\partial w_k}{\partial v_j}\frac{\partial}{\partial w_k}$

$=\sum_k\frac{\partial w_k}{\partial u_i}\frac{\partial}{\partial w_k}=\frac{\partial}{\partial u_i}(\Psi\circ\Phi)$

$=(\Psi\circ\Phi)_*\left(\frac{\partial}{\partial u_i}\right)$

true for $\forall i$ $\Rightarrow$ $\Psi_*\circ\Phi_*=(\Psi\circ\Phi)_*.$