MATH 4033 • Spring 2018 • Calculus on Manifolds
Problem Set #1 • Regular Surfaces • Due Date: 25/02/2018, 11:59PM (Optional)

Remark: *All assignments are optional but you are strongly recommended to work on them to keep yourself on track. You are welcome to submit any part of your homework to the Canvas system by the deadline. Follow the instructions posted on Canvas. The instructor and TA will give you some feedback as soon as possible.*

1. Consider a smooth map $\mathsf{F}(u,v) : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ where $\mathcal{U}$ is an open set. Denote:

$$\mathsf{F}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

Show that the following are equivalent:

(a) $\dfrac{\partial \mathsf{F}}{\partial u} \times \dfrac{\partial \mathsf{F}}{\partial v} \neq 0$ for any $(u,v) \in \mathcal{U}$.

(b) $\left\{\dfrac{\partial \mathsf{F}}{\partial u}, \dfrac{\partial \mathsf{F}}{\partial v}\right\}$ are linearly independent for any $(u,v) \in \mathcal{U}$.

(c) The Jacobian matrix:

$$\frac{\partial(x,y,z)}{\partial(u,v)} := \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$

has a trivial null-space for any $(u,v) \in \mathcal{U}$.

(d) For any $(u,v) \in \mathcal{U}$, at least one of the following Jacobian matrices is invertible:

$$\frac{\partial(x,y)}{\partial(u,v)} \quad \frac{\partial(y,z)}{\partial(u,v)} \quad \frac{\partial(z,x)}{\partial(u,v)}$$

(e) The matrix:

$$[g] := \begin{bmatrix} \mathsf{F}_u \cdot \mathsf{F}_u & \mathsf{F}_u \cdot \mathsf{F}_v \\ \mathsf{F}_v \cdot \mathsf{F}_u & \mathsf{F}_v \cdot \mathsf{F}_v \end{bmatrix}$$

is positive definite[1] for any $(u,v) \in \mathcal{U}$.

2. Let A be a 3×3 matrix with real entries $[a_{ij}]$. Consider the set

$$\Sigma := \{\mathsf{x} \in \mathbb{R}^3 : \mathsf{x} \cdot A\mathsf{x} = 1\}.$$

Here $\mathsf{x} \in \mathbb{R}^3$ is regarded as a column vector.

(a) Show that Σ is a regular surface whenever $\Sigma \neq \emptyset$.

(b) Suppose further that $A = P^T D P$ for some orthogonal matrix P (i.e. $P^T P = I$) and diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

i. Show that Σ is diffeomorphic to $\mathbb{S}^2$ if $\lambda_i > 0$ for all i.

ii. Show that Σ is diffeomorphic to the cylinder $x^2 + y^2 = 1$ if $\lambda_1, \lambda_2 > 0$, and $\lambda_3 = 0$.

[1]A symmetric matrix being positive definite means all of its eigenvalues are positive.

3. Let $\mathbb{S}^2$ be the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$. Suppose $f:\mathbb{S}^2\to(0,\infty)$ is a smooth, positive-valued function. Consider the set Σ defined by:

$$\Sigma := \{f(\mathsf{x})\,\mathsf{x} : \mathsf{x}\in\mathbb{S}^2\}.$$

   (a) Suppose $\mathsf{F}(u,v):\mathcal{U}\to\mathbb{S}^2$ is a smooth local parametrization of $\mathbb{S}^2$. Show that:

$$\begin{aligned}&\mathsf{G}:\mathcal{U}\to\Sigma\\&(u,v)\mapsto f\big(\mathsf{F}(u,v)\big)\,\mathsf{F}(u,v)\end{aligned}$$

   is a smooth local parametrization of Σ. Hence, show that Σ is a regular surface.

   (b) Let $\mathsf{F}_i(u,v):\mathcal{U}_i\to\mathbb{S}^2$, where $i=1,2$, be two overlapping smooth local parametrizations of $\mathbb{S}^2$, and $\mathsf{G}_i:\mathcal{U}_i\to\Sigma$ be the parametrization of Σ induced by F_i. Show that $\mathsf{G}_1^{-1}\circ\mathsf{G}_2=\mathsf{F}_1^{-1}\circ\mathsf{F}_2$.

