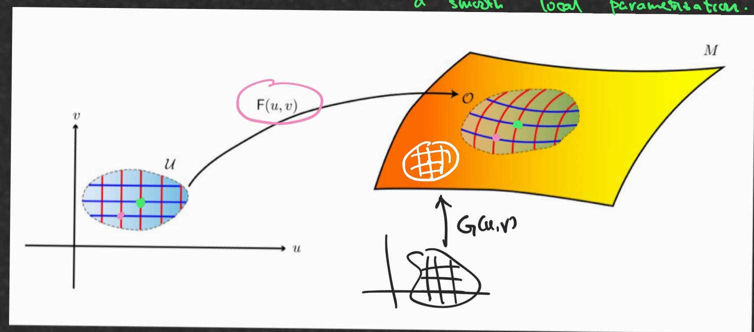
MATH 4033
LECTURES 01-03
WEEK 01

Chapter 1: Regular Surfaces

M is a regular surface if every point can covered by a smooth local parametrization.



$\vec{F}(u,v): \mathbb{R}^2 \to \mathbb{R}^3$

Sphere ($\rho = 1$)
$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

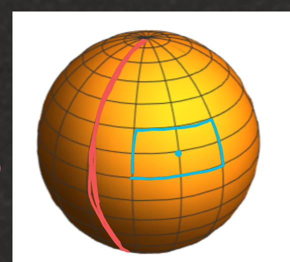
Ex. $F(\varphi_1,\theta_1) = F(\varphi_2,\theta_2) \Rightarrow \begin{cases}\varphi_1 = \varphi_2 \\ \theta_1 = \theta_2\end{cases}$

$F(\varphi,\theta) = (x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta))$
$= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

$\vec{F}(\theta,\varphi): \underbrace{(0,\pi)}_{\varphi} \times \underbrace{(0,2\pi)}_{\theta} \to \mathbb{R}^3$

local parametrization

1-1, onto

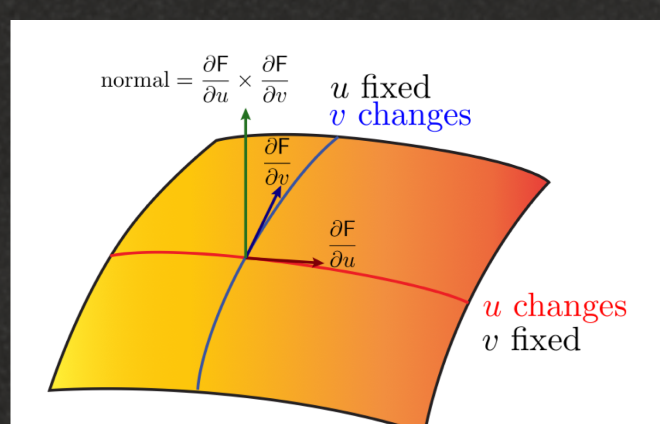


Definition 1.1 (smooth local parametrization)

$\vec{F}(u,v): \underset{\text{open}}{\mathcal{U}} \subset \mathbb{R}^2 \to \mathbb{R}^3$

is a smooth local parametrization def

1. $\vec{F}: \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ is C^∞.
2. $\vec{F}: \mathcal{U} \to \vec{F}(\mathcal{U})$ is a homeomorphism
   ($\vec{F}$ is continuous and $\vec{F}: \mathcal{U} \to \vec{F}(\mathcal{U})$ is invertible and $\vec{F}^{-1}$ is continuous.)
3. $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0 \quad \forall (u,v) \in \mathcal{U}$.



$F(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

$\frac{\partial F}{\partial \varphi} = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

$\frac{\partial F}{\partial \theta} = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$$\frac{\partial F}{\partial\varphi}\times\frac{\partial F}{\partial\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$= (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, \cos\varphi\sin\varphi)$

$\left|\frac{\partial F}{\partial\varphi}\times\frac{\partial F}{\partial\theta}\right| = (\sin^4\varphi + \cos^2\varphi\sin^2\varphi)^{\frac{1}{2}}$

$= \sin\varphi \neq 0 \Leftarrow \varphi \in (0,\pi)$

$f(x,y): \mathbb{R}^2 \to \mathbb{R}$, C^∞.

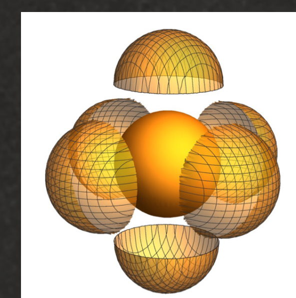
$\{(x,y,f(x,y)) : (x,y) \in \mathbb{R}^2\} \leftarrow$ graph of f.

$\vec{F}(x,y) = (x,y,f(x,y)): \mathbb{R}^2 \to \mathbb{R}^3$.

