MATH 4033
LECTURES 07-09
WEEK 03

$\mathbb{R}^n$ Metric spaces

$d(x,y)$ ✓

$B_\varepsilon(\vec{x}) = \{y : d(x,y) < \varepsilon\}$.

$x_n \to x$ as $n \to \infty$.
$\overset{def}{\iff}$ $\forall$ open set $\mathcal{U} \ni x$,
$\exists N$ s.t. whenever $n \geq N$,
$x_n \in \mathcal{U}$

topological space $\approx$ a space where we can talk about "open sets".

X. $\mathcal{J}$ = collection of subsets of X

e.g. $X = \{1,2,3\}$. $\mathcal{J} = \{\{1\}, \{2,3\}, \{1,3\}, \ldots\}$ ← topology of X.

① $\phi, X \in \mathcal{J}$
② $\mathcal{U}_1, \ldots, \mathcal{U}_N \in \mathcal{J} \Rightarrow \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_N \in \mathcal{J}$
③ $\mathcal{U}_\alpha \in \mathcal{J} \Rightarrow \bigcup_\alpha \mathcal{U}_\alpha \in \mathcal{J}$.

$f: \mathbb{R} \to \mathbb{R}$ f is continuous on $\mathbb{R}$
$\iff f^{-1}(\mathcal{U})$ is open $\forall$ open set $\mathcal{U} \subset \mathbb{R}$.

$f: (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$
f is continuous on $X \overset{def}{\iff} \forall \mathcal{U} \in \mathcal{J}_Y$, $f^{-1}(\mathcal{U}) \in \mathcal{J}_X$.

Hausdorff ⟶

second countable
$(X, \mathcal{J}_X)$ $\exists\, \mathcal{O}_1, \mathcal{O}_2, \ldots \in \mathcal{J}_X$ (countable)
s.t. $\forall \mathcal{U} \in \mathcal{J}_X$
$\exists \{i_k\}$ s.t. $\mathcal{U} = \bigcup_{k=1}^\infty \mathcal{O}_{i_k}$

X
$\mathcal{J}_X = \{\varnothing, X\}$.

Definition (topological manifold)

M is a topological n-manifold
if ① M is a topological space (with topology ...) ← ① M is Hausdorff and 2nd countable
② $\forall p \in M$, $\exists\, \mathcal{O}$ open s.t. $\exists\, \mathcal{U} \subset \mathbb{R}^n \xrightarrow{F} \mathcal{O}_p \subset M$ homeomorphism

$\mathcal{U} \subset \mathbb{R}^3$
$\mathcal{J}_\Sigma = \{\mathcal{U} \cap \Sigma : \mathcal{U} \text{ open in } \mathbb{R}^3\}$.
$\Sigma \subset \mathbb{R}^3$ regular surface

Definition (smooth n-manifold)

$(M, \mathcal{J})$ is a topological n-manifold.
$\{F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{O} \subset M\}$ (open, open)
+ any overlapping parametrizations F and G have C^∞ transition maps $G^{-1} \circ F$ and $F^{-1} \circ G$.
⟶ C^∞ (smooth) n-manifold

e.g. any regular surface in $\mathbb{R}^3$ is a smooth 2-manifold.

$z = \sqrt{x^2+y^2}$ is a C^∞ manifold.

$F(x,y) = (x, y, \sqrt{x^2+y^2}) : \mathbb{R}^2 \to$ Cone.

homeo. $\mathcal{U} \subset \mathbb{R}^2$. ✗

$\cong S^1 \times S^1$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

M^m, N^n are C^∞ smooth manifolds.
$F(\vec{u})$, $G(\vec{v})$

$M^m \times N^n = \{(x,y) : x \in M, y \in N\}$.

$(F \times G)(\vec{u}, \vec{v}) = (F(\vec{u}), G(\vec{v})) \in M \times N$.
(in $\mathbb{R}^{m+n}$, in $\mathbb{R}^m \times \mathbb{R}^n$)

$(\tilde{F} \times \tilde{G})^{-1} \circ (F \times G)(\vec{u}, \vec{v})$
$= (\tilde{F} \times \tilde{G})^{-1}(F(\vec{u}), G(\vec{v}))$
$= (\underbrace{\tilde{F}^{-1} \circ F(\vec{u})}_{C^\infty}, \underbrace{\tilde{G}^{-1} \circ G(\vec{v})}_{C^\infty})$ ← $(\tilde{F} \times \tilde{G})(\tilde{F}^{-1} \circ F(\vec{u}), \tilde{G}^{-1} \circ G(\vec{v})) = (F(\vec{u}), G(\vec{v}))$

$\Rightarrow M \times N$ is a C^∞ $(m+n)$-manifold.