   (c) Show that $\mathbb{S}^2$ and Σ are diffeomorphic. Write down the diffeomorphism explicitly.

   (d) Define a map $\Phi:\mathbb{R}^3\backslash\{(0,0,0)\}\to\mathbb{R}^3\backslash\{(0,0,0)\}$ by:

$$\Phi(\mathsf{x})=\frac{\mathsf{x}}{|\mathsf{x}|^2}.$$

   Denote $\Sigma^*:=\Phi(\Sigma)$, i.e. the image of Σ under the map Φ.

      i. Explain why Σ^* is also a regular surface.
      ii. Let $\phi:\Sigma\to\Sigma^*$ be the restriction of Φ on Σ. Show that ϕ is a diffeomorphism.
      iii. Show that for any $p\in\Sigma$, the tangent map $(\phi_*)_p:T_p\Sigma\to T_{\phi(p)}\Sigma^*$ at p is:

$$(\phi_*)_p(V)=\frac{|p|^2\,V-2(p\cdot V)p}{|p|^4}$$

      where $|p|$ is the norm of p in $\mathbb{R}^3$, and $p\cdot V$ is the usual dot product of p and V in $\mathbb{R}^3$.

4. Let F_+ and F_- be the stereographic parametrizations of the unit sphere $\mathbb{S}^2$ as discussed in Example 1.5 of the lecture notes. Here we regard $\mathbb{C}$ as $\mathbb{R}^2$ by identifying $z=u+iv\in\mathbb{C}$ with $(u,v)\in\mathbb{R}^2$. Then, $\mathsf{F}_+:\mathbb{C}\to\mathbb{S}^2\backslash\{(0,0,1)\}$ and its inverse can be expressed as:

$$\mathsf{F}_+(z)=\left(\frac{2\mathrm{Re}(z)}{|z|^2+1},\frac{2\mathrm{Im}(z)}{|z|^2+1},\frac{|z|^2-1}{|z|^2+1}\right)\qquad \mathsf{F}_+^{-1}(x_1,x_2,x_3)=\frac{x_1+x_2 i}{1-x_3}$$

Here we use (x_1,x_2,x_3) for coordinates of $\mathbb{R}^3$ instead of (x,y,z) to avoid notation conflicts.

   (a) Consider the south-pole stereographic parametrization $\mathsf{F}_-:\mathbb{C}\to\mathbb{S}^2\backslash\{(0,0,-1)\}$. Find the explicit expressions of $\mathsf{F}_-(z)$, where $z\in\mathbb{C}$, and $\mathsf{F}_-^{-1}(x_1,x_2,x_3)$, where $(x_1,x_2,x_3)\in\mathbb{S}^2\backslash\{(0,0,-1)\}$.

   (b) Verify that:

$$\mathsf{F}_-^{-1}\circ\mathsf{F}_+(z)=\frac{1}{\bar{z}}\qquad \mathsf{F}_+^{-1}\circ\mathsf{F}_-(z)=\frac{1}{\bar{z}}.$$

   State the domains on which they are defined.

(c) Consider the complex-valued function $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$ such that $\alpha\delta \neq \beta\gamma$. Define a map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by:

$$\Phi(p) := \begin{cases} \mathsf{F}_+(\alpha/\gamma) & \text{if } p = (0,0,1) \\ (0,0,1) & \text{if } p = \mathsf{F}_+(-\delta/\gamma) \\ \mathsf{F}_+ \circ f \circ \mathsf{F}_+^{-1}(p) & \text{otherwise} \end{cases}$$

i. Show that Φ is bijective.

ii. Find an explicit expression of each of the following:

$$\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_+(z) \qquad \mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_+(z) \qquad \mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_-(z) \qquad \mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-(z)$$

State the domain of each of them.

iii. Show that Φ is smooth at the point $(0,0,1)$.

iv. Show that tangent map Φ_* at $(0,0,1)$ is invertible.