1. ✓
2. 1-1 trivial.
   $\vec{F}^{-1}(x,y,z) = (x,y)$ continuous ✓
3. $\frac{\partial F}{\partial x} \times \frac{\partial F}{\partial y} = (\ \cdot\ ,\ \cdot\ , \pm 1) \neq 0$. ✓

$\vec{F}_1(x,y) = (x,y,\sqrt{1-x^2-y^2})$
$: \{(x,y) : x^2+y^2 < 1\} \to \mathbb{R}^3$

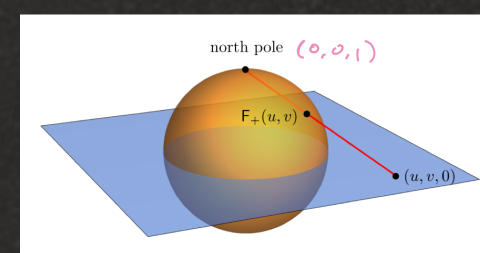
$\vec{F}_2(x,y) = (x,y,-\sqrt{1-x^2-y^2})$



$\vec{F}_+ : \underset{(u,v)}{\mathbb{R}^2} \to \mathbb{R}^3$

$\vec{F}_+(\mathbb{R}^2)$

$=$ sphere $\backslash \{$north pole$\}$.



$\vec{r}(t) = (0,0,1) + t(u,v,-1)$
$= (tu, tv, 1-t)$
$x^2+y^2+z^2 = 1$

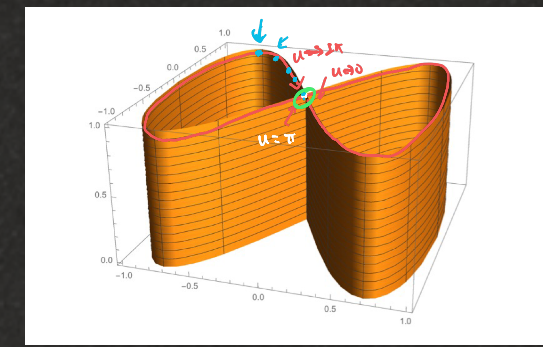
$(tu)^2 + (tv)^2 + (1-t)^2 = 1$
$t^2(u^2+v^2) + 1 - 2t + t^2 = 1$
$t(u^2+v^2+1) = 2$
$\Rightarrow t = \frac{2}{u^2+v^2+1} \rightsquigarrow \vec{r}(t) = \underbrace{\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)}_{\vec{F}_+(u,v)}$

$(x,y,z) = \vec{F}_+(u,v)$
$x = 2u/(u^2+v^2+1)$, $z = \frac{u^2+v^2-1}{u^2+v^2+1} \Rightarrow u^2+v^2 = \frac{1+z}{1-z}$, $u = \frac{x}{1-z}$, $v = \frac{y}{1-z}$
$y = 2v/(u^2+v^2+1)$

$\Rightarrow \vec{F}_+^{-1}(x,y,z) = \underbrace{\left(\frac{x}{1-z}, \frac{y}{1-z}\right)}_{\text{continuous on}}$

$\vec{F}_+: \mathbb{R}^2 \to S^2 \backslash \{(0,0,1)\}$ North pole.

$\vec{F}_+^{-1}: S^2 \backslash \{(0,0,1)\} \to \mathbb{R}^2$



$\vec{F}(u,v): (0,2\pi)\times(-\infty,\infty) \to \mathbb{R}^3$
$= (\sin u, \sin 2u, v)$

$F(u_n,v_n) \to$ ●

$\exists (u_n,v_n) \in (0,2\pi)\times\mathbb{R}$ div.

$\vec{F}(u_n,v_n)$ converges.

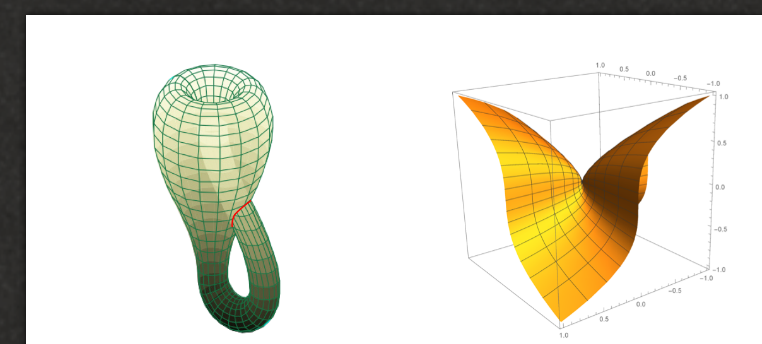
$\vec{F}^{-1}(\vec{F}(u_n,v_n))$

$\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 1$ level set of g.

Theorem 1.6

$\Sigma = \{g(x,y,z) = c\} \neq \emptyset$.
If $\nabla g \neq 0 \ \forall (x,y,z) \in \Sigma$, then Σ is a regular surface.