Cylinder $\cong \mathbb{R} \times S^1$

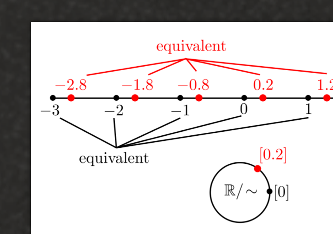
$(0, 2\pi) \ni \theta \mapsto e^{i\theta}$ $e^{i\varphi}$ $\varphi \in (-\pi, \pi)$

$F(z, \theta) = (z, e^{i\theta}) \in \mathbb{R} \times S^1$.

Klein bottle

Quotient set

$\mathbb{R}$
$x \sim y \overset{def}{\iff} x - y \in \mathbb{Z}$.
$[x] := \{y \in \mathbb{R} : x \sim y\}$.
$[1] = \{1, 2, 3, \ldots, 0, -1, -2, -3, \ldots\}$
$[2] = \{1, 2, 3, 4, \ldots, 0, -1, -2, -3, -4, \ldots\}$ } $[1] = [2]$.
$[1.5] = \{1.5, 2.5, 3.5, \ldots, 0.5, -0.5, -1.5, \ldots\} = [2.5]$



$[0] = [1] = [2] = \ldots$

$[0] = [1]$

$\mathbb{R}/\sim = \{[x] : x \in \mathbb{R}\} \cong$ ○

$\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$.

$(x,y) \sim (x', y') \overset{def}{\iff} \begin{cases} x - x' \in \mathbb{Z} \\ y - y' \in \mathbb{Z}. \end{cases}$
$(1,3) \sim (2,4) \sim (0,1)$
$(0.1, \pi) \sim (-0.9, \pi+3)$

$= \mathbb{R}^2/\sim = \{[(x,y)] : (x,y) \sim (x', y')\}$

$\mathbb{R}^2/\sim$?
$(x,y) \sim (x', y') \iff \begin{cases} y - y' \in \mathbb{Z} \\ \text{if } y - y' \text{ is even} \Rightarrow x - x' \in \mathbb{Z} \\ \text{if } y - y' \text{ is odd} \Rightarrow x + x' \in \mathbb{Z}. \end{cases}$

$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$
$\mathbb{R}^{n+1} \setminus \{0\} = \{(x_1, \ldots, x_{n+1}) : x_i \in \mathbb{R}\}$, not all zero.
$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$
$\overset{def}{\iff} \exists \lambda \in \mathbb{R} \setminus \{0\}$ s.t.
$(x_1, \ldots, x_{n+1}) = \lambda (y_1, \ldots, y_{n+1})$.

$(1, 0, 5) \sim (\frac{1}{2}, 0, \frac{5}{2})$
$[(1,0,5)] = [(\frac{1}{2}, 0, \frac{5}{2})] \neq [(1, 0, \frac{5}{2})]$

x_3, x_2, x_1

$\mathbb{RP}^2$

= Klein bottle

Möbius strip

$\mathbb{R}^2$ $\xrightarrow{\pi}$ $\mathbb{R}^2/\sim$

$\pi(x,y) := [(x,y)]$.

$F_{(0,0)} := \pi|_{(0,1)\times(0,1)} \to \subset \mathbb{R}^2/\sim$ 1-1.

$F_{(\frac{1}{2},0)} := \pi|$

$\{F_{(a,b)} := \pi| : \to \mathbb{R}^2/\sim\}$.

$F_{(0,0)}$, $F_{(\frac{1}{2},0)}^{-1} \circ F_{(0,0)}$, $F_{(\frac{1}{2},0)}$
$[(\frac{1}{2}, \frac{1}{2})]$

$F_{(\frac{1}{2},0)}^{-1} \circ F_{(0,0)}(x,y) = \begin{cases} (x+1, y) & 0 < x < \frac{1}{2} \\ (x,y) & \frac{1}{2} < x < 1 \end{cases}$ (is C^∞).

