MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #2 • Abstract Manifolds • Due Date: 11/03/2018, 11:59PM

Instructions: *"Outsource" all topological issues to MATH 4225. Make fair use of the word "similarly" to reduce your workload.*

1. Suppose Σ is a n-dimensional topological manifold in $\mathbb{R}^{n+1}$ (where $n \geq 2$) equipped with a family of local parametrizations $\{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to \Sigma\}$ covering the whole Σ and satisfying the following conditions:

   1. Each F_α is smooth as a map from $\mathcal{U}_\alpha$ to $\mathbb{R}^{n+1}$
   2. Each F_α is a homeomorphism onto its image $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$.
   3. Regarding $(x_1, \ldots, x_{n+1}) = \mathsf{F}_\alpha(u_1, \ldots, u_n)$, the Jacobian matrix

   $$\frac{\partial(x_1, \ldots, x_{n+1})}{\partial(u_1, \ldots, u_n)}$$

   at every $(u_1, \ldots, u_n) \in \mathcal{U}_\alpha$ has a trivial null-space.

   (a) Show that Σ is an n-dimensional smooth manifold.

   (b) Show further that Σ is a submanifold of $\mathbb{R}^{n+1}$.

2. The complex projective plane $\mathbb{CP}^1$ is defined as follows:

   $$\mathbb{CP}^1 := \{[z_0 : z_1] : (z_0, z_1) \neq (0,0)\}.$$

   Here z_0, z_1 are complex numbers, and we declare $[z_0 : z_1] = [w_0 : w_1]$ if and only if $(z_0, z_1) = \lambda(w_0, w_1)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

   (a) Show that $\mathbb{CP}^1$ is a smooth manifold of (real) dimension 2.

   (b) Show that $\mathbb{CP}^1$ and the sphere $\mathbb{S}^2$ are diffeomorphic. [Hint: consider stereographic projections]

3. Consider the following equivalence relation $\sim$ defined on $\mathbb{R}^2$:

   $$(x, y) \sim (x', y') \iff (x', y') = \big((-1)^n x + m,\, y + n\big) \text{ for some integers } m \text{ and } n.$$

   (a) Sketch an edge-identified square to represent the quotient space $\mathbb{R}^2/\sim$.

   (b) Consider the two parametrizations of $\mathbb{R}^2/\sim$:

   $$\begin{aligned} \mathsf{G}_1 : (0,1) \times (0,1) &\to \mathbb{R}^2/\sim \\ (x,y) &\mapsto [(x,y)] \end{aligned} \qquad \begin{aligned} \mathsf{G}_2 : (0,1) \times (0.5,1.5) &\to \mathbb{R}^2/\sim \\ (x,y) &\mapsto [(x,y)] \end{aligned}$$

   Find the transition map $\mathsf{G}_2^{-1} \circ \mathsf{G}_1$.

   (c) Write down a diffeomorphism between $\mathbb{R}^2/\sim$ and the Klein bottle K in $\mathbb{R}^4$ described in Example 2.16.

4. Consider the following subset of $\mathbb{R}^2 \times \mathbb{RP}^1$

   $$M = \left\{ \Big((x_1, x_2), [y_1 : y_2] \Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \;\middle|\; x_1 y_2 = y_1 x_2 \right\}$$

(a) Show that M is a smooth 2-manifold by considering the following parametrizations:

$$\mathsf{F}(u_1, u_2) = \Big((u_1 u_2,\, u_2), [u_1 : 1]\Big)$$
$$\mathsf{G}(v_1, v_2) = \Big((v_1,\, v_1 v_2), [1 : v_2]\Big)$$

(b) Consider the two projection maps $\pi_1 : M \to \mathbb{R}^2$ and $\pi_2 : M \to \mathbb{RP}^1$ defined by:

$$\pi_1\Big((x_1, x_2), [y_1 : y_2]\Big) = (x_1, x_2)$$
$$\pi_2\Big((x_1, x_2), [y_1 : y_2]\Big) = [y_1 : y_2]$$

i. Show that $\pi_1^{-1}(p)$ is either a point, or diffeomorphic to $\mathbb{RP}^1$.
ii. Show that π_2 is a submersion.

5. The tangent bundle TM of a smooth n-manifold M is the disjoint union of all tangent spaces of M, i.e.

$$TM := \bigcup_{p \in M} \{p\} \times T_p M = \{(p, V_p) : p \in M \text{ and } V_p \in T_p M\}.$$

(a) Show that TM is a smooth $2n$-manifold. [Again, skip the topological parts, but show detail work of the differentiable parts.]
(b) Show that the map $\pi : TM \to M$ defined by $\pi(p, V_p) := p$ is a submersion.
(c) Define the subset $\Sigma_0 := \{(p, 0_p) \in TM : p \in M\}$ where 0_p is the zero vector in $T_p M$. This set Σ_0 is called the zero section of the tangent bundle. Show that Σ_0 is a smooth n-manifold diffeomorphic to M, and that it is a submanifold of TM.
(d) Now suppose M is just a C^k-manifold (where $k \geq 2$), then TM is a $C^{\text{what?}}$-manifold?

6. A *Lie group* G is a smooth manifold such that multiplication and inverse maps

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned} \qquad\qquad \begin{aligned} \nu : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are both smooth (C^∞) maps. As an example, $\mathrm{GL}(n, \mathbb{R})$ is a Lie group since it is an open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, hence it can be globally parametrized using coordinates of $\mathbb{R}^{n^2}$. The multiplication map is given by products and sums of coordinates in $\mathbb{R}^{n^2}$, hence it is smooth. The inverse map is smooth too by the Cramer's rule $A^{-1} = \frac{1}{\det(A)}\mathrm{adj(A)}$ and that $\det(A) \neq 0$ for any $A \in \mathrm{GL}(n, \mathbb{R})$.

(a) Recall that $T_{(e,e)}(G \times G)$ can be identified with $T_e G \oplus T_e G = \{(X, Y) : X, Y \in T_e G\}$.
i. Show that the tangent map of μ at (e, e) is given by:

$$(\mu_*)_{(e,e)}(X, Y) = X + Y.$$

ii. Show that μ is a submersion at (e, e).

(b) Show that the tangent map of ν at e is given by:

$$(\nu_*)_e(X) = -X.$$

[Hint for part (a): when taking partial derivative $\frac{\partial f}{\partial u}$ at $(u, v) = (u_0, v_0)$, it is OK to substitute $v = v_0$ first, and then differentiate $f(u, v_0)$ by u. It is possible to prove (b) using the result from (a)i and the manifold chain rule in an appropriate way.]