Example: $g(x,y,z) = x^2+y^2+z^2$
$\nabla g = (2x, 2y, 2z) = \vec{0} \Leftrightarrow (x,y,z) = (0,0,0)$.
$\underset{\text{unit sphere}}{\Sigma := \{g = 1\}}$, $(0,0,0) \notin \Sigma$. $\Rightarrow \Sigma$ is a regular surface.



(a) Klein Bottle has a self-intersection. (b) $F(u,v) = (u^3, v^3, uv)$ fails condition (3).

$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)_p \neq 0$

WLOG, assume $\frac{\partial g}{\partial z}(p) \neq 0$.
$\nabla g(p)$ NOT horizontal.
$\Rightarrow$ tangent plane at p NOT vertical.
Around p, Σ is locally a graph over (x,y) of $f(x,y)$.

$\nabla g(p) \neq \vec{0}$
$F(x,y) = (x,y,f(x,y))$
$\Sigma : \{g(x,y,z) = c\}$

Prop. 1.8 $\Sigma = \{g(x,y,z) = c\} \neq \emptyset$.
If $\nabla g \neq 0 \ \forall (x,y,z) \in \Sigma$.
$F(u,v): \mathcal{U} \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$.
1. C^∞ ✓
3. $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0 \ \forall (u,v) \in \mathcal{U}$.

2'. $F: \mathcal{U} \to \mathcal{O} := F(\mathcal{U})$ bijective

$\Rightarrow F^{-1}: \mathcal{O} \to \mathcal{U}$ is continuous.

eg. $F(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi): \underset{\varphi}{(0,\pi)} \times \underset{\theta}{(0,2\pi)} \to \Sigma$

$F(u,v)$, $\mathcal{O} = F(\mathcal{U})$, $g^{-1}(c)$
$\pi(x,y,z) = (x,y)$
$F(u,v) = (x(u,v), y(u,v), z(u,v))$

Given: $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq \vec{0}$. at p

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$= \det\frac{\partial(y,z)}{\partial(u,v)}\hat{i} - \det\frac{\partial(x,z)}{\partial(u,v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(u,v)}\hat{k}$

$J f(p) \neq 0$

WLOG $\det\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ at p.

$\pi \circ F(u,v)$
$= (x(u,v), y(u,v))$

$\to (\pi\circ F)^{-1}$ exists locally. and is C^∞.

$F^{-1} = \underset{\text{conti.}}{(\pi\circ F)^{-1}} \circ \underset{\text{conti.}}{\pi} \leftarrow$ because $(\pi\circ F)\circ F^{-1} = \pi$
$\Rightarrow F^{-1}$ continuous. □

$\mathcal{O}_\alpha \cap \mathcal{O}_\beta$

M, $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$, F_α, F_β

$\underset{\subset \mathbb{R}^2}{\mathcal{U}_\alpha}$, $F_\alpha^{-1}(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$, $F_\beta^{-1}\circ F_\alpha$, $\mathcal{U}_\beta$, u_1, u_2, v_1, v_2

Goal: Show $F_\beta^{-1}\circ F_\alpha : F_\alpha^{-1}(\mathcal{O}_\alpha\cap\mathcal{O}_\beta) \to F_\beta^{-1}(\mathcal{O}_\alpha\cap\mathcal{O}_\beta)$. is C^∞ for regular surfaces in $\mathbb{R}^3$.

Prop. 1.11 transition maps for regular surfaces are C^∞.

$F_\alpha(u_1,u_2) = (x(u_1,u_2), y, z)$
$\frac{\partial F_\alpha}{\partial u_1} \times \frac{\partial F_\alpha}{\partial u_2} \neq 0$. at p

normal at p
$\frac{\partial F_\beta}{\partial v_1} \times \frac{\partial F_\beta}{\partial v_2} \neq 0$.

$\pi(x,y,z) = (x,y)$
$F_\beta(v_1,v_2) = (x(v_1,v_2), \ldots)$

WLOG Assume $\det\frac{\partial(x,y)}{\partial(u_1,u_2)} \neq 0$ at p.

$\det\frac{\partial(x,y)}{\partial(v_1,v_2)} \neq 0$ at p.

$\pi\circ F_\alpha(u_1,u_2)$
$= (x(u_1,u_2), y(u_1,u_2))$
$\overset{\text{IFT}}{\Rightarrow} \pi\circ F_\alpha$ is locally smoothly invertible near p.

$\overset{\text{IFT}}{\Rightarrow} \pi\circ F_\beta$ is locally smoothly invertible near p.

$F_\beta^{-1}\circ F_\alpha = (\pi\circ F_\beta)^{-1}\circ(\pi\circ F_\alpha)$
$F_\alpha^{-1}\circ F_\beta = (\pi\circ F_\alpha)^{-1}\circ(\pi\circ F_\beta)$ $\Big\}$ C^∞ at p.