Similarly. $F_{(a,b)}^{-1} \circ F_{(a',b')}(x,y) = \begin{cases} \text{translation 1} \\ \text{translation 2} \end{cases}$

$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$
$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$
$\overset{def}{\iff} \exists \lambda \in \mathbb{R} \setminus \{0\}$ s.t. $(x_1, \ldots, x_{n+1}) = \lambda (y_1, \ldots, y_{n+1})$
$[(x_1, \ldots, x_{n+1})] =: [x_1 : \cdots : x_{n+1}]$
$[1 : 0 : \pi] = [\frac{1}{\pi} : 0 : 1]$

$\mathbb{RP}^2$

$F_i : \mathbb{R}^n \to \mathcal{O}_i \subset \mathbb{RP}^n$
$F_i(u_1, \ldots, u_n) := [u_1 : \cdots : u_{i-1} : \underset{i}{1} : u_i : \cdots : u_n]$

e.g.
$F_1(u_1, \ldots, u_n) = [1 : u_1 : u_2 : \cdots : u_n] \in \mathcal{O}_1 = \{[x_1 : \cdots : x_{n+1}] : x_1 \neq 0\}$
$F_2(v_1, \ldots, v_n) = [v_1 : 1 : v_2 : \cdots : v_n] \in \mathcal{O}_2 = \{[x_1 : \cdots : x_{n+1}] : x_2 \neq 0\}$

$F_2^{-1} \circ F_1(u_1, \ldots, u_n)$: $F_1^{-1}(\mathcal{O}_1 \cap \mathcal{O}_2) \to F_2^{-1}(\mathcal{O}_1 \cap \mathcal{O}_2)$ $\{u_1 \neq 0\}$
$= F_2^{-1}([1 : u_1 : u_2 : \cdots : u_n]) = F_2^{-1}([\frac{1}{u_1} : 1 : \frac{u_2}{u_1} : \cdots : \frac{u_n}{u_1}])$
$= (\frac{1}{u_1}, \frac{u_2}{u_1}, \ldots, \frac{u_n}{u_1})$ is C^∞ on $F_1^{-1}(\mathcal{O}_1 \cap \mathcal{O}_2)$.

If $[1 : u_1 : \cdots : u_n] = [1 : \tilde{u}_1 : \cdots : \tilde{u}_n]$
$\Rightarrow u_i = \tilde{u}_i$ $\forall i$.

$\mathbb{RP}^n$ is a C^∞ n-manifold.

Similarly $\mathbb{CP}^n$ is a C^∞ $2n$-manifold.
Einstein manifold.
Kähler manifold.
algebraic geom

← stereographic $F_1 : \mathcal{U} \to \mathcal{O}$.
$F_2 : \mathcal{U}_2 \to \mathcal{O}_2$
$(u_1, u_2) \mapsto (u_1, u_2, \sqrt{1 - u_1^2 - u_2^2})$
$F_3(\theta, \varphi)$

$\{F_1, F_2, F_3(\theta,\varphi), F_4, \ldots\} = \mathcal{D}$
atlas. Differential structure.

$\mathbb{R}^2$ $(\mathbb{R}^2, \mathcal{D})$
$\{F_0(x,y) = (x,y) : \mathbb{R}^2 \to \mathbb{R}^2\}$ atlas $(\mathbb{R}^2, \mathcal{D}')$.
$\{F_1(r,\theta) = (r\cos\theta, r\sin\theta) : \mathbb{R}_+ \times (0, 2\pi)\}$
$F_1^{-1} \circ F_0$. inverse function theorem
$F_0^{-1} \circ F_1$ trivial C^∞

$\mathcal{D} = \{F_0, F_1, \ldots\}$

$F_0^{-1} \circ G(x,y)$
$= F_0^{-1}(x, y + |x|)$
$= (x, y + |x|)$.

$G(x,y) = (x, y + |x|) : \mathbb{R}^2 \to \mathbb{R}^2$ 1-1, onto.
$\mathcal{D}' = \{G, \ldots\}$

$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$
$\mathbb{C}^{n+1} \setminus \{0\} \ni (z_1, \ldots, z_{n+1}) \sim (w_1, \ldots, w_{n+1})$
$\overset{def}{\iff} \exists \lambda \neq 0$, $\lambda \in \mathbb{C}$ s.t.
$(z_1, \ldots, z_{n+1}) = \lambda (w_1, \ldots, w_{n+1})$.

$\mathbb{CP}^1 = S^2$?
$[z_1 : z_2]$
$[1 : i] = [i, -1] = [\frac{1}{2}, \frac{i}{2}]$

$\{[z_1 : z_2]\}$ < $\{[z_1 : z_2] : z_1 \neq 0\}$ = $[1 : \frac{z_2}{z_1}]$ = $\{[1 : w] : w \in \mathbb{C}\}$
$\{[0 : z_2]\} = \{[0 : 1]\}$ ← ∞ ($z_2 \neq 